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TEACHER NOTICING OF STUDENT THINKING: AN ANALYSIS OF A TEACHER'S INTERPRETATION OF MATHEMATICS PROBLEM SOLVING[1]

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Abstract: *One area of particular difficulty in mathematics learning is the transition from arithmetic to algebra. In this paper, we report on results from a project whose goal was to produce mathematical tasks and strategies to support the development of holistic relational thinking in problem solving among middle school students. Based on a teacher's written journals, we focus on her noticing of student work while solving problems provided in our project. Our results show that the teacher would benefit from paying closer attention to students who generally succeed in mathematical problem solving, and from strengthening her ability to interpret a wider range of possible representations, especially those that do not conform to the representation usually used.*

Keywords: *teacher noticing, mathematics problem solving, relational thinking, algebra.*

1. Introduction

One area of particular difficulty in mathematics learning is the transition from arithmetic to algebra. In elementary school, problems are solved using arithmetic, i.e., concrete numbers and operations on them; in secondary school, however, they are solved using algebra, i.e., equations, inequalities, functions, knowledge of structures and relationships (Cai & Knuth, 2011; Kieran, 1989; Schmidt & Bednarz, 2002). Within the arithmetic-algebra transition, problems become mathematically more complex. Students, who have not developed more sophisticated arithmetic strategies and

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continue to rely on the concrete numerical data and sequential operations on them, experience severe difficulties. One of the reasons for this difficulty is that numerical-sequential strategies are not effective in solving problems with a “disconnected” structure (Bednarz & Janvier, 1996), i.e., problems where “no direct bridging can be established between the known quantities” (p. 123) to obtain the unknown quantities. As these students move on to secondary school, their problem-solving difficulties are compounded with difficulties in algebra. Thus, at the age of 12-14 years, many students experience difficulties in mathematics, failing to solve complex mathematical problems, and struggling with algebra. Relational thinking about the whole structure that focuses on the relations among quantities in a problem and that promotes holistic relational thinking (Polotskaia, 2015) – is offered as an operational framework to helping students develop algebraic thinking.

Elementary school teachers are often ill-prepared to help students develop strategies for solving more complex and disconnected problems; they, too, tend to approach problems in a numerical-sequential manner. Special professional-development programs that focus on relational thinking in problem solving have been identified as effective in introducing much needed change in instruction. One such program, based on Thompson’s distinction between “numerical” and “quantitative” thinking and the concept of “relationally complex situations” (Thompson, 1993) has been successfully tried in the USA (Brown, 2012). We have successfully implemented a similar program in Quebec working with elementary school teachers on an Equilibrated Development Approach to teaching solving arithmetic problems to 6 to12 year-old children (Polotskaia, Savard, & Fellus, 2019). Our current project aimed at expanding the developed approach to problem solving within the context of transition from arithmetic to algebra.

2. Research goals

The leading perception in our project is that holistic relational thinking is an effective problem-solving tool. Therefore, our objective was to produce new instructional materials (such as a system of mathematical tasks) and strategies (such as specific types of teacher intervention) to support the development of holistic relational thinking in problem solving among middle school students, especially those struggling with mathematics, and thus facilitate the latter’s transition from arithmetic to algebra.

In this paper, we focus on one of our teachers’ noticing student work while solving problems provided in our project. Our analysis is based on Jacobs Lamb, and Philipp (2010) conceptualization of teacher noticing of students’ mathematical understanding. According to the authors, three interrelated skills are defined: a) attending to specific mathematical elements in students’ answers; b) interpreting students’ mathematical understanding considering the mathematical elements identified in students’ answers; and c) deciding how to respond on the basis of students’ mathematical understanding.

2. Methods

During a period of a two-year project, we worked closely with two teacher-participants to design a collection of word problems, the easiest of which requiring one arithmetic operation and the most difficult requiring one algebraic equation with multiple operations for their solution. We used the problem-solving cycle model (Savard, 2008; Polotskaia, 2015) to design specific teaching strategies. The first year of the research project was devoted to the design and class testing of the various problem-solving activities. This period also allowed the teacher-participants to appropriate the approach to teaching relationally—an approach which was new for them by their admission. During the second year, the teacher-participants used the relational teaching approach and the developed teaching tools in an autonomous manner. They regularly reported on their in-class observations related to the implementation of the approach. Drawing on Mason's (2011) definition of noticing as: "a collection of practices designed to sensitize oneself so as to notice opportunities in the future in which to act freshly rather than automatically out of habit" (p. 35), we frame noticing as a teacher's AHA moments as points of transition from automated behaviour to focused attention to specific student behaviours.

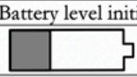
While we video recorded the lessons implemented by one of the teachers, the other teacher participant, a female teacher, recorded her implementation of relational thinking by keeping a reflection journal following the use of each of the word problems specially designed for the purposes of this project. The journal entries included a presentation of all of the students' work followed by comments made by the teacher and a series of hypotheses regarding how the lesson structure promoted or restricted student reasoning. In this paper, we analyse these journal entries in order to understand what the teacher notices with regards to students' thinking on the tasks developed within our research project. In short, we aimed at answering the following research question: What is the nature of the teacher's noticing of student thinking with regards to algebraic problems?

We present our results based on two illustrative examples of problems that we used during the project. The first is a simple problem that is comprised of two different structures: comparison and part-part-whole. Throughout the article we refer to this problem as the simple problem. The second one is a multi-step problem that requires students to analyse multiple situations and the restrictions imposed by the word problem. This problem deals with multiple proportions and ratios, and we refer to it as the complex problem. Figures 1 and 2 show each of the problems.

Jean-François is a stylist and he has recently renovated his boutique. He analyzed his expenses and realized that he had spent 3 times more in furniture and decoration than in his work materials. In total, he spent \$ 7,500.00. How much did he spend in work materials?

Fig. 1. *Simple problem*

During a school trip, a group of students are waiting for their flight at the airport. Arriving at the destination, Mr. Girard, their teacher, will need a GPS for 10 minutes to find the way to the hotel. Mr. Girard asks his students if he can borrow one of their cell phones. To make sure that the GPS will work long enough, the phone must be at least 60% charged. In addition, as the phones will be roaming, the cost of the data must not exceed $ 15. Four students propose to use their phone. They have 30 minutes to recharge their device before boarding. Here is information about the students' devices:

Student	Data cost for the GPS	Battery level initially	Information about charging
Benoît	$ 3.25 every 5 minutes	 30%	Every one hour, half of the battery is recharged.
Jean-Sébastien	$ 17		Every 15 minutes, the battery recharges 2 bars.
Éliane	$ 8		A complete charge takes 45 minutes.
Maryse	$ 0.50 per minute		1/8 of the battery is recharged every 10 minutes.

Among the students who propose using their phone, which one can Mr. Girard choose to: 1) make sure the charge of the battery will be sufficient after the recharging period and, 2) the cost of the data for the GPS will be less than $ 15?

Fig. 2. *Complex problem*

3. Results

Based on the theoretical framework proposed by Jacobs Lamb, and Philipp (2010), two main themes emerged in our data, the first one being what the teacher notices with regards to the mathematical procedure during the problem, and the second being the teacher's noticing of the representations done by the students in order to assist in their reasoning. We also analysed the moments in which the teacher provided general or no comments, which comprised another theme in our analysis. We start by introducing such theme. Table 1 shows a summary of the findings.

3.1. General or no comments

For a considerable number of students, the teacher did not provide any comment regarding their mathematical reasoning (either the procedure or the representation). In fact, the teacher did not write anything for over 17% of the students, indicating that she did not notice anything worth of mention. In fact, all of these students happened to be the ones who received full marks on their tasks, indicating that nothing was wrong in their mathematical procedure or representation.

Teacher noticing of student work in mathematics problems Table 1

Themes		Simple Problem	Complex Problem
Noticing of math procedure	Comment if it is wrong	11.8%	0.0%
	Describe student work	47.1%	35.3%
	Analyze student work	11.8%	29.4%
Noticing of representation	Comment if it is right or wrong	47.1%	82.4%
	Describe student work	5.9%	11.8%
	Analyze the student work	11.8%	29.4%
General comment		5.9%	41.2%
No comment at all		17.6%	0.0%

Another portion of the students, mostly in the complex problem, received comments from the teacher about general aspects of their learning, not necessarily connected to the mathematics of the problem. These comments were usually about the student engagement in the task, or their usual performance in class. For example, one of the students did not express themselves well in class during discussions, for which the teacher noted in her journal: “Needs help, it’s a student who does not express himself”.

3.2. Teacher noticing of mathematical procedure

While talking about the mathematical procedure carried out by the students who were solving both types of problems, the teacher noticed three different aspects. First, she noticed whether the procedure was correct or not, which happened for over 11% of the students. For instance, one of the students received a comment stating a “procedural error”, with no detail of which procedure was done wrong.

The second aspect noticed by the teacher is related to the specific description of student work. As mentioned before, most students who received written comments in their work were students that did not get full marks in their assessment, therefore the teacher noticed and described the procedural mistake carried out by the student. These comments vary in details, and include the following: “two different trials: 7500 ÷ 4 and 7500 ÷ 3, none is solved”; “subtracted minutes from %”.

A third category of comments with respect to mathematical procedure was comprised of analyses that the teacher proposed to interpret the student work. Most often, these analyses came with a description of the work, but also included ideas about what led the student to make such mistakes. For instance, while explaining that a student “subtracted minutes from %”, the teacher added that the student has “difficulties to treat two types of information”.

3.3. Teacher noticing of mathematical representation

This theme generally follows the same structure of the previous one, but it describes what the teacher noticed with regards to the students’ visual representations of the problems. As part of this research project, we emphasized the importance of visual representation of additive and multiplicative structures in algebraic problem solving. Identifying the proper representation of quantities can help students model mathematics problems and develop their algebraic reasoning.

We can see that the vast majority of the comments were generated by an assessment of whether the representation was right or wrong. In other words, whenever the representation designed by the students matched with what the teacher expected, the student received a comment as “divided by 5 despite a well-made scheme” or “excellent scheme, but wrong choice of operation”. If the representation did not meet the criteria, or sometimes if the student did not draw a visual representation, the teacher wrote “inappropriate scheme” or “no scheme”.

For a small number of students, the teacher provided a clear description of the visual representation, such as “difficulty to place the numbers that represent the whole and the part”. For another group of students, the teacher provided an analysis of what their representation mean (either mathematically or in the context of the problem): “this scheme clearly replaces the rule of three, which was not well understood previously”. These analyses also included moments in which the teacher recognizes that she does not understand what the student did: “scheme difficult to understand, I don’t know which reasoning the student used”, or “I question myself whether it would have been better to use a comparative scheme”.

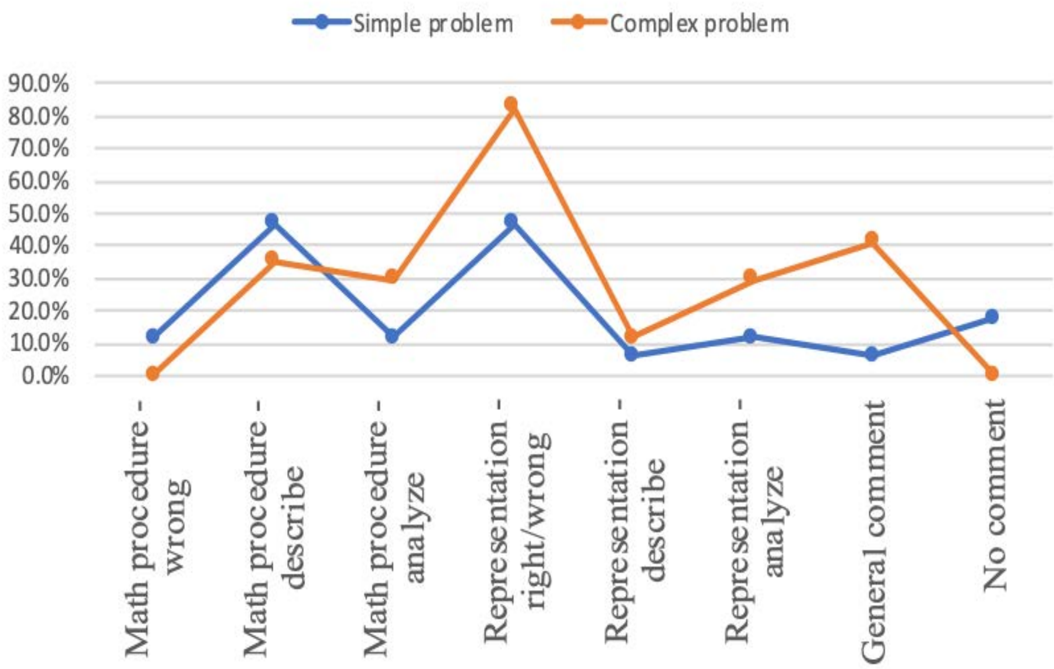


Fig. 3. *Teacher noticing of student work in the simple and complex problems*

4. Discussion

As presented in our results, our participant teacher mostly noticed aspects related to the mathematical procedure the representation carried out by the students. Attending to evidence when analysing student work is an important competence in developing high quality teaching practices and assessing student work. In that sense, our teacher participant seems to have developed a specific ability to describe mathematical procedures, more so than she did for the visual representations. We find this information consistent with our research project. After all, although the project intended to strengthen students' competence in mathematics through problem solving and representation, our teacher participant (as well as the other collaborating teacher) had to go through a learning process in order to implement the usage of visual representations of the relational aspects in the word problems (mostly linear diagrams). Since research shows that teaching practices and teacher noticing are correlated, it is understandable that noticing would reflect current teaching practices (more focused on procedure).

When analysing our participant's noticing of mathematical procedure, we can see that the teacher provided a nuanced understanding with regards to identifying the mistakes students made. The teacher does not provide any comments or interpretation whenever students get the answer correctly. In doing so, the teacher runs the risk of not assessing students' thinking to the same degree of depth as that of struggling students. This engenders what we believe to be a remedial approach to teaching, in which attention is dedicated to addressing misconceptions only when they already become an explicit obstacle to learning (in this case, explicated by students' failure to solve a problem).

Noticing the visual representation was less nuanced. The teacher mostly provided an assessment of whether the work was correct or incorrect. When a student provided a visual representation that did not correspond to what the teacher expected, the teacher struggled to interpret and make sense of what happened. For example, figure 4 shows the work of a student whose visual representation of the simple problem the teacher failed to interpret. The student makes a comparison diagram (two quantities represented in distinct lines) and seems to have understood "3 times more" and "7,500" as the difference between the two quantities. Therefore, the student seems to divide 7,500 by 3 in order to find each of the "times". In other words, the student seems to have interpreted "3 times more" as being 3 identical jumps that total 7,500. Finally, the student fails to find the amount spent on materials (even though they correctly identified the quantity in the diagram by using a "?").

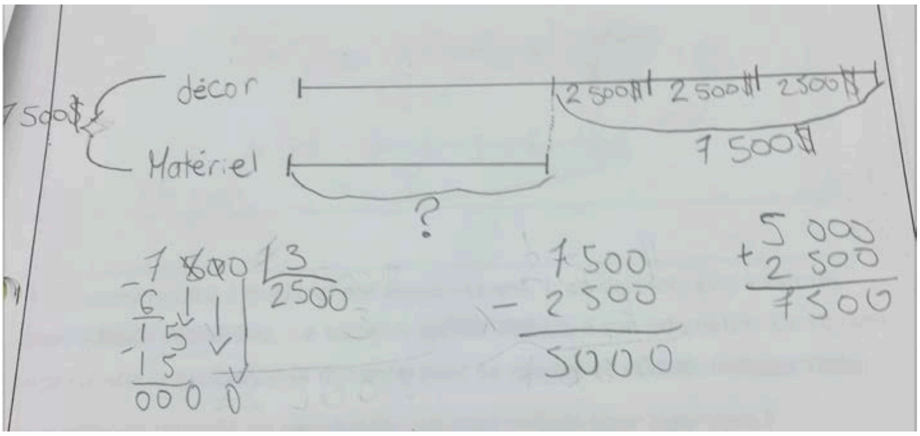


Fig. 4. *A student work that the teacher was not able to interpret*

When students succeeded in the final answer, the participant teacher seemed to take for granted that the reasoning was correct. However, it is in our understanding that the written work might not have corresponded to an appropriate reasoning, and the student could show signs of struggle despite a correct final answer. For example, figure 5 represents the work of a student who got full marks in the simple problem by making a comparison diagram. As expected, the teacher did not provide any comment on the student's reasoning since the work received full marks. However, one could further explore the student's understanding of the two quantities. The student divided 7,500 by 4 and that was the final answer. Without having identified each quantity in the diagram, we cannot guarantee that the student understandood that $1,375 represents the quantity being asked in the problem (amount spent in materials) or if they simply carried out the division and gave it as an answer.

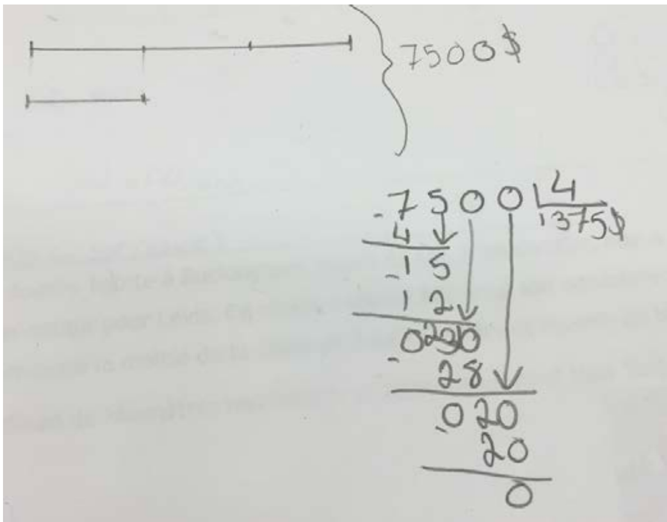


Fig. 5. *A student work that received full marks*

5. Conclusions

In this paper, we aimed at understanding the nature of a teacher's noticing of student written work through journal entries provided by one teacher participant in our multi-stage project. What we noticed is that the teacher participant's ability to notice student reasoning has two distinct dimensions, mathematical procedure and visual representation. We concluded that each of these dimensions, although certainly connected, encompasses different needs in terms of professional development.

While the noticing of mathematical procedure seems to be stronger and more nuanced, we believe the teacher would benefit from paying closer attention to students who generally succeed in mathematical problem solving. In terms of noticing the representations that students produce, we believe the teacher would benefit from strengthening her ability to interpret a wider range of possible representations, especially those that do not conform to the representation usually used.

In both situations, working on anticipating student thinking and preparing to respond seem to be a viable option for professional development. Future research might be interested in eliciting participating teachers' thinking with regards to student reasoning through written work. Mathematics problem solving requires teaching practices that address not only the procedure but all the dimensions involved in the process. In this project, we focused on how the teacher notices the visual representations in addition to the procedure. Future research might also emphasize different aspects of problem solving and how a teacher reacts (noticing, responding, discussing, etc.) to student thinking regarding these different aspects.

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