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A model for determining the utility function using Fuzzy numbers

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Abstract: *This paper proposes an interactive model for determining the utility function of the decision maker. The model is based on the estimation of the certainty equivalent, which is estimated by the decision maker as a trapezoidal fuzzy number.*

Key-words: *utility function**,** fuzzy numbers, St. Petersburg paradox*

1. Introduction

The concept of utility, which is used in the economic decision-making theory to compare different evaluation variants, is defined as a subjective measure depending on the decision-making factor.

In order to model an individual's behaviour regarding certain risky choices, the *utility function* was introduced. The construction of the utility function has to satisfy the rationality axioms defined in 1947 by *von Neumann* and *Morgenstern* in their *Theory of Games and Economic Behavior* and presented in Ionescu et al. (1999).

But the expectation utility theory actually begins with *Daniel Bernoulli,* who, in a memo addressed to the Science Academy in Saint Petersburg in 1738, criticized the criterion of the mathematic expectation introduced by *Pascal* and proposed a new decision rule by introducing the utility function, a derivative and strictly increasing function, which should measure the individual's satisfaction.

The *utility* function expresses the decision maker's estimation as regards the risk and the interest in the additional marginal gains that can be achieved.

The concept of *utility* is based on the risk aversion usually manifested by the decision maker and his level of wealth.

Generally, in many economic models, the argument for the utility function is the capital (income, profit) of the given individual.

It is supposed that he makes decisions that bring about the maximization of the value of the mathematical expectation of his utility.

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2. Properties of the utility function

Bernoulli's utility concept has expanded in the economic reasoning by considering the model of the rational and economic man *(REM - Rational Economic Man)*.

The utility function U has to reflect the individuals' "rational" preferences, which implies particular characteristics (Aftalion et al., 1998).

Every individual prefers to have as big capital/wealth as possible, symbolized by W. Then, if $W_2 > W_1$, we obviously have $U(W_2) > U(W_1)$. Mathematically, the utility function has to be increasing and derivable, so $U' > 0$.

The utility function has to reflect the individual's attitude regarding the risk. A priori, there can be three attitudes: *risk propensity, risk neutrality* and *risk aversion* (Stoleriu, 2004).

We call *lottery* (experiment) the entity $L = (X, Y; \pi)$ where X is the maximum gain with the probability π, whereas the minimum gain is Y with the probability $1 - \pi$.

As regards the attitude towards risk, a decision-maker can be (Stoleriu 2010, 15):
- risk-averse (risk aversion) if he prefers the expected value of the lottery to the detriment of the lottery:

$$\pi \cdot U(X) + (1-\pi) \cdot U(Y) \leq U(\pi \cdot X + (1-\pi) \cdot Y), \quad (\forall)\pi \in [0,1], \quad (\forall) X, Y \qquad (1)$$

- risk-seeker (risk seeking) if he prefers the winning lottery given by the expected value of the lottery:

$$\pi \cdot U(X) + (1-\pi) \cdot U(Y) \geq U(\pi \cdot X + (1-\pi) \cdot Y), \quad (\forall)\pi \in [0,1], \quad (\forall) X, Y \qquad (2)$$

- indifferent (neutral relative to the risk) if:

$$\pi \cdot U(X) + (1-\pi) \cdot U(Y) = U(\pi \cdot X + (1-\pi) \cdot Y), \quad (\forall)\pi \in [0,1], \quad (\forall) X, Y \qquad (3)$$

The graphs of the utility functions in relation to the three attitudes towards risk are shown in figure 1.

In practice, most individuals have risk aversion in normal circumstances. Thus, out of two investments that have the same mathematical profitability expectation, the less risky one will be chosen (Lixăndroiu 2005, 200-205).

We consider the lottery $L = (W + h, W - h; 0.5)$. If he agrees to gamble, his fortune can become $(W + h)$ with a 0.5 probability if he wins, and $(W - h)$ with a 0.5 probability if he loses. If he prefers not to gamble, he will keep his initial capital W.

Therefore,

$$U(W) > 0.5 \cdot U(W + h) + 0.5 \cdot U(W - h) \qquad (4)$$

The relation (4) can be written:

$$U(W) - U(W-h) > U(W+h) - U(W) \tag{5}$$

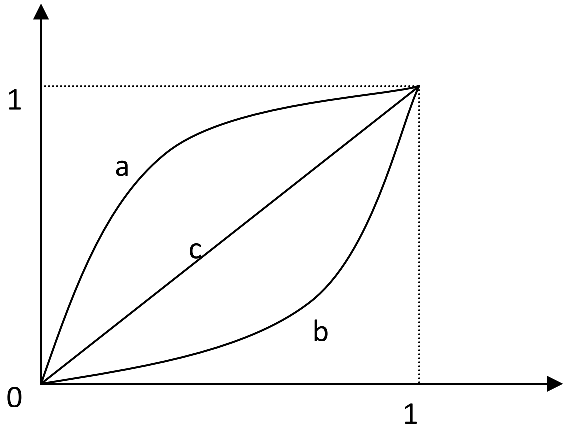


Fig. 1. *The utility function and the attitude towards risk*
a – risk aversion, b – risk propensity, c - risk neutrality

It results that the bigger W becomes, the less the utility function U increases. Therefore, the function is concave. Mathematically, if the function U is twice derivable, $U'(W)$, the slope, is decreasing and $U''(W)$ is negative. This function characterizes decision makers who are risk-averse.

If the function $U(W)$ is known, two measures of risk can be calculated:

- *the coefficient of absolute risk aversion:*

$$A(W) = -\frac{U''(W)}{U'(W)} \tag{6}$$

- *the coefficient of relative risk aversion:*

$$R(W) = -\frac{W \cdot U''(W)}{U'(W)} \tag{7}$$

The two measures of the risk $A(W)$ and $R(W)$ are known as *Arrow-Pratt's indices.* Their reverses are called *risk tolerance coefficients.*

The best known utility functions are (Quittard-Pinon 2003, 133): the quadratic, logarithmic, power functions and the general class of the *CARA* functions *(Constant Absolute Risk Aversion)*, *CRRA (Constant Relative Risk Aversion)* and *HARA (Hyperbolic Risk Aversion).*

Quadratic	$U(W) = a \cdot W - b \cdot W^2$
Logarithmic	$U(W) = \ln(W)$
Exponential	$U(W) = -\exp(-a \cdot W)$
Power	$U(W) = W^{\gamma}, \gamma < 1$
HARA	$U(W) = \frac{1-\gamma}{\gamma} \cdot \left(\frac{\beta \cdot W}{1-\gamma} + \eta\right)^{\gamma}$, $\beta, \gamma \neq 1$, η real parameters

Table 1. *The main utility functions*

For example, in the case of the logarithmic function, we have $U' > 0, U'' < 0,$ and $A'(W) < 0$ signifies a decreasing *absolute risk aversion,* while $R(W) = 1$ signifies a constant *relative risk aversion.*

3. Fuzzy numbers

The use of *fuzzy sets* in economic modelling allows the consideration of insufficiently defined, less rigid phenomena. For the members of these sets, there are several intermediate degrees of belonging, between full membership and non-membership.

It has been shown that fuzzy numbers can be obtained as a special case of fuzzy sets. Fuzzy numbers have a greater expressive power than interval numbers due to the capability of gradation on various levels.

The Triangular Fuzzy Number

We consider the function $F_A : A \rightarrow [0,1]$, with $A = [a_1, a_2] \subset R$ and having a maximum in the point $(a_M, 1)$, defined as (Bojadziev 1995, 35), (Gherasim 2005, 14):

$$F_A(x) = \begin{cases} F_A^s(x), & a_1 \leq x \leq a_M \\ F_A^d(x), & a_M \leq x \leq a_2 \end{cases} \tag{8}$$

The fuzzy number *A* is defined by the function (8), called the belonging function.

A triangular fuzzy number *A*, represented by the triad (a_1, a_M, a_2), is defined by the membership function:

$$\alpha = F_A(x) = \begin{cases} \dfrac{x-a_1}{a_M - a_1}, & a_1 \le x \le a_M \\ \dfrac{x-a_2}{a_M - a_2}, & a_M \le x \le a_2 \\ 0, & x \notin [a_1, a_2] \end{cases} \tag{9}$$

where $[a_1, a_2]$ is the support range, and the point $(a_M, 1)$ is the peak (figure 2).

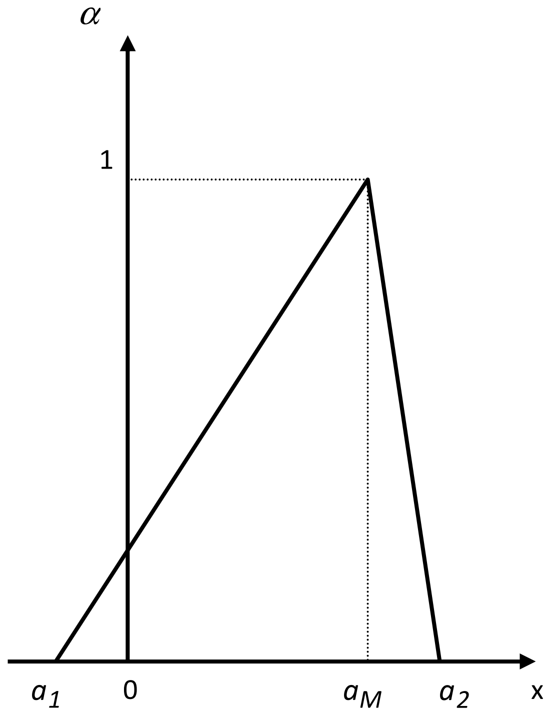


Fig.2. *The Triangular Fuzzy Number*

The Trapezoidal Fuzzy Number

A trapezoidal fuzzy number *A* is represented by the quadruplet $(a_1, a_{M1}, a_{M2}, a_2)$ and is defined by the membership function (Bojadziev 1995, 45), (Gherasim 2005, 16):

$$\alpha = F_A = \begin{cases} \dfrac{x-a_1}{a_{M1}-a_1}, & a_1 \le x \le a_{M1} \\ 1, & a_{M1} \le x \le a_{M2} \\ \dfrac{x-a_2}{a_{M2}-a_2}, & a_{M2} \le x \le a_2 \\ 0, & x \notin [a_1, a_2] \end{cases} \tag{10}$$

The graphical representation of the trapezoidal fuzzy number $A\ (a_1, a_{M1}, a_{M2}, a_2)$ is shown in figure 3.

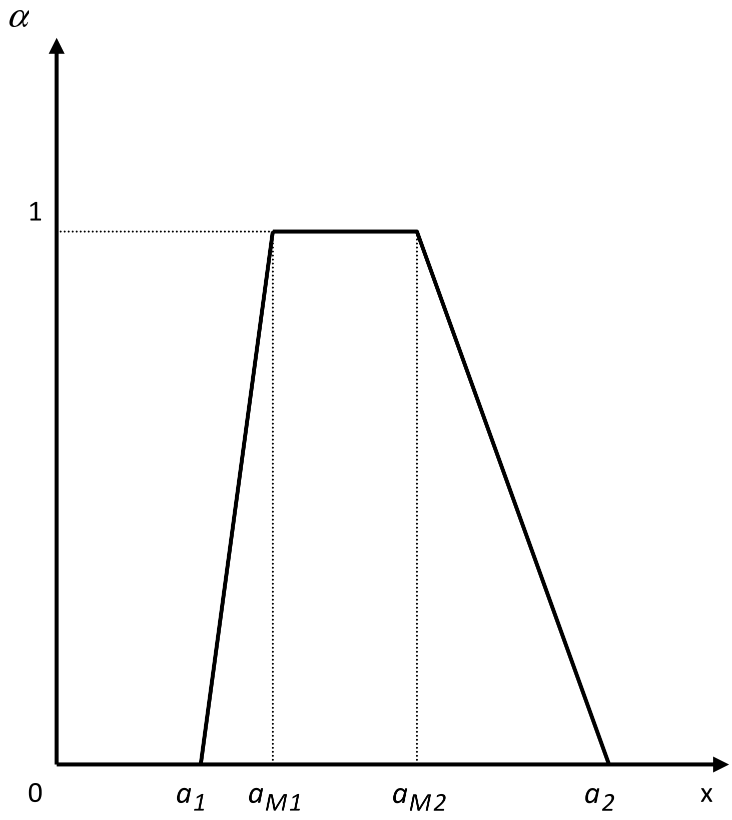


Fig. 3. *The Trapezoidal Fuzzy Number*

Remark. If in the case of a trapezoidal fuzzy number we have $a_{M1} = a_{M2}$, it becomes a triangular fuzzy number.

The real number associated to a trapezoidal fuzzy number is defined:

$$\langle \tilde{a} \rangle = \frac{a_1 + a_{M1} + a_{M2} + a_2}{4} \tag{11}$$

4. A model for determining the utility function

An important issue is to establish *the utility function of the decision maker*. The most important methods for determining the utility function are those based on lotteries. In what follows, we present a model which is based on estimating *the certainly equivalent* (Filip 2002, 252-254) and which uses trapezoidal fuzzy numbers to express the decision maker' estimates.

The utility function is specific to the decision maker and depends on his initial wealth (W_0), the importance of the action, the propensity towards risk.

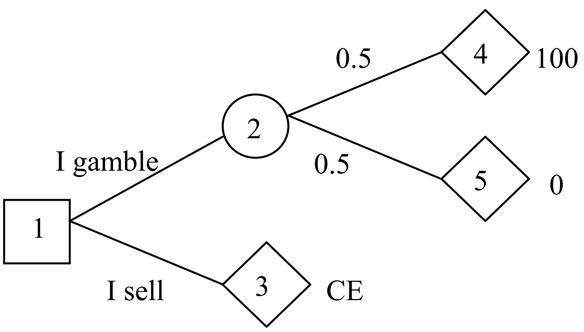
In the case of the decision maker with risk aversion, the mathematical expected value of the final wealth utility $E[U(W_0 + x)]$ is different from the utility of the final expectation wealth $U(W_0 + E[x])$.

We define the *certainty equivalent (CE)* as the value of the inverse function of the mathematical expectation of the final wealth utility $E[U(W_0 + x)]$:

$$EC = U^{-1}\{\, E[U(W_0 + x)]\,\} \tag{12}$$

The model involves the following steps (Lixăndroiu 2014, 26-28):

Step 1. We consider the reference lottery $L = (100, 0; 0.5)$. The decision tree is:



The extreme points of the utility function are determined:

$U(100) = 1 \qquad U(0) = 0$

Step 2. The decision maker is asked which is the amount for which he would sell the lottery ticket to another person. Depending on the financial situation (state of wealth), the decision maker indicates the *certainly equivalent value (CE)* as a trapezoidal fuzzy number. E.g. $CE_1 = (15,18,22,25)$. The real number is associated with: $\langle CE_1 \rangle = \frac{15+18+22+25}{4} = 20.$

Step 3. The utility of the alternative *"I sell"* is calculated, which is equal to the current expected value of the lottery:
$U(20) = 0.5 \cdot U(100) + 0.5 \cdot U(0) = 0.5$
Thus, we obtained a first point $A_1(CE_1, U(CE_1))$ belonging to the utility function, i.e. $A_1(20, 0.5)$.

Step 4. The reference lottery L is redefined by establishing two new lotteries: $L_{21} = (100, 20; 0.5)$ and $L_{22} = (20, 0; 0.5)$. Now a new point on the utility function chart can be determined for each lottery. For example, the decision maker indicates the value of the *certainly equivalent* for the lottery L_{21} as a trapezoidal fuzzy number $CE_2 = (30,35,42,53)$. The real associated number is:
$\langle CE_{21} \rangle = \frac{30+35+42+53}{4} = 40.$

We obtain: $U(40) = 0.5 \cdot U(100) + 0.5 \cdot U(20) = 0.5 + 0.5 \cdot 0.5 = 0.75$ and so we have a new point on the chart.

The same is valid if, for the lottery L_{22}, the decision maker indicates the certainly equivalent as a trapezoidal fuzzy number $CE_{22} = (3,4,7,10)$. The real associated number is: $\langle CE_{22} \rangle = \frac{3+4+7+10}{4} = 6.$

We obtain: $U(6) = 0.5 \cdot U(20) + 0.5 \cdot U(0) = 0.5 \cdot 0.5 + 0 = 0.25$ and the new point on the graph will be $A_3(6, 0.25)$

Step 5. Step no. 4 is resumed and the decision maker is required to establish the certainly equivalent for four new lotteries: $L_{31} = (100, 40; 0.5)$, $L_{32} = (40, 20; 0.5)$, $L_{33} = (20, 6; 0.5)$ and $L_{34} = (6, 0; 0.5)$. For example, for lottery L_{31}, the decision maker indicates the certainly equivalent value as a trapezoidal fuzzy number $CE_{31} = (45,50,57,68)$. The associated real number is: $\langle CE_{31} \rangle = \frac{45+50+57+68}{4} = 55.$

We obtain: $U(55) = 0.5 \cdot U(100) + 0.5 \cdot U(40) = 0.5 \cdot 1 + 0.5 \cdot 0.75 = 0.875$ and the new point on the graph will be $A_4(55, 0.875)$.

The process continues until a sufficient number of points is determined to produce an analytical expression for the utility function.

For example, using the software *MathCAD* (Scheiber et al. 1994, 88-90), the interpolation polynomial with spline cubic functions was built for the points:

$(0, 0), (6, 0.25), (20, 0.5), (40, 0.75), (55, 0.875), (100, 1)$, thereby achieving the utility function of the decision maker.

Given *n* points (x_i, y_i), the problem of the interpolation with spline cubic functions consists in the construction of some three-degree polynomials as:

$$P_i(x) = c_{0i} + c_{1i} \cdot (x - x_i) + c_{2i} \cdot (x - x_i)^2 + c_{3i} \cdot (x - x_i)^3,$$
$$i = 1, 2, \ldots, n-1, \quad x \in [x_i, x_{i+1}]$$

Thus, the values of the utility function can be determined for other values of the argument, for example:

$U(10) = 0.354$ $U(30) = 0.628$, $U(70) = 0.941$, $U(80) = 0.966$, $U(90) = 0.983$.

With the *11* values deducted for the utility function, the chart by points of the utility function was traced. This is shown in figure 4.

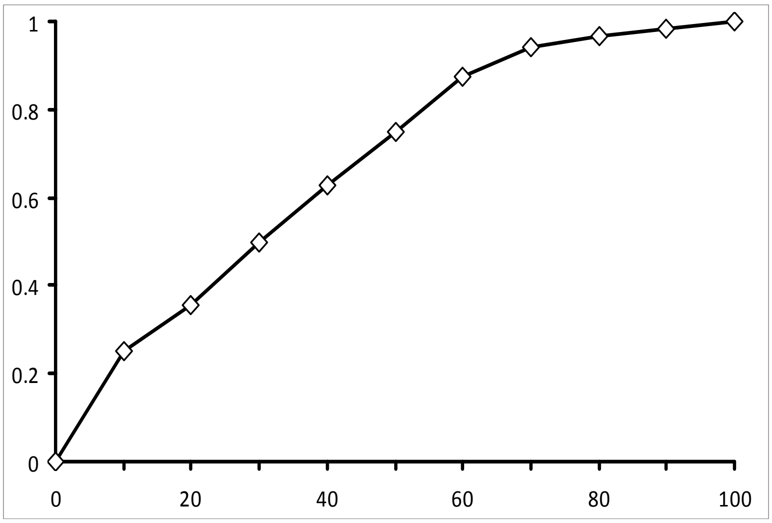


Fig. 4. *The utility function of the decision maker*

The concave form of the utility function shows the decision maker's risk aversion.

5. Conclusions

The use of fuzzy trapezoidal numbers in the presented interactive model for the determination of the utility function of a decision maker allows a less rigid approach to a difficult concept – the individual utility level. The technique used gives mathematical precision to a cognitive human process and allows the utility function to be established using the interpolation polynom with spline cubic functions.

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