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A PROFIT MAXIMIZATION METHOD USING POST OPTIMAL ANALYSIS IN LINEAR PROGRAMMING

Ciprian RUSESCU[1]

Abstract: *The competition for market survival has become fiercer over the years. The decision-makers are continuously challenged to maximize the operational efficiency of their businesses, especially when it comes to production structure. A good example is the product – mix problem, highlighting how to choose between numerous possible products or quantities to be manufactured while also considering resource constraints. Linear Programming (LP) represents the best method to model such a scenario, targeting the optimal operational point for all decisional variables. The present paper covers the mixed product scenario double perspective, firstly to determine the optimal operational points and secondly to provide additional information on further improvement scenarios of these points by changing the initial problem constraints. The decision makers are advised to efficiently increase the current production level, taking advantage of the under-utilized capacity of specific constraints in order to prolong the accessibility of those already fully utilized. As a direct consequence, units will be more efficiently used and hence allowing the aggregate operational profit growth.*

Keywords*: Linear programming, sensitivity analysis, post optimally analysis, positive sensitivity analysis, Lagrange multiplier*

1. Introduction

Linear programming is a well-known concept in the economic / industrial environment, where decisional factors are continuously facing the weary challenge of utilizing limited resources in an optimal manner. An interesting affirmation (Luenberger and Ye, 2008) associates its strength with the capacity of being applied in various real scenarios, offering also different methods of solving specific problems.

The main goal of specific fast-solving algorithms utilization (as simplex) is to provide the optimal solution (in case there is any). Once highlighted, it has to be analysed / tested for practical reasons, offering managerial insights that will lead to a best policy implementation. However, deciders expectations related to linear programming are mostly not limited to optimal solution delivery. Due to the economic environment unpredictability, the identification of output sensitivity related to the input parameter

[1] "Danubius" International University, Galați. România, ciprian.rusescu@univ-danubius.ro

changes proved to be much more useful than knowing each decision variable / objective function value. If the sensitivity degree is high enough and there are reliable reasons to believe that the model is representative for a real situation, they may be interested in a thorough solution output analysis. The most frequent scenario involves changes in resources level (right side term of each constraint) and considering the fact that the solution requires input parameters constancy, the subject of sensitivity analysis comes into play.

A wide range of companies are using such a modelling solution to solve different kinds of practical problems. In these coordinates, sensitivity analysis role is to obtain specific information regarding the influence of the input data variation on the decisional process. In case changes appear in the activity costs or resources availability, it needs to determine their influence over the spending schedule / supplying program, in order to assume the best decision. Another frequent scenario involves the imminent introduction of additional constraints / activities, after a stringent analysis of such a choice impact over the aggregate results. Linear programming also offers a perfect occasion to introduce the "what-if" analysis principle (Vakilifard, Esmalifalak et al., 2013), delivering powerful tools for post-optimal analysis.

High-performance information programs have been lately developed based on specific algorithms. The vast majority of available packages created to solve linear programming problems do not have such limitations, offering valuable information regarding the solution sensitivity to specific data variations (sensitivity / post optimality analysis). Sensitivity analysis, which starts once the initial problem solution was delivered, consists in a set of chained activities targeted to highlight the solution sensitivity degree at initial assumption changes. Developed around an optimal basis, the simplex method is properly treated in the specific literature, being introduced in various papers (Dantzig, 1963; Gal, 1979). It also has a tremendous importance in real situations, where parameter values can be estimated (Dahiya andVerma 2007).

2. Literature Review

Starting from the definition proposed by Sung and Park (1988), Yang (1990) distinguished two types of sensitivity analysis. The first focuses on identifying the characteristic region within which the optimal basis remains unchanged despite perturbations introduced into the problem (the stability region). The second, referred to as Positive Sensitivity Analysis (PSA), aims to determine the specific region within which the nonnegative variables comprising the optimal solution remain nonnegative even after disruptive factors are introduced. As per Park, Kim et al. (2004), PSA expressly highlights the bounded region within positive elements of optimal solutions remains superior to zero.

Other different treatment possibilities of sensitivity analysis are represented by the tolerance approach (Wendell, 1984 &1985) and global approach (Wagner, 1995). The goal of the tolerance approach is to provide the maximum percent data variation, good enough to keep the initial optimal solution structure unchanged, whilst the global sensitivity analysis aims to determine the key variability factors. A merged analysis of these two scenarios offers interesting answers related to the linear programming key properties (Borgonovo et al., 2018)

Based on the “optimal partitions” concept, another promising method of parametric analysis was developed. Initially referring to two complementary but also disjointed subsets of the restriction set method (Adler & Monteiro, 1992), the method was generalized four years later (Monteiro & Mehrotra, 1996). The same concept represents the basis for building another sensitivity analysis method within which cost coefficients and resources availability are simultaneously changing (Greenberg, 2000). The Yang's or Adler & Monteiro's method involves additional computation for interior-point methods in order to determine optimal solution / optimal partition. However, the connections between PSA and sensitivity analysis using optimal bases / partition are eventually highlighted (Boyd and Vandenberghe 2004).

The sensitivity analysis, subsequent to the simplex method application, is rightly developed around on optimal base, involving an insignificant computational effort. The method has been discussed in a wide range of papers so far (Dantzig, 1963; Higgle and Wallace, 2003; Ahmed et al, 2021; etc.), being also transposed in many linear programming codes. The degeneracy scenario may offer incomplete information due to alternative optimal bases (Baker and Evans, 1982; Knolmayer, 1984; Jansen et al., 1997, Kim, Park et al., 2004).

Linear programming represents a practical analysis tool for optimal resources allocation, mainly in underdeveloped countries economies. The mathematical algorithm was conceived by the mathematician George Dantzig (1947), in his attempt to plan various range of U.S. Air Force activities, considering limited resources. Seven years later, Lemke offers the dual simplex method, as a perfect answer at his primal version (1951) - a rigorous mathematic tool aiming to solve linear programming problems (Momoh, El-Hawary et al. 1999). In 1979, Khachiyan and Kozlov present the first polynomial algorithm for linear equations system, whilst Karmarkar (1984) suggests a specific projective method, strong enough to set the linear program’s polynomial-time solvability, thus starting the research related to the interior point method. Dantzig himself advocates for this approach, considered the right tool for solving a wide range of economic / industrial problems.

Equation systems represent an appropriate tool of linear algebra, utilized for optimizing the resources allocation problem, ensuring adequate support to financial decision-making problems, especially in the software implementation context. Therefore, linear programming became a popular optimization technique, aimed at decisional variables values able to optimize a unique set objective (profit maximization / cost minimization) based on specific constraints.

3. Methodology

The present paper presents a linear programming subject, aiming to optimize a maximum objective, described as a linear function. The process is restricted by the presence of certain constraints (linear as well), also respecting the economic rationality criteria - meaning nonnegativity assumption of considered variables. The chosen method for such an optimization process is the Lagrange method, requiring first of all, certain specific assumptions. An initial maximization function should be firstly set, $f(x_i), i = \overline{1,n}$ gathered with the presence of some particular constraints $g_j(x_i), j = \overline{1,m}$, conceiving this way a new one, called Lagrangean:

$$L(x_1, x_2, \ldots\ldots, x_n) = f(x_1, x_2, \ldots\ldots, x_n) - \sum_{j=1}^{m} \lambda_j \cdot g_j(x_1, x_2, \ldots\ldots, x_n) \quad (1)$$

The λ_j parameters are called Lagrange multipliers and from the economic perspective they represent a measure of objective function sensitivity to the constraint parameter variation (shadow price). In the maximization problem, a specific shadow price / dual variable can be associated to each constraint, its interpretation being reported only at the present situation (debated problem), offering additional information for economic analysis / increasing system efficiency.

Nested functions, meaning the objective $f(x_i)$ gathered with constraint function set $g(x_i)$ are supposed to be continuous and differentiable. To solve the problem and find out the minimum local point (x_i^*), it is necessary to define first order condition (for each Lagrangean variable), as shown bellow:

$$\frac{\partial L(x_i)}{\partial x_i} = 0 \rightarrow \partial f(x_i) + \sum_{j=1}^{m} \lambda_j \cdot \partial g_j(x_i) = 0 \qquad \frac{\partial L(x_i)}{\partial \lambda_j} = 0 \rightarrow g_j(x_i) = 0 \quad (2)$$

In order to solve the current system restrictions (2) and find the aggregate solution - including optimal point x_i^* / shadow prices λ_j^*, the dual problem feasibility conditions have to be considered (3), as well as the complementary slackness conditions (4):

$$g_j(x_i) \geq 0 \quad \rightarrow \quad \lambda_j \geq 0, (\forall)\, i = \overline{1,n}, j = \overline{1,m} \quad (3)$$

$$\lambda_j g_j(x_i) = 0, (\forall)\, i = \overline{1,n}, j = \overline{1,m} \quad (4)$$

After optimal solution highlighting, we can go on by testing its sensitivity at various parameters value changes. There are two options in this respect, first one aimed to see how the influence of different parameters values changes over the final result, by solving the initial problem repeatedly, for different inputs (preferably by computer). The plurality of the considered scenarios increases the precision method, the main shortcoming in this case being represented by the long time needed in order to test the wide range of possible changes. The second one is represented by the specific post – optimality method, which is used once the linear programming problem solution is revealed. For example, different changes to the constraint's right-side terms can be applied to improve the optimal profit value. In each case, the sensitivity region is represented by the values range within which the optimal quantities values can fluctuate, without changing the solution variable structure. A direct consequence is generally the optimal solution value / feasible region change, except for the constraints redundancy scenario.

Case study

The first goal of our case study is to set the optimal production mix of a certain firm, manufacturer of two different products, whose technological process involves the passing through three different departments. The main purpose is represented by the profit maximization, considering specific constraints related to each department available time, (expressed in hours / half-year):

$$(\max) f = 12x_1 + 16x_2 \tag{5}$$

As previously mentioned, the objective reflects the business activity potential, reported at the market selling price. Each constraint expresses the necessity of fitting the department's available time, reported to the specific processing period for any production unit. In economic terms, the main question is how to ensure the output manufacture, in order not to exceed each section specific limited time (1200 hours / 1500 hours / 1700 hours):

$$\begin{cases} 2x_1 + 5x_2 \leq 1200 \\ 3x_1 + 2x_2 \leq 1500 \\ 4x_1 + 3x_2 \leq 1700 \end{cases} \tag{6}$$

Any included variable should imperiously fulfil the nonnegativity restriction, in order to respect the economic rationality criteria.

Specific manufactured quantities ($x_1^* = 350$ units / $x_2^* = 100$ units) are highlighted by solving the maximization problem, as well as the highest possible profit level ($f_{max} =$ 5.800 $) in the mentioned initial hypothesis: (Table 1)

The result of the basic mixed products problem Table 1

Shadow price	$\lambda_1 = 2$ $\lambda_2 = 0$ $\lambda_3 = 2$
x^*	$x_1^* = 350$ $x_2^* = 100$
f_{max}	$f = 5800$ $

Source: Author's own research

The solution feasible areas, including the optimal point within it, are presented in Figure 1. Other interesting information is delivered by the second constraint redundancy, meaning the lack of contribution in reducing the admissibility field in the attempt to reach the optimal point. Mathematically speaking, the shadow price of this constraint should be zero, whilst any positive value reflects the impact of one unit fluctuation over the general objective function. In other words, a more efficient improvement of the profit function can be achieved by expanding the constraint having the highest shadow price level.

The zero value shadow price of the second constraint reflects the fact that the corresponding time resource is not fully utilized in the manufacturing process. In such a scenario, the second goal of the current case study is revealed, meaning the resources utilization improvement. Hence, the established objective is to examine how a semiannual reduction in the second section time influences aggregate performance under the condition that the availability of all remaining components is concurrently enhanced.

a) We will suppose initially that the first equation is not expandable (the specific time must remain unchanged) and we expand the third one instead, whilst releasing the entire unused capacity of the second constraint. The new optimization problem can be written as follows:

$$
\begin{aligned}
&(\max) f = 12x_1 + 16x_2 \\
&2x_1 + 5x_2 \leq 1200 \\
&3x_1 + 2x_2 \leq 1500 - A \\
&4x_1 + 3x_2 \leq 1700 + k \cdot A
\end{aligned}
\tag{7}
$$

The similarities with the initial problem are obvious, the novelty consisting in a different steps optimization for the expandable constraint. In this respect **A** units of unused time will be released in second constraint, the third one being proportionally expanded, while assuming that the second constraint time worthiness for one unit equals the worthiness of ***k*** units in the third.

A further analysis of the optimal solution of problem (5) reveals that the maximal value of **A** parameter is 250. Four different scenarios of the optimization problem (7) are considered, $k = 0.5, k = 1, k = 1.5$ and $k = 2$, whilst for **A** parameter we assume $A = 30, A = 40, A = 50, A = 60$ and $A = 70$ values.

Figure 1 also captures the profit optimization for the mentioned scenarios. Starting from the basic problem optimal point (profit value 5800 $) four different linear evolutions are revealed (one for each scenario). The optimal point in $k = 1$ case reflects the value at which the shadow prices of both constraints match, so that any variation of ***A*** value activates one of them, whilst the other becomes redundant instantly.

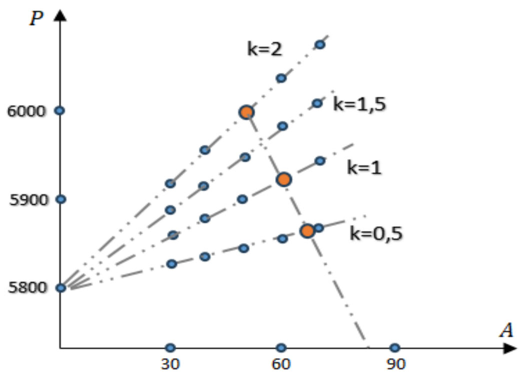

Fig 1. *Optimal function value for different values of **A** and **k** parameters*

b) The second assumed scenario requires the first equation expandability, the available time for the third one remaining unchanged. The optimization problem is described below:

$$(\max) f = 12x_1 + 16x_2 \tag{8}$$

$$2x_1 + 5x_2 \leq 1200 + k \cdot A$$

$$3x_1 + 2x_2 \leq 1500 - A$$

$$4x_1 + 3x_2 \leq 1700$$

The situation is not quite different from the previous one, considering existing similarities – time worthiness for one unit equals the worthiness of $\boldsymbol{k}$ units in the first restriction. Although the optimal solution improvement process will lead to different values (the expanded constraint is not the same), the objective function of the equational system (8) follows exactly the same path, meaning Figure 1 perfectly reflects the present scenario as well.

c) The present discussion cannot be ended without analysing the total expandability scenario, the released time unit of the second constraint being counterbalanced by a proportional extension, split between the other two constraints. In other words, the $\boldsymbol{k}$ unit worthiness will be replaced by $\boldsymbol{k_1}$ units in the first restriction and others $\boldsymbol{k_2}$ units in the third ($\boldsymbol{k_1} + \boldsymbol{k_2} = \boldsymbol{k}$).

$$(\max) f = 12x_1 + 16x_2 \tag{9}$$

$$2x_1 + 5x_2 \leq 1200 + k_1 \cdot A$$

$$3x_1 + 2x_2 \leq 1500 - A$$

$$4x_1 + 3x_2 \leq 1700 + k_2 \cdot A$$

Things are slightly different here, due to the k_1 and k_2 complementarity. In these circumstances, the profit growth will be reflected by the steepest trajectory in Figure 1, considering $k_1 + k_2 = 2$ as assumed hypothesis. The same assumption guarantees problem (9) profit level constancy as long as A parameter value remains unchanged, despite the complementary variation of k_1 and k_2.

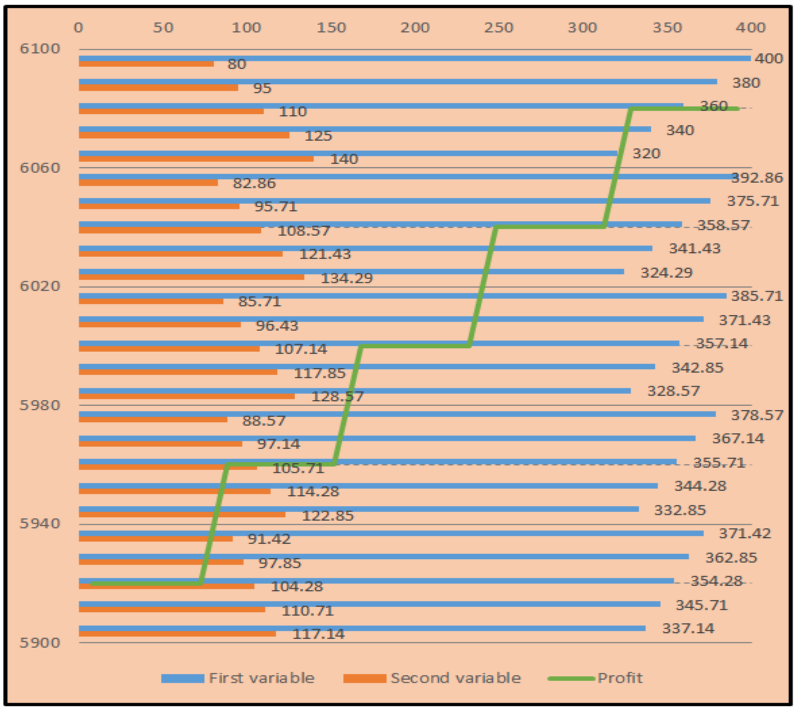

Fig. 2. *Profit evolution in the complementary extension scenario*

A more specific analysis of the current scenario can be realized based on Figure 2. Horizontal lines within it express the optimal solution structure, whilst the broken ones highlight the profit level evolution. In addition to the aggregate vertical variation limits, four horizontal thresholds delineate the specific production structures capable of maintaining a constant profit level, with any crossing of these boundaries indicating a transition to a different profit tier.

The numerical approach highlights other interesting aspects, the starting point being represented by the linearity of the price evolution trajectory. As long as parameter A value remains unchanged, the complementary evolution of k_1 and k_2 involves opposite direction changes of the optimal solution components values. On the other hand, it is worth emphasizing the modification steps constancy, the objective function structure being directly responsible for their proportionality level (the first component's decrease / growth causes a 75% increase / lowering of the second one's value).

4. Findings

The rational utilization of specific resources represents a very sensitive aspect, becoming more and more thorny as time passes. Many countries in the world have been affected by the improper resources usage, the optimal operating point being this way negatively affected for various industries. Microeconomically, long-term resource planning for each production unit becomes critical, and the mention of the limits within available resources becomes a must. Management may further improve the optimal operating point by utilizing available resources accordingly. No matter the row material necessity, machinery usage, or labour force targeted, a small level variation will generally improve the final result.

Such an analysis was conducted in the present study, explaining how exactly the maximum level of specific constraints can vary, in order to positively impact the aggregate result. The managerial team can equilibrate the resources, by reducing the excess at one level (selling machineries or transferring as reserve), whilst increasing the other, already fully utilized (deciding eventually the acquisition of new equipment). The efficient utilization of time resources would improve the objective function, ensuring operational profit growth.

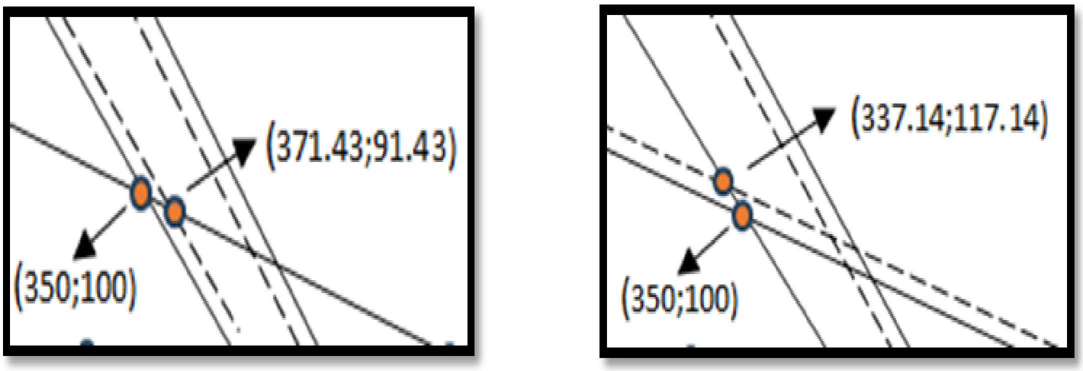

Fig. 3. *Optimal solution improvement in the unique constraint extension scenario*

To solve this aspect, linear programming principles are invoked, treating different scenarios related to ***A*** and **k** parameters various values. As per Figure 3, the decrease of the redundant constraint right-hand term does not move the optimal point. In exchange, the proportional relaxation of any other constraint changes the maximal solution, whose new value guarantees the objective improvement. More precisely, the third constraint extension push left the optimal solution, whilst the first extension shifts the maximum point on the right side. The post-optimal level certainly depends on the ***k*** parameter considered values. If both constraints are simultaneously expanded, the parallel support lines of the optimal solution improvement trajectories include previously determined value pairs, one from each scenario (Figure 4). Mathematically speaking, such an extension can be written as a linear combination whose extreme cases (one constant zero value) cannot be excluded - once enabled, the optimal point moves only along each nonredundant constraint graphical approach. This kind of approach can be easily extended at many other restriction types, possibly encountered in various industries.

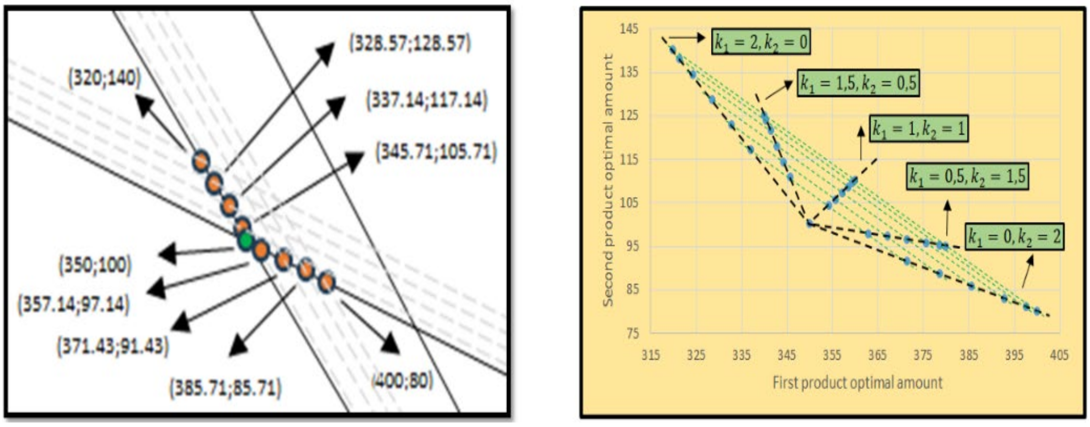

Fig. 4. *Optimal solution improvement in the complementary extension scenario*

5. Conclusions

The company's management should permanently adapt the decisional process at the unpredictable economic context, especially in terms of the production structure line, as the main factor for objective improvement. A manufacture trying to determine an optimal product mix structure represents a very suitable example, the Linear Programming (LP) usually representing the perfect tool for such a process modelling. The only drawback is represented by its hypothesis regarding the parameter constancy unlike the real economic environment, always dynamic and unpredictable. In order to avoid major troubles, the company's management must know in advance the impact of a resource level variation / manufacturing process modification / raw material cost fluctuation on the final result. This kind of research is well-known in the specific literature as sensitivity analysis / post-optimality analysis, targeting the influence of various economic parameter changes on the primal optimal solutions.

Software packages related to linear programming generally not only approach the basic problem but also provide information regarding the optimal solution sensitivity to specific data variations. That is exactly what was done in the previous chapter: using LP's general framework, we highlighted the area (interval) within optimal solution structure remains unchanged at the modification of the right-hand side constraint term. Such information proves to be of overwhelming importance in practice, where parameter values can be estimated.

Our present research, aimed at the post optimal analysis of the profit maximization model, may help company management to assume right decisions, in order to face the resources level modification. More precisely, deciding factors might try to release unused production capacity of some constraints, expanding the other's capacity accordingly (with resources already fully utilized). If the aggregate production time decreases on a certain department can be achieved by reducing the usage times of various machineries / possibly removing some of them from use, the increase in the available time requires an extremely rigorous analysis. In order not to put pressure on costs, the management may decide to overload the existing equipment, but in the long run such a choice may prove to be a disaster. The most efficient solution in this case would be the acquisition of new machineries, providing this way the additional time required, without endangering the existing operating capacity. The management will also decide regarding the investment's opportunity at a given time or its postponement, respectively.

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