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SOME NEW RESULTS ON NORMAL BH-ALGEBRAS

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Abstract

The concept of BH-algebra was introduced in 1998 by Y. B. Jun, E. H. Roh and H. S. Kim as a generalization of BCH/BCI/BCK-algebras. The same authors launched the idea of a normal BH-algebra at the end of the last century. This paper discusses the concept of atoms in normal BH-algebras. Thus, some conditions were found that they ensure the existence of such elements. Also, the extension of a normal BH-algebra A to the normal BH-algebra $A \cup \{a\}$ is considered so that the element a is an atom in $A \cup \{a\}$.

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1 Introduction

Y. Imai, K. Iséki and S. Tanaka introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([5, 6, 7]). Q. P. Hu and X. Li introduced in [4] a class of BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Next, in 1984, Y . Komori introduced in [13] the concept of BCC-algebras, to solve some problems on BCK-algebras. Y. B. Jun, E. H. Roh, and H. S. Kim introduced the concepts of BH-algebras and normal BH-algebras, which are generalizations of BCH/BCI/BCK-algebras ([8]). This type of abstract algebra is the focus of interest of many researchers (see, for example [1, 9, 10, 11, 15]).

The concept of atoms in BCC-algebras was introduced in 1995 by W. A. Dudek and X. Zhang ([2]). Then, the concept of atoms in BZ-algebras is discussed in [3] by W. A. Dudek, X. Zhang and Y. Wang. This phenomenon in pseudo BH-algebras was discussed in 2015 by Y. B. Jun and S. S. Ahn ([10]).

In this paper, we introduce the concept of atoms in normal BH-algebras and analyze its properties. It was done in a slightly different way than in the above-mentioned article [10]. Some conditions which confirm the existence of such elements were found. Two necessary and sufficient criteria were found for an element to be an atom in a BH-algebra. Also, some important properties of atoms in normal BH-algebras are described. In addition to the previous one, the set L(A) of

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all atoms of the normal BH-algebra is an anti-chain and the set $L(A) \cup \{0\}$ of the normal BH*-algebra A is a subalgebra. (The determination of the concept of BH*-algebras it can be found in [14].) In addition to the previous one, an extension of a normal BH-algebra A to the normal BH-algebra $A \cup \{a\}$ is designed so that the element a is an atom in $A \cup \{a\}$.

2 Preliminaries

The concept of BH-algebras was introduced 1998 in [8] by Y. B. Jun, E. H. Roh and H. S. Kim.

Definition 1 ([8], Definition 2.1). *A BH-algebra is a non-empty set A with a constant 0 and a binary operation $\cdot$ satisfying the following axioms:*
(BH1) $(\forall x \in A)(x \cdot x = 0)$,
(BH2) $(\forall x \in A)(x \cdot 0 = x)$,
(BH3) $(\forall x, t \in A)((x \cdot y = 0 \ \wedge \ y \cdot x = 0) \implies x = y)$.

Example 1 ([8], Example 2.2(a)). *Let $A = \{0, 1, 2, 3\}$ and let the operation in A be determined as follows*

$\cdot$	0	1	2	3
0	0	3	0	2
1	1	0	0	0
2	2	2	0	0
3	3	3	1	0

It is easy to verify that $(A, \cdot, 0)$ is a BH-algebra. □

Example 2 ([8], Example 2.2(b)). *Let $\mathbb{R}$ be the field of all real numbers and define the operation $\cdot$ as follows*

$$x \cdot y = \begin{cases} 0 & \text{for } x = 0, \\ \frac{(x-y)^2}{x} & \text{for } x \neq 0, \end{cases}$$

where $-$ is the usual substraction of real numbers. Then it is easy to check that $(\mathbb{R}, \cdot, 0)$ is a BH-algebra. □

Definition 2 ([8], Definition 3.1). *Let A be a BH-algebra and J a subset of A. I is called a BH-ideal of A if it satisdies*
(J0) $0 \in J$,
(J1) $(\forall x, y \in A)((x \cdot y \in J \ \wedge \ y \in J) \implies x \in J)$.
A BH-ideal J in a BH-algebra A said to be proper if $J \neq A$.

Definition 3 ([15], Definition 3.3). *A BH-algebra A is said to be normal if it satisfying the following condition:*
(BH4) $(\forall x, y \in A)(0 \cdot (x \cdot y) = (0 \cdot x) \cdot (0 \cdot y))$,
(BH5) $(\forall x, y \in A)((x \cdot y) \cdot x = 0 \cdot y)$,
(BH6) $(\forall x, y \in A)((x \cdot (x \cdot y)) \cdot y = 0)$.

Example 3 ([15], Example 3.6)**.** *Let $A = \{0, 1, 2, 3\}$ and let the operation in A be determined as follows*

$\cdot$	0	1	2	3
0	0	1	0	0
1	1	0	1	1
2	2	1	0	0
3	3	1	3	0

Then $(A, \cdot, 0)$ is a normal BH-algebra. The subsets $J_1 = \{0, 2\}$ and $J_2 = \{0, 2, 3\}$ are BH-ideals in A. The subset $K = \{0, 1\}$ is not a BH-ideal in A because, for example, $2 \cdot 1 = 1 \in K$ and $1 \in K$ but $2 \notin K$ holds. □

Definition 4 ([14])**.** *A BH-algebra A is called a BH*-algebra if it satisfies the identity*

(BH7) $(\forall x, y \in A)((x \cdot y) \cdot x = 0)$.

For this class of BH-algebras it holds

Proposition 1 ([11], Lemma 2.4)**.** *Let A be a BH^*-algebra. Then the following identity holds:*

(BH5") $(\forall x \in A)(0 \cdot x = 0)$

For this class BH-algebras also the following applies: (BH4) is a tautology.

The following Example shows that a normal BH-algebra can satisfy the condition (BH7).

Example 4 ([14])**.** *Let $A = \{0, 1, 2\}$ and let the operation defined by*

$\cdot$	0	1	2
0	0	0	0
1	1	0	2
2	2	0	0

Then $(A, \cdot, 0)$ is a BH-algebra but it is also a normal BH-algebra.* □

3 The main results

In order to simplify the writing, we use the function $\varphi : A \longrightarrow A$, defined as follows $(\forall x \in A)(\varphi(x) := 0 \cdot x)$. From (BH1), we get $\varphi(0) = 0$. The formulas (BH4) and (BH5) now have forms:

(BH4') $(\forall x, y \in A)(\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y))$
(BH5') $(\forall x, y \in A)(\varphi(y) = (x \cdot y) \cdot x)$.

So, this function is a homomorphism of normal BH-algebras. Such a case is not in the paper [3] because there this function is not a homomorphism.

Let us introduce the relation $\leqslant$ on a BH-algebra A as follows

(1) $(\forall x, y \in A)(x \leqslant y \iff y \cdot x = 0)$.

It is obvious that this relation is reflexive (it follows from (BH1)) and antisymmetric (it follows from (BH3)).

However, if the BH-algebra A were transitive ([12], Definition 2.4), then the relation $\leqslant$ would be a partial order relation on A.

Remark 1. *The previously introduced relation in (normal) BH-algebras is somewhat different from the analogous relation introduced in* [10] *(pp. 208).*

Example 5. *Let* $A = \{0, 1, 2, 3\}$ *as in the Example 3. In this normal BH-algebra* A*, the relation* $\leqslant$ *is given by*

$$\leqslant = \{(0,0),(1,1),(2,2),(3,3),(2,0),(3,2)\}.$$

Using this technique, it can be concluded that any BH-ideal J in a BH-algebra A has the following property:

(J2) $(\forall x, y \in A)((y \leqslant x \wedge y \in J) \implies x \in J)$.

Indeed, if $x, y \in A$ are elements in A such that $y \leqslant x$ and $y \in J$, then holds $x \cdot y = 0 \in J$ and $y \in J$. Thus $x \in J$ according (J1).

Recall that a nonempty subset S of a (normal) BH-algebra A is called a subalgebra if

$$(\forall x, y \in A)((x \in S \wedge y \in S) \implies x \cdot y \in S).$$

Let J be a BH-ideal in a normal BH-algebra A that satisfies the additional condition (BH7) and let $x, y \in A$ be arbitrary elements such that $x \in J$ and $y \in J$. Then from $x \leqslant x \cdot y$ and $x \in J$ it follows $x \cdot y \in J$ by (J2). Therefore, J is a subalgebra in A.

It can be concluded that a normal BH-algebra A in relation to the relation $\leqslant$, determined in this way, has the following properties:

Proposition 2. *Let* A *be a normal BH-algebra. Then the following holds:*

(2) $(\forall x, y \in A)((x \leqslant 0 \wedge y \leqslant 0) \implies x \cdot y \leqslant 0)$

(3) $(\forall x, y \in A)(x \leqslant y \implies \varphi(x) \leqslant \varphi(y))$,

(4) $(\forall x, y \in A)(y \leqslant x \cdot (x \cdot y))$,

(5) $(\forall y \in A)(y \leqslant \varphi^2(y))$,

(6) $(\forall x \in A)(\varphi(x) = \varphi^3(x))$,

(7) $(\forall x, y \in A)(x \leqslant \varphi^2(y) \implies \varphi^2(x) = \varphi^2(y))$.

Proof. (2) Let $x \leqslant 0$ and $y \leqslant 0$. Then $0 \cdot x = 0$ and $0 \cdot y = 0$. Thus

$$0 = 0 \cdot 0 = (0 \cdot x) \cdot (0 \cdot y) = 0 \cdot (x \cdot y)$$

by (BH4'). This means $x \cdot y \leqslant 0$.

(3) Let $x, y \in A$ be such that $x \leqslant y$. This means $y \cdot x = 0$. Then

$$0 = \varphi(0) = \varphi(y \cdot x) = \varphi(y) \cdot \varphi(x)$$

by (BH4). Hence $\varphi(x) \leqslant \varphi(y)$.

(4) Claim (4) is a direct consequence of (BH6).

(5) Claim (5) is obtained from (5) for $x = 0$.

(6) If we put $x = \varphi^2(x)$ and $y = x$ in (BH5), we get $\varphi(x) = (\varphi^2(x) \cdot x) \cdot \varphi^2(x)$. From here we get $\varphi(x) = 0 \cdot \varphi^2(x) = \varphi^3(x)$ since $\varphi^2(x) \cdot x = 0$ according to (5).

(7) Let $x, y \in A$ be such that $x \leqslant \varphi^2(y)$. This means $\varphi^2(y) \cdot x = 0$. Then, according to (6), we have

$$\varphi(y) = \varphi^3(y) = 0 \cdot \varphi^2(y) = (\varphi^2(y) \cdot x) \cdot \varphi^2(y) = \varphi(x).$$

Thus $\varphi^2(y) = \varphi^2(x)$. □

Assertion (2) in the previous proposition can be understood in the following way: For any normal BH-algebra A, the set $K(A) := \{x \in A : x \leqslant 0\}$ is a subalgebra of A.

In what follows, we use the terminology from the paper [15]. Let A be a BH-algebra. The set $M(A) = \{x \in A : \varphi^2(x) = x\}$ is called a medial part of A and an element of $M(A)$ is called a medial element of A. In [15] it was shown that $0 \in M(A)$ and so $M(A)$ is non-empty. $M(A)$ is not a subalgebra of A, in general case. But, if the BH-algebra A satisfies the additional condition (BH4), then $M(A)$ is a subalgebra in A ([15], Theorem 3.1). Also, in that case, the subset $Ker(\varphi) = \{x \in A : \varphi(x) = 0\}$ is an BH-ideal and a subalgebra in A ([15], Theorem 3.2).

In addition to the previous one, in the normal BH-algebra A, the subset $M(A)$ is a subalgebra in A ([15], Theorem 3.1). Indeed, let $x, y \in M(A)$. This means $\varphi^2(x) = x$ and $\varphi^2(y) = y$. On the other hand, since $\varphi^2(x \cdot y) = \varphi^2(x) \cdot \varphi^2(y)$ follows from (BH4'), we have $\varphi^2(x \cdot y) = x \cdot y$. Hence, $x \cdot y \in M(A)$.

In 2015, Y. B. Jun and S. S. Ahn introduced the concept of pseudo atoms in pseudo BH-algebras ([10], Definition 3.12). Our determination of the concept of atoms in normal BH-algebras is somewhat different from the aforementioned determination. We introduce the concept of atoms in a normal BH-algebra by the following definition.

Definition 5. *An element $a\,(\neq 0)$ of a normal BH-algebra A is said to be an atom in A if the following holds*

(A) $(\forall\, x \in A)(a \leqslant x \implies (x = a \vee x = 0))$.

We denote the family of all atoms in the BH-algebra A by $L(A)$.

The following proposition gives a sufficient condition for the element $a \in A$ to be an atom in A.

Proposition 3. *Let A be a normal BH-algebra and let $a \in A$. If the subset $\{0, a\}$ is a BH-ideal in A, then a is an atom in A.*

Proof. Let $x \in A$ be arbitrary element such that $a \leqslant x$. Then $x \cdot a = 0 \in \{0, a\}$ and $a \in \{0, a\}$. Thus $x \in \{0, a\}$ by (J1). This means $x = 0$ or $x = a$. So, the element a is an atom in A. □

Also, the validity of the following statement can be proved:

Proposition 4. *Let $a\,(\neq 0)$ be an isolated element in a BH-algebra A, that is, let a be not comparable to any element in A except itself. Then a is an atom in A.*

Proof. Let $x \in A$ be an arbitrary element. Then $x \cdot a \neq 0$ and $a \cdot x \neq 0$. It is obvious that $x \neq a$ because, otherwise, $x \cdot a = 0$ and $a \cdot x = 0$ which contradicts the assumption. If $x = 0$, we would have $x \neq a$. We have $x \neq 0$, $x \neq a$ and $a \nleqslant x$, so, therefore, we can create the implication $(x \neq 0 \wedge x \neq a) \implies a \nleqslant x$. This proves that an isolated element in a BH-algebra A is an atom in A because the last implication is the contraposition of the implication (A). □

Example 6. *Let $A = \{0, 1, 2, 3\}$ as in the Example 3. Since $\{0, 2\}$ is a BH-ideal in A (see Example 3), element 2 is an atom in A in accordance with Proposition 3. Element 1 is an isolated element in A, so, according to Proposition 4, it is also an atom in A.* □

The previous example shows that since the subset $\{0, 1\}$ is not a BH-ideal in A, we conclude that the converse of the Proposition 3 does not have to be valid.

The following proposition gives another sufficient condition for recognizing whether an element is an atom in the (normal) BH-algebra.

Proposition 5. *Let A be a normal BH-algebra. If the element $a\,(\neq 0)$ of A satisfies the identity*

$$(\forall\, x, y \in A)(x \cdot (x \cdot (a \cdot y)) = a \cdot y),$$

then a is an atom on A.

Proof. Let $x \in A$ be such that $a \leqslant x$. Then $x \cdot a = 0$ and

$$a = a \cdot 0 = x \cdot (x \cdot (a \cdot 0)) = x \cdot (x \cdot a) = x \cdot 0 = x.$$

Hence a is an atom in A. □

Some more information about the set $L(A)$ is given by the following theorems:

Theorem 1. *The set $L(A)$ is an anti-chain.*

Proof. Let $a, b \in L(A)$ be such that $a \neq b$ and assume that $a \leqslant b$ holds. Then it would be $a = b$ or $b = 0$ which is impossible. Therefore, it must be $a \nleqslant b$. Analogously, it can be obtained that $b \nleqslant a$ holds. □

Theorem 2. *Let a, b be an atom in a BH-algebra A. Then the following holds:*

(8) $(\forall\, x \in A)(x \cdot (x \cdot a) = a \vee x \cdot (x \cdot a) = 0)$,

(9) $\varphi^2(a) = a \vee \varphi^2(a) = 0$.

Proof. Let a, b be atoms in a BH-algebra A.

(8) Since, according to (5), we have $(\forall\, x \in A)(a \leqslant x \cdot (x \cdot a))$, from here we get $x \cdot (x \cdot a) = a$ or $x \cdot (x \cdot a) = 0$ because a is an atom in A..

(9) According to (5), $a \leqslant \varphi^2(a)$ is valid, so we get $\varphi^2(a) = a$ or $\varphi^2(a) = 0$ since a is an atom in A. □

However, the converse of statement (8) is also valid:

Theorem 3. *Let A be a BH-algebra. If the condition (8) holds for the element $a \in A$, then a is an atom in A.*

Proof. Let $x \in A$ be such that $a \leqslant x$. This means $x \cdot a = 0$. If we include this equality in (8), we get $x \cdot 0 = a$ or $x \cdot 0 = 0$. So, with respect to (BH2), we get $x = a$ or $x = 0$. Hence, a is an atom in A. □

Also, the converse of statement (9) is valid:

Theorem 4. *Let A be a BH-algebra. If the condition* (9) *holds for the element $a \in A$, then a is an atom in A.*

Proof. Let $x \in A$ be such that $a \leqslant x$. Then $\varphi^2(a) = \varphi^2(x)$ according to (7). This $x \leqslant \varphi^2(x) = \varphi^2(a) = a$ if $\varphi^2(a) = a$. Now suppose that $\varphi^2(a) = 0$ holds. Then $x \leqslant \varphi^2(x) = \varphi^2(a) = 0$ implies $x \leqslant 0$ which gives $0 \cdot x = 0$. Since, according to (BH2), we have $0 \cdot x = x$, we get $x = 0$. This proves that a is an atom in A. □

Some of the mutual relations of atoms in the normal BH-algebra are described by the following statement as a consequence of the previous theorem.

Corollary 1. *Let a, b be atoms in a normal BH-algebra A. Then*

$$b \cdot (b \cdot a) = a \ \vee \ b \cdot a \leqslant b.$$

Proof. The proof is a direct consequence of the assertion (8). □

Example 7. *Let $A = \{0, 1, 2, 3, 4\}$ and let the operation in A be determined as follows*

$\cdot$	0	1	2	3	4
0	0	1	0	0	4
1	1	0	1	1	4
2	2	1	0	0	4
3	3	1	3	0	4
4	4	1	4	4	0

Then $(A, \cdot, 0)$ is a normal BH-algebra. In A, the relation $\leqslant$ is given by

$$\leqslant = \{(0,0), (1,1), (2,2), (3,3), (2,0), (3,2), (4,4)\}.$$

The subset $J_1 = \{0, 2\}$ is a BH-ideals in A. Elements 1 and 4 are isolated elements in A. Therefore, $L(A) = \{1, 2, 4\}$. On the other hand, for 2 we have $\varphi^2(2) = 0$ so $2 \notin M(A)$. □

The previous example shows that $M(A) \neq L(A)$ in the general case.

In the following example, we give a normal BH-algebra in which all elements, except 0, are atoms in it.

Example 8. *Let $A = \{0, 1, 2, 3\}$ and let the operation in A be determined as follows*

$\cdot$	0	1	2	3
0	0	1	2	3
1	1	0	2	3
2	2	1	0	3
3	3	1	2	0

Then $(A, \cdot, 0)$ is a normal BH-algebra in which all elements are isolated. Therefore $L(A) = A \setminus \{0\}$. □

In the normal BH-algebra that additionally satisfies the condition (BH7), something more can be said about the structure of the set $L(A)$. For simplicity of expression, we recognize this class of BH-algebras under the name 'normal BH*-algebra'. Example 4 shows that such a class of algebras exists.

Theorem 5. *If A is a normal BH*-algebra, then the set $L(A)\cup\{0\}$ is a subalgebra in A.*

Proof. Let $a, b \in L(A)$ be atoms in A. Then holds $a \leqslant ab$ by (BH7). Thus $a \cdot b = a \in L(A)$ or $a \cdot b = 0 \in L(A) \cup \{0\}$ since a is an atom in A. Besides, in this algebra, also the following is valid: $x \cdot 0 = x \in L(A) \cup \{0\}$ and $0 \cdot x = 0$ for any $x \in L(A) \cup \{0\}$. □

Example 9. *Let $A = \{0, 1, 2, 3\}$ and let the operation in A be determined as follows*

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	3	2	0

Then $(A, \cdot, 0)$ is a normal BH-algebra. The order in this algebra is given as follows $\leqslant=\{(0,0), (1,0), (2,0), (3.0), (4,0), (1,1), (3,1), (4,1), (2,2), (4,2), (3,3), (4,3), (4,4)\}$. Subsets $\{0,1\}$ and $\{0,2\}$ are BH-ideals in A. That is why elements 1 and 2 are atoms in this algebra. Therefore $L(A) = \{1,2\}$. In addition to the previous one, the following equalities illustrate the previous theorem: $1\cdot 2 = 1$ and $2\cdot 1 = 2$. The subset $L(A)\cup\{0\} = \{0,1,2\}$ is a subalgebra in A. Besides, it is obvious that for the subsets $V(1) = \{x \in A : x \leqslant 1\}$ and $V(2) = \{x \in A : x \leqslant 2\}$ the following holds $V(1) \cap V(2) = \{4\}$.* □

We finish this report by creating an extension of the normal BH-algebra $(A, \cdot, 0)$ to the normal BH-algebra $(A \cup \{a\}, *, 0)$ so that the element a is an atom in $A \cup \{a\}$.

Theorem 6. *Let A be a normal BH-algebra and let $a \notin A$. We can design the normal BH-algebra $(A \cup \{a\}, *, 0)$ such that the element a is an atom in $(A \cup \{a\}, *, 0)$.*

Proof. Let us put $B = A \cup \{a\}$ and design the operation $*$ on B as follows:

$$x * y = \begin{cases} x \cdot y & \text{for } x \in A \ \wedge \ y \in A, \\ x & \text{for } x \in A \ \wedge \ y = a, \\ \varphi(y) & \text{for } x = a \ \wedge \ y \in A \setminus \{0\}, \\ a & \text{for } x = a \ \wedge \ y = 0, \\ 0 & \text{for } x = a \ \wedge \ y = a. \end{cases}$$

It should be checked whether the structure, created in this way, satisfies the axioms (B1)-(B6). This can be verified by replacing one or both variables in formulas (BH1)-(BH6) with the variable a. This type of direct verification is of a technical nature, so we omit it. For the sake of illustration, we will show some of those procedures. For $x = a$, we have:

(BH4): $0 \cdot (a \cdot y) = 0 \cdot \varphi(y) = \varphi^2(y)$ and $(0 \cdot a)(0 \cdot y) = 0 \cdot \varphi(y) = \varphi^2(y)$.

(BH5): If $y \neq 0$, we have $(a \cdot y) \cdot a = \varphi(y) \cdot a = \varphi(y)$.

(BH6): If $y \neq 0$, we have $(a \cdot (a \cdot y)) \cdot y = \varphi^2(y) \cdot y = 0$ by (5).

It is easy to see that a is an atom in B. Indeed, if $x \in B$ is such that $a \leqslant x$, then $x * a = 0$. On the other hand, since $x * a = x$, we get $x = 0$, which proves that a is an atom in B. □

The following example illustrates the previous theorem:

Example 10. *Let $A = \{0, 1, 2, 3\}$ be as in Example 3. According to the Example 6, we have $L(A) = \{1, 2\}$. We put $B = A \cup \{a\}$ and determine the operation $*$ as follows*

$\cdot$	0	1	2	3	a
0	0	1	0	0	0
1	1	0	1	1	1
2	2	1	0	0	2
3	3	1	3	0	3
a	a	1	0	0	1

Then $(B, *, 0)$ is a normal BH-algebra. The relation $\leqslant$ is given by

$$\leqslant = \{(0,0), (2,0), (3,0), (1,1), (2,2), (3,3), (3,2), (2,a), (3,a), (a,a)\}.$$

Besides this atom in the normal BH-algebra B and element 1 is an atom in B because 1 is an isolated element in B. Therefore, $L(B) = \{1, a\}$. Let us show that a is an atom in B. Let us take $x \in B$ such that $a \leqslant x$. Then $x \cdot a = 0$. Since, on the other hand, $x \cdot a = x$, we conclude that $x = 0$ which proves that a is an atom in B.

4 Final comments

This report makes a contribution to research of BH-algebras. It analyzes the concept of atoms in a normal BH-algebras and in a normal BH*-algebra. The author believes that the presented results about (normal) BH-algebras and their atoms increase our knowledge about this class of logical algebras. An open possibility remains to investigate the properties of the structure which is called the normal BH*-algebra here. In addition to the previously mentioned, one could seek an answer to the question: Under what conditions is $L(A) \cup \{0\} = A$?

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