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A GENERAL FIXED POINT THEOREM FOR A PAIR OF MAPPINGS SATISFYING A MIXED IMPLICIT RELATION IN S - METRIC SPACES

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Abstract

The purpose of this paper is to extend the notion of mixed implicit relation by [19] to S - metric spaces and to prove a general fixed point theorem for a pair of mappings in S - metric spaces, generalizing some results by [3], [10], [15] and other papers.

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1 Introduction

Let X be a nonempty set and $f, g : X \mapsto X$. A point $x \in X$ such that $fx = gx = w$ is said to be a point of coincidence of f and g and w is called a point of coincidence of f and g. By $\mathcal{C}(f, g)$ we denote the set of all coincidence points of f and g.

Jungck [8] introduced the notion of weakly compatible mappings.

Definition 1 ([8]). *Let X be a nonempty set. Two mappings $f, g : X \mapsto X$ are said to be weakly compatible if $fgx = gfx$ for all $x \in \mathcal{C}(f, g)$.*

Theorem 1 ([8]). *Let f and g be weakly compatible self mappings of a nonempty set X. If f and g have a unique point of coincidence w, then w is the unique common fixed point of f and g.*

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In [2], Alber and Guerre - Delabriere introduced the notion of weakly contractive mappings in Hilbert spaces as a generalization of contractive mappings and established a fixed point theorem.

In [23], Rhoades extend this idea in Banach spaces and proved existence of fixed point of weakly contractive mappings. In [4], Choudhury introduced the concept of weakly C - contractive mappings in metric spaces.

Definition 2 ([4]). *A mapping $T : X \mapsto X$, where (X, d) is a complete metric space, is said to be weakly C - contractive if for all $x, y \in X$,*

$$d(Tx, Ty) \le \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \phi(d(x, Ty), d(y, Tx)),$$

where $\phi : [0, \infty)^2 \mapsto [0, \infty)$ is a continuous function such that $\phi(x, y) = 0$ if and only if $x = y = 0$.

A generalization of metric space, named D - metric spaces, is introduced in [5], [6].

Mustafa and Sims [13], [14] proved that most of the claims concerning the fundamental topological structures from D - metric spaces are incorrect and introduced an appropriate notion of metric space, named G - metric space. In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings in G - metric spaces.

In [12], Mustafa initiated the study of fixed points for weakly compatible mappings in G - metric spaces. Recently, in [24], the authors introduced a generalization of G - metric spaces, named S - metric spaces.

Recently, in [7], the authors proved that the notion of S - metric space is not a generalization of G - metric or vice versa. Hence, the notion of G - metric space and S - metric space are independent.

Other results in the study of fixed points in S - metric spaces are obtained in [15], [20], [21] and in other papers. Quite recently, some results for fixed points for four mappings in S - metric spaces are obtained in [10] and [25].

Some results for weakly compatible mappings in S - metric spaces are obtained in [10] and [26].

Several fixed point theorems and common fixed point theorems in metric spaces have been unified in [16], [17] and other papers considering a general condition by an implicit function.

The method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, ultra - metric spaces, convex metric spaces, Hilbert spaces, probabilistic metric spaces, intuitionistic metric spaces, partial metric spaces, weak partial metric spaces, dislocated metric spaces, for single - valued mappings, hybrid pairs of mappings and multi-valued mappings. With this method, the proof of existence of fixed points is more simple.

The notion of mixed implicit relation in metric spaces is recently published in [19].

2 Preliminaries

Definition 3 ([24]). *Let X be a nonempty set. An S - metric on X is a function $S : X^3 \mapsto \mathbb{R}_+$ such that for all $x, y, z, a \in X$:*
$(S_1) : S(x, y, z) = 0$ if and only if $x = y = z$,
$(S_2) : S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.
The pair (X, S) is called a S - metric space.

Example 1. *Let $X = \mathbb{R}$ and $S(x, y, z) = |x - z| + |y - z|$. Then, (X, S) is a S - metric space and $S(x, y, z)$ is said to be the usual S - metric on $\mathbb{R}$.*

Lemma 1 ([24]). *If S is a S - metric on a nonempty set X, then*

$$S(x, x, y) = S(y, y, x) \text{ for all } x, y \in X.$$

Definition 4 ([23], [24]). *Let (X, S) be a S - metric space. For $r > 0$ and $x \in X$, we define the open ball $B_S(x, r)$ with center x and radius r:*

$$B_S(x, r) = \{y \in X : S(x, x, y) < r\}.$$

The topology induced by the S - metric on X is the topology generated by the base of all open balls in X.

Definition 5 ([23], [24]). *a) A sequence $\{x_n\}$ in (X, S) converges to $x \in X$, denoted $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, if $S(x_n, x_n, x) \to 0$ as $n \to \infty$.*
b) A sequence $\{x_n\}$ in (X, S) is a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.
c) The space (X, S) is complete if every Cauchy sequence in (X, S) is a convergent sequence.

Lemma 2 ([23], [24]). *Let (X, S) be a S - metric space. If $x_n \to x$ and $y_n \to y$, then $S(x_n, x_n, y_n) \to S(x, x, y)$.*

Lemma 3 ([24]). *Let (X, S) be a S - metric space and $\{x_n\}$ a convergent sequence. Then $\lim_{n\to\infty} x_n$ is unique.*

Definition 6 ([9]). *An altering distance is a function $\psi : [0, \infty) \mapsto [0, \infty)$ such that:*
(ψ_1) : ψ is continuous and nondecreasing,
(ψ_2) : $\psi(t) = 0$ if and only if $t = 0$.

We denote by Ψ the set of all altering distances and by Φ the set of all continuous functions $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) = 0$ if and only if $t = 0$.

The notion of ϕ - weak contraction in S - metric spaces is defined in [11].

Definition 7 ([11]). *A self mapping T of an S - metric space is said to be ϕ - weak contractive if $S(Tx, Tx, Ty) = S(x, x, y) - \phi(S(x, x, y))$.*

Theorem 2 ([11]). *If T is a ϕ - weak contractive on a S - metric space, then T has a unique fixed point.*

Quite recently, a generalization of this theorem is proved in [3].

Definition 8 ([3])**.** *Let (X,S) be a S - metric space and $f:X\mapsto X$ such that*

$$\psi\left(S\left(fx,fy,fz\right)\right)\leq\psi\left(M\left(x,y,z\right)\right)-\phi\left(M\left(x,y,z\right)\right),$$

where

$$M\left(x,y,z\right)=\max\left\{\begin{array}{c}S\left(x,x,y\right),S\left(x,x,fx\right),S\left(y,y,fy\right),S\left(z,z,z\right),\\ \alpha S\left(fx,fx,fy\right)+\left(1-\alpha\right)S\left(fy,fy,fz\right)\end{array}\right\},$$

for all $x,y,z\in X$, $\alpha\in(0,1)$, $\psi\in\Psi$, $\phi\in\Phi$. Then f is called a (α,ψ,ϕ) - generalized weak contractive map.

Theorem 3 ([3])**.** *Let (X,S) be a complete S - metric space and f be a (α,ψ,ϕ) - generalized weak contractive map. Then f has a unique fixed point u and f is continuous at u.*

Remark 1. *1) If $y\to x$ and $z\to y$, then*

$$M\left(x,x,y\right)=\max\left\{\begin{array}{c}S\left(x,x,y\right),S\left(x,x,fx\right),S\left(y,y,fy\right),\\ \alpha S\left(fx,fx,fx\right)+\left(1-\alpha\right)S\left(fy,fy,y\right)\end{array}\right\}.$$

2) By our opinion, the proof of continuity in Theorem 3 is not correct because $S\left(x_n,x_n,fx_n\right)$ is not $S\left(x_n,x_n,x_{n+1}\right)$.

The following lemma is used in the proof of Theorem 3.

Lemma 4 ([3, Lemma 1.26])**.** *Let (X,S) be a S - metric space and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty}S(x_n,x_n,x_{n+1})=0$. If $\{x_n\}$ is not a Cauchy sequence, then there exists an $\varepsilon>0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k>m_k>k$ such that $S\left(x_{m_k},x_{m_k},x_{n_k}\right)\geq\varepsilon$, $S\left(x_{m_k-1},x_{m_k-1},x_{n_k}\right)<\varepsilon$ and*

(1) $\lim_{n\to\infty}S\left(x_{m_k},x_{m_k},x_{n_k}\right)=\varepsilon$,

(2) $\lim_{n\to\infty}S\left(x_{m_k-1},x_{m_k-1},x_{n_k}\right)=\varepsilon$,

(3) $\lim_{n\to\infty}S\left(x_{m_k},x_{m_k},x_{n_k-1}\right)=\varepsilon$,

(4) $\lim_{n\to\infty}S\left(x_{m_k-1},x_{m_k-1},x_{n_k-1}\right)=\varepsilon$.

The following theorem is proved in [10].

Theorem 4. *Let (X,S) be a S - metric space and $f,g:X\mapsto X$ be two mappings such that f is g - weak contractive map. Assume that:*

1) $f\left(X\right)\subset g\left(X\right)$,

2) $g\left(X\right)$ *is a complete subspace of* (X,S),

3) f *and* g *are weakly compatible.*

Then f and g have a unique common fixed point.

3 Mixed implicit relations

Let $\mathcal{F}_{MX}$ be the set of all lower semi - continuous functions $F:\mathbb{R}_+^6\mapsto\mathbb{R}$ such that:

(F_1) : F is nonincreasing in variable t_6,
(F_2) : For all $u>0, v\geq 0$, $F(u,v,v,u,0,2u+v)\leq 0$ implies $u<v$,
(F_3) : $F(t,t,0,0,t,t)\geq 0,\ \forall t>0$.

In the following examples property (F_1) is obviously.

Example 2. *$F(t_1,...,t_6)=t_1-k\max\{t_2,t_3,t_4,t_5,t_6\}$, where $k\in\left[0,\frac{1}{3}\right)$.*

(F_2) : Let $u>0, v\geq 0$ and $F(u,v,v,u,0,2u+v)=u-k\max\{u,v,2u+v\}\leq 0$. If $u\geq v$, then $u(1-3k)\leq 0$, a contradiction. Hence $u<v$.
(F_3) : $F(t,t,0,0,t,t)=t(1-k)\geq 0,\ \forall t>0$.

Example 3. *$F(t_1,...,t_6)=t_1-k\max\left\{t_2,t_3,t_4,\dfrac{t_5+t_6}{3}\right\}$, where $k\in[0,1)$.*

(F_2) : Let $u>0, v\geq 0$ and $F(u,v,v,u,0,2u+v)=u-k\max\left\{u,v,\frac{2u+v}{3}\right\}\leq 0$. If $u\geq v$, then $u(1-k)\leq 0$, a contradiction. Hence $u<v$.
(F_3) : $F(t,t,0,0,t,t)=t(1-k)\geq 0,\ \forall t>0$.

Example 4. *$F(t_1,...,t_6)=t_1^2-t_1(at_2+bt_3+ct_4)-ct_5t_6$, where $a,b,c\geq 0$, $a+b+c<1$ and $a+d<1$.*

(F_2) : Let $u>0, v\geq 0$ and $F(u,v,v,u,0,2u+v)=u^2-u(av+bv+cu)\leq 0$. If $u\geq v$, then $u^2[1-(a+b+c)]\leq 0$, a contradiction. Hence $u<v$.
(F_3) : $F(t,t,0,0,t,t)=t^2[1-(a+d)]\geq 0,\ \forall t>0$.

Example 5. *$F(t_1,...,t_6)=t_1^2-at_1t_2-bt_3t_4-ct_5t_6$, where $a,b,c\geq 0$, $a+b<1$ and $a+c<1$.*

(F_2) : Let $u>0, v\geq 0$ and $F(u,v,v,u,0,2u+v)=u^2-auv-buv\leq 0$. If $u\geq v$, then $u^2[1-(a+b)]\leq 0$, a contradiction. Hence $u<v$.
(F_3) : $F(t,t,0,0,t,t)=t^2[1-(a+c)]\geq 0,\ \forall t>0$.

Example 6. *$F(t_1,...,t_6)=t_1^3-at_1t_2t_3-bt_2t_3t_4-ct_3t_4t_5-dt_4t_5t_6$, where $a,b,c,d\geq 0$ and $a+b<1$.*

(F_2) : Let $u>0, v\geq 0$ and $F(u,v,v,u,0,2u+v)=u^3-auv^2-buv^2\leq 0$. If $u\geq v$, then $u^3[1-(a+b)]\leq 0$, a contradiction. Hence $u<v$.
(F_3) : $F(t,t,0,0,t,t)=t^3\geq 0,\ \forall t>0$.

Example 7. *$F(t_1,...,t_6)=t_1-at_2-bt_3-ct_4-dt_5-et_6$, where $a,b,c,d,e\geq 0$, $a+b+c+3e<1$ and $a+d+e<1$.*

(F_2) : Let $u>0, v\geq 0$ and $F(u,v,v,u,0,2u+v)=u-av-bv-cu-e(2u+v)\leq 0$. If $u\geq v$,then $u[1-(a+b+c+3e)]\leq 0$, a contradiction. Hence $u<v$.
(F_3) : $F(t,t,0,0,t,t)=t[1-(a+b+c+3e)]\geq 0,\ \forall t>0$.

Let $\mathcal{G}_{MX}$ be the set of all continuous functions $G:\mathbb{R}_+^5\mapsto\mathbb{R}$ such that $G(s_1,s_2,s_3,s_4,s_5)=0$ if and only if $s_1=s_2=...=s_5=0$.

Example 8. $G(s_1,...,s_5) = \max\{s_1, s_2, ..., s_5\}$.

Example 9. $G(s_1,...,s_5) = \max\left\{s_1, s_2, s_3, \frac{s_4+s_5}{3}\right\}$.

Example 10. $G(s_1,...,s_5) = \max\left\{s_1, s_2, s_3, \frac{s_4+s_5}{3}\right\}$.

Example 11. $G(s_1,...,s_5) = \max\left\{s_1, \frac{s_2+s_3}{2}, \frac{s_4+s_5}{3}\right\}$.

Example 12. $G(s_1,...,s_5) = s_1^2 + s_2^2 + s_3^2 + s_4^2 + s_5^2$.

Example 13. $G(s_1,...,s_5) = \frac{s_1}{1+s_2} + \frac{s_2}{1+s_3} + \frac{s_3}{1+s_4} + \frac{s_4}{1+s_5} + \frac{s_5}{1+s_1}$.

Example 14. $G(s_1,...,s_5) = s_1 + \frac{s_2+s_3+s_4+s_5}{2}$.

Example 15. $G(s_1,...,s_5) = s_1 + s_2 + s_3 + s_4 + s_5$.

Example 16. $G(s_1,...,s_5) = as_1 + bs_2 + s_3 + s_4 + s_5$, *where* $a, b \geq 0$, $a, b < 1$.

Definition 9. *A function* $\phi(t_1,...,t_6) = F(t_1,...,t_6) + G(t_1,...,t_5)$, *where* $F \in \mathcal{F}_{MX}$ *and* $G \in \mathcal{G}_{MX}$ *is called a mixed implicit function.*

4 Main results

Theorem 5. *Let* (X, S) *be a* S *- metric space and* $f, g : X \mapsto X$ *two self mappings such that for all* $x, y \in X$

$$\begin{array}{c} F\left(\begin{array}{c} S(fx,fx,fy), S(gx,gx,gy), S(gx,gx,fx), \\ S(gy,gy,fy), S(gy,gy,fx), S(gx,gx,fy) \end{array}\right) + \\ G\left(\begin{array}{c} S(gx,gx,gy), S(gx,gx,fx), \\ S(gy,gy,fy), S(gy,gy,fx), S(gx,gx,fy) \end{array}\right) \leq 0 \end{array} \tag{1}$$

for some $F \in \mathcal{F}_{MX}$ *and some* $G \in \mathcal{G}_{MX}$.

If $f(X) \subset g(X)$ *and* $g(X)$ *is a complete subspace of* X, *then* f *and* g *have a point of coincidence.*

Moreover, if f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point of X. Since $f(X) \subset g(X)$, there exists $x_1 \in X$ such that $fx_0 = gx_1$. Continuing this process, we define the sequence $\{x_n\}$ in X such that $fx_n = gx_{n+1}$. If there exists $n \in \mathbb{N}$ such that $fx_n = fx_{n+1}$, then $fx_n = gx_{n+1} = fx_{n+1} = z$ and z is a point of coincidence of f and g. Suppose that $fx_n \neq fx_{n+1}$, for all $x \in X$. Hence, $gx_n \neq gx_{n+1}$. By (1) for $x = x_{n-1}$ and $y = x_n$ we obtain

$$\begin{array}{c} F\left(\begin{array}{c} S(fx_{n-1},fx_{n-1},fx_n), S(gx_{n-1},gx_{n-1},gx_n), S(gx_{n-1},gx_{n-1},fx_{n-1}), \\ S(gx_n,gx_n,fx_n), S(gx_n,gx_n,fx_{n-1}), S(gx_{n-1},gx_{n-1},fx_n) \end{array}\right) + \\ G\left(\begin{array}{c} S(gx_{n-1},gx_{n-1},gx_n), S(gx_{n-1},gx_{n-1},fx_{n-1}), \\ S(gx_n,gx_n,fx_n), S(gx_n,gx_n,fx_{n-1}), S(gx_{n-1},gx_{n-1},fx_n) \end{array}\right) \leq 0, \end{array}$$

$$F\left(\begin{array}{c} S\left(gx_n,gx_n,gx_{n+1}\right),S\left(gx_{n-1},gx_{n-1},gx_n\right),S\left(gx_{n-1},gx_{n-1},gx_n\right), \\ S\left(gx_n,gx_n,gx_{n+1}\right),0,S\left(gx_{n-1},gx_{n-1},gx_{n+1}\right)\end{array}\right)+ \\ G\left(\begin{array}{c} S\left(gx_{n-1},gx_{n-1},gx_n\right),S\left(gx_{n-1},gx_{n-1},gx_n\right), \\ S\left(gx_n,gx_n,gx_{n+1}\right),0,S\left(gx_{n-1},gx_{n-1},gx_{n+1}\right)\end{array}\right)\leq 0. \tag{2}$$

By Lemma 1 and (S_2) we obtain

$$\begin{array}{l} S\left(gx_{n-1},gx_{n-1},gx_{n+1}\right)=S\left(gx_{n+1},gx_{n+1},gx_{n-1}\right) \\ \leq 2S\left(gx_{n+1},gx_{n+1},gx_n\right)+S\left(gx_{n-1},gx_{n-1},gx_n\right) \\ =2S\left(gx_n,gx_n,gx_{n+1}\right)+S\left(gx_{n-1},gx_{n-1},gx_n\right).\end{array} \tag{3}$$

By (2) we obtain

$$F\left(\begin{array}{c} S\left(gx_n,gx_n,gx_{n+1}\right),S\left(gx_{n-1},gx_{n-1},gx_n\right),S\left(gx_{n-1},gx_{n-1},gx_n\right), \\ S\left(gx_n,gx_n,gx_{n+1}\right),0,2S\left(gx_n,gx_n,gx_{n+1}\right)+S\left(gx_{n-1},gx_{n-1},gx_n\right)\end{array}\right)+ \\ G\left(\begin{array}{c} S\left(gx_{n-1},gx_{n-1},gx_n\right),S\left(gx_{n-1},gx_{n-1},gx_n\right), \\ S\left(gx_n,gx_n,gx_{n+1}\right),0,S\left(gx_{n-1},gx_{n-1},gx_{n+1}\right)\end{array}\right)\leq 0. \tag{4}$$

Since

$$G\left(\begin{array}{c} S\left(gx_{n-1},gx_{n-1},gx_n\right),S\left(gx_{n-1},gx_{n-1},gx_n\right), \\ S\left(gx_n,gx_n,gx_{n+1}\right),0,S\left(gx_{n-1},gx_{n-1},gx_{n+1}\right)\end{array}\right)>0,$$

then

$$F\left(\begin{array}{c} S\left(gx_n,gx_n,gx_{n+1}\right),S\left(gx_{n-1},gx_{n-1},gx_n\right),S\left(gx_{n-1},gx_{n-1},gx_n\right), \\ S\left(gx_n,gx_n,gx_{n+1}\right),0,2S\left(gx_n,gx_n,gx_{n+1}\right)+S\left(gx_{n-1},gx_{n-1},gx_n\right)\end{array}\right)<0,$$

which implies by (F_2) that

$$S\left(gx_n,gx_n,gx_{n+1}\right)<S\left(gx_{n-1},gx_{n-1},gx_n\right).$$

Hence, the sequence $\{S\left(gx_n,gx_n,gx_{n+1}\right)\}$ is a nonincreasing positive sequence, which implies that $S\left(gx_n,gx_n,gx_{n+1}\right)$ is convergent to a limit $r\geq 0$. Suppose that $r>0$. Then by (4) and (3) we obtain

$$F\left(r,r,r,r,0,3r\right)\leq 0.$$

By (F_2) we obtain that $r<r$, a contradiction. Hence

$$r=\lim_{n\to\infty}S\left(gx_n,gx_n,gx_{n+1}\right)=0.$$

We prove that $\{gx_n\}$ is a Cauchy sequence in X. Suppose that $\{gx_n\}$ is not a Cauchy sequence. By Lemma 4, there exists $\varepsilon>0$ and the sequences $\{m_k\},\{u_k\}$ and $S\left(gx_{m_k},gx_{m_k},gx_{n_{u_k}}\right)\geq\varepsilon$ and $S\left(gx_{m_{k-1}},gx_{m_{k-1}},gx_{m_k}\right)<\varepsilon$ satisfying conditions $(1)-(4)$.

By (1) for $x = x_{m_k-1}$ and $y = x_{u_k-1}$ we obtain

$$F\left(\begin{array}{c} S\left(fx_{m_k-1}, fx_{m_k-1}, fx_{u_k-1}\right), S\left(gx_{m_k-1}, gx_{m_k-1}, gx_{u_k-1}\right), \\ S\left(gx_{m_k-1}, gx_{m_k-1}, fx_{m_k-1}\right), S\left(gx_{u_k-1}, gx_{u_k-1}, fx_{u_k-1}\right), \\ S\left(gx_{u_k-1}, gx_{u_k-1}, fx_{m_k-1}\right), S\left(gx_{m_k-1}, gx_{m_k-1}, fx_{u_k-1}\right) \end{array}\right) + G\left(\begin{array}{c} S\left(gx_{m_k-1}, gx_{m_k-1}, gx_{u_k-1}\right), S\left(gx_{m_k-1}, gx_{m_k-1}, fx_{m_k-1}\right), \\ S\left(gx_{u_k-1}, gx_{u_k-1}, fx_{u_k-1}\right), S\left(gx_{u_k-1}, gx_{u_k-1}, fx_{m_k-1}\right), \\ S\left(gx_{m_k-1}, gx_{m_k-1}, fx_{u_k-1}\right) \end{array}\right) \leq 0,$$

$$F\left(\begin{array}{c} S\left(gx_{m_k}, gx_{m_k}, gx_{u_k}\right), S\left(gx_{m_k-1}, gx_{m_k-1}, gx_{u_k-1}\right), \\ S\left(gx_{m_k-1}, gx_{m_k-1}, gx_{m_k}\right), S\left(gx_{u_k-1}, gx_{u_k-1}, gx_{u_k}\right), \\ S\left(gx_{u_k-1}, gx_{u_k-1}, gx_{m_k}\right), S\left(gx_{m_k-1}, gx_{m_k-1}, gx_{u_k}\right) \end{array}\right) + G\left(\begin{array}{c} S\left(gx_{m_k-1}, gx_{m_k-1}, gx_{n_k-1}\right), S\left(gx_{m_k-1}, gx_{m_k-1}, gx_{m_k}\right), \\ S\left(gx_{u_k-1}, gx_{u_k-1}, gx_{u_k}\right), S\left(gx_{u_k-1}, gx_{u_k-1}, gx_{m_k}\right), \\ S\left(gx_{m_k-1}, gx_{m_k-1}, gx_{u_k}\right) \end{array}\right) \leq 0. \tag{5}$$

By Lemma 1,

$$S\left(gx_{u_k-1}, gx_{u_k-1}, gx_{m_k}\right) = S\left(gx_{m_k}, gx_{m_k}, gx_{u_k-1}\right).$$

Letting n tend to infinity in (5), we obtain by Lemma 4

$$F\left(\varepsilon, \varepsilon, 0, 0, \varepsilon, \varepsilon\right) + G\left(\varepsilon, 0, 0, \varepsilon, \varepsilon\right) \leq 0.$$

Since $G\left(\varepsilon, 0, 0, \varepsilon, \varepsilon\right) > 0$, then $F\left(\varepsilon, \varepsilon, 0, 0, \varepsilon, \varepsilon\right) < 0$, a contradiction of (F_3).

Hence, $\{gx_n\}$ is a Cauchy sequence in $g\left(X\right)$. Since $g\left(X\right)$ is complete, there exists t such that $\lim_{n\to\infty} gx_n = t \in g\left(X\right)$. Hence, there exists $p = g\left(t\right)$. We prove that $fp = gp$. By (1) for $x = x_{n-1}$ and $y = p$ we obtain

$$F\left(\begin{array}{c} S\left(fx_{n-1}, fx_{n-1}, fp\right), S\left(gx_{n-1}, gx_{n-1}, gp\right), S\left(gx_{n-1}, gx_{n-1}, fx_{n-1}\right), \\ S\left(gp, gp, fp\right), S\left(gp, gp, fx_{n-1}\right), S\left(gx_{n-1}, gx_{n-1}, fp\right) \end{array}\right) + G\left(\begin{array}{c} S\left(gx_{n-1}, gx_{n-1}, gp\right), S\left(gx_{n-1}, gx_{n-1}, fx_{n-1}\right), \\ S\left(gp, gp, fp\right), S\left(gp, gp, fx_{n-1}\right), S\left(gx_{n-1}, gx_{n-1}, fp\right) \end{array}\right) \leq 0.$$

Letting n tend to infinity we obtain

$$F\left(S\left(gp, gp, fp\right), 0, 0, S\left(gp, gp, fp\right), 0, S\left(gp, gp, fp\right)\right) + G\left(0, 0, S\left(gp, gp, fp\right), 0, S\left(gp, gp, fp\right)\right) \leq 0.$$

Since $G\left(0, 0, S\left(gp, gp, fp\right), 0, S\left(gp, gp, fp\right)\right) > 0$, then

$$F\left(S\left(gp, gp, fp\right), 0, 0, S\left(gp, gp, fp\right), 0, S\left(gp, gp, fp\right)\right) < 0.$$

By (F_2), $S\left(gp, gp, fp\right) < 0$, a contradiction. Hence, $gp = fp = t$ and t is a point of coincidence of f and g.

We prove that t is the unique point of coincidence for f and g. Suppose that there exists $z = fw = gw$. By (1) for $x = p$ and $y = w$ we obtain

$$F\begin{pmatrix} S(fp, fp, fw), S(gp, gp, gw), S(gp, gp, fp), \\ S(gw, gw, fw), S(gw, gw, fp), S(gp, gp, fw) \end{pmatrix} + \\ G\begin{pmatrix} S(gp, gp, gw), S(gp, gp, fp), \\ S(gw, gw, fw), S(gw, gw, fp), S(gp, gp, fw) \end{pmatrix} \leq 0.$$

By Lemma 1 we obtain

$$F(S(t,t,z), S(t,t,z), 0, 0, S(t,t,z), S(t,t,z)) + \\ G(S(t,t,z), 0, 0, S(t,t,z), S(t,t,z)) \leq 0.$$

Hence

$$F(S(t,t,z), S(t,t,z), 0, 0, S(t,t,z), S(t,t,z)) < 0,$$

a contradiction of (F_3) if $S(t,t,z) > 0$. Hence $S(t,t,z) = 0$ which implies $z = t$ and t is the unique point of coincidence of f and g.

Moreover, if f and g are weakly compatible, then by Theorem 1, f and g have a unique common fixed point. □

If $\psi(t) = t$, $F(t_1, ..., t_6) = t_1 - \max\left\{t_2, t_3, t_4, \dfrac{t_5 + t_6}{3}\right\}$ and $G(s_1, ..., s_5) = \max\left\{s_1, \frac{s_2+s_3}{2}, \frac{s_4+s_5}{3}\right\}$, by Theorem 5 we obtain

Corollary 1. *Let (X, S) be a S - metric space and $f, g : (X, S) \mapsto (X, S)$ such that for all $x, y \in X$*

$$\begin{aligned} &S(fx, fx, fy) \\ &\quad - \max\left\{S(gx, gx, gy), S(gx, gx, fx), S(gy, gy, fy), \frac{S(gy, gy, fx) + S(gx, gx, fy)}{3}\right\} \\ &\quad + \max\left\{S(gx, gx, gy), \frac{S(gx, gx, fx) + S(gy, gy, fy)}{2}, \frac{S(gy, gy, fx) + S(gx, gx, fy)}{3}\right\} \\ &\quad \leq 0. \end{aligned} \tag{6}$$

If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a point of coincidence.

Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Example 17. *Let $X = \mathbb{R}$ and $S(x, y, z) = |z - x| + |z - y|$. Then, by Example 1, (X, S) is a complete S - metric space. Let*

$$fx = 1, \ gx = 2x - 1.$$

Then $f(X) = \{1\}$ and $g(X) = \mathbb{R}$. Hence, $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X. If $fx = gx$, then $\mathcal{C}(f, g) = \{1\}$ and $fg1 = gf1 = 1$. Hence, f and g are weakly compatible.

On the other hand, $S(fx, fx, fy) = 0$. Therefore, for all $x, y \in X$, (6) is trivially.

By Corollary 1, f and g have a unique common fixed point $x = 1$.

Remark 2. *Combining Examples 2-7 with Examples 8-16, by Theorem 5 we obtain new particular results.*

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