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CONVERGENCE OF THIRD ORDER NEWTON-LIKE METHOD IN RIEMANNIAN MANIFOLDS

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Abstract

In this article, we present semilocal convergence of third order Newton-like method in Riemannian manifolds. We study convergence analysis with Lipschitz continuity condition and by using recurrence relations of the method. Finally, two numerical examples are given to illustrate the effectiveness of our results.

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1 Introduction

Many problems in engineering and technology field can be solved through nonlinear equation

$$\mathfrak{G}(x) = 0, \tag{1}$$

where $\mathfrak{G}$ is a nonlinear operator defined in an open convex subset Ω of a Banach space B into itself. To determine the roots of (1) has attracted the attention of mathematicians in the field of pure and applied mathematics. The exact solution of (1) is difficult to find so that we use iterative methods to solve these equations. One study the convergence of iterative methods usually based on semilocal and local convergence analysis. If the convergence analysis which uses information around a solution and estimates the radius of convergence ball, then it is said to be local convergence where as if the convergence analysis tells information around an initial point, then it is said to be semilocal convergence. There are several articles that can be found in literature which were devoted to study so many

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iterative methods in Banach spaces[3, 4, 5]. The third order Newton-like method [14] in Banach space to solve (1) is defined as:

$$\left.\begin{aligned} y_n &= x_n - \mathfrak{G}'(x_n)^{-1}\mathfrak{G}(x_n), \\ x_{n+1} &= x_n - 2[\mathfrak{G}'(x_n) + \mathfrak{G}'(y_n)]^{-1}\mathfrak{G}(x_n), \text{ for each } n = 0, 1, 2, \ldots, \end{aligned}\right\} \quad (2)$$

where $\mathfrak{G}'(x_n)$ is first Fréchet derivative in Ω. Recently, there has been a growing interest in studying iterative methods in Riemannian manifolds, since there are many numerical problems in manifolds that arise in many contexts [7, 8]. Some higher order iterative methods in manifolds have been studied in [6, 2, 1]. In this article, we extend the third order Newton-like method (2) in Riemannian manifolds to find the singular point of a vector field. We study the convergence theorem under Lipschitz continuity condition on the second order covariant derivative of a vector field and by using recurrence relations of the method.

The article is divided into six sections as follows: Section 1 is introduction. In Section 2, we introduce some basic results of differential geometry. In Section 3, we present recurrence relations for third order Newton-like method in Riemannian manifolds. In Section 4, we establish existence and uniqueness theorem of our method. In Section 5, two numerical examples are given. In Section 6, some brief conclusions are given.

2 Preliminaries

In this section, we introduce some basic results of differential geometry (for more details see [10, 13, 12]).

Let $\mathbb{K}$ be a real n dimensional Riemannian manifold. The tangent space of $\mathbb{K}$ at a is denoted by $T_a\mathbb{K}$. The inner product $\langle .,. \rangle_a$ on $T_a\mathbb{K}$ induces the norm $\|.\|_a$. The tangent bundle of $\mathbb{K}$ is denoted by $T\mathbb{K}$ and is defined by

$$T\mathbb{K} := \{(a, v); a \in \mathbb{K} \; and \; v \in T_a\mathbb{K}\} = \bigcup_{a \in Z} T_a\mathbb{K}.$$

Let $a, t \in \mathbb{K}$, and $\varrho : [0, 1] \to \mathbb{K}$ be a piecewise smooth curve joining a and t. Then the arc length of ϱ is defined by $l(\varrho) = \int_0^1 \|\varrho'(x)\| dx$, and the Riemannian distance from a to t is defined by $d(a, t) = \inf_\varrho l(\varrho)$, where the infimum is taken over all the piecewise smooth curves ϱ connecting a and t. Let $\chi(\mathbb{K})$ be the set of all vector fields of class C^∞ on $\mathbb{K}$ and $D(\mathbb{K})$ the ring of real-valued functions of class C^∞ defined on $\mathbb{K}$. An affine connection ∇ on $\mathbb{K}$ is a mapping

$$\begin{aligned} \nabla : \chi(\mathbb{K}) \times \chi(\mathbb{K}) &\to \chi(\mathbb{K}) \\ (X, \mathfrak{F}) &\mapsto \nabla_X \mathfrak{F} \end{aligned}$$

that satisfies the following properties

$$\text{(i) } \nabla_{fX + g\mathfrak{F}}\mathfrak{V} = f\nabla_X\mathfrak{V} + g\nabla_{\mathfrak{F}}\mathfrak{V}.$$

$$\text{(ii) } \nabla_X(\mathfrak{F} + \mathfrak{V}) = \nabla_X\mathfrak{F} + \nabla_X\mathfrak{V}.$$

$$\text{(iii) } \nabla_X(f\mathfrak{F}) = f\nabla_X\mathfrak{F} + X(f)\mathfrak{F},$$

where $X,\ \mathfrak{F},\ \mathfrak{V} \in \chi(\mathbb{K})$ and $f\ , g \in D(\mathbb{K})$. Let $\mathfrak{F}$ be a vector field of class C^1 on $\mathbb{K}$, the covariant derivative of $\mathfrak{F}$ is determined by the connection ∇ which defines on each $a \in \mathbb{K}$, a linear application of $T_a\mathbb{K}$ itself

$$\begin{aligned} D\mathfrak{F}(a) : T_a\mathbb{K} &\to T_a\mathbb{K} \\ v \mapsto D\mathfrak{F}(a)(v) &= \nabla_X \mathfrak{F}(a), \end{aligned}$$

where X is a vector field satisfies $X(a) = v$. A parametrized curve $\varrho : I \subseteq \mathbb{R} \to \mathbb{K}$ is a geodesic at $p_0 \in I$, if $\nabla_{\varrho'(p)}\varrho'(p) = 0$ at the point p_0. If ϱ is a geodesic for all $p \in I$, then we say ϱ is a geodesic. If $[x, y] \subseteq I$, then ϱ is a geodesic segment joining $\varrho(x)$ to $\varrho(y)$. A basic property of geodesic is that, $\varrho'(p)$ is parallel along $\varrho(p)$ therefore $\|\varrho'(p)\|$ is constant. Let $U(a, s)$ and $U[a, s]$ be an open and a closed geodesic ball with centre a and radius s respectively. By the Hopf-Rinow theorem, if $\mathbb{K}$ is complete metric space, then for any $a, t \in \mathbb{K}$ there exists a geodesic ϱ called minimizing geodesic joining a to t with

$$l(\varrho) = d(a, t).$$

If $v \in T_a\mathbb{K}$ then there exists a unique minimizing geodesic ϱ such that $\varrho(0) = a$ and $\varrho'(0) = v$. The point $\varrho(1)$ is called the image of v by the exponential map at a, i.e.

$$\exp_a : T_a\mathbb{K} \to \mathbb{K},$$

such that $\exp_a(v) = \varrho(1)$ and $\varrho(p) = \exp_a(pv)$ for any $p \in [0, 1]$. Let ϱ be a piecewise smooth curve. Then for any $x, y \in \mathbb{R}$, the parallel transport along ϱ is denoted by $R_{\varrho,\cdot,\cdot}$ and given by

$$\begin{aligned} R_{\varrho,x,y} : T_{\varrho(x)}\mathbb{K} &\to T_{\varrho(y)}\mathbb{K} \\ v &\mapsto V(\varrho(y)), \end{aligned}$$

where V is the unique vector field along ϱ such that $\nabla_{\varrho'(p)}V = 0$ and $V(\varrho(x)) = v$. Let $j \in \mathbb{N}$ and $\mathfrak{F}$ be a vector field of class C^k. Then the covariant derivative of order j of $\mathfrak{F}$ is denoted by $D^j\mathfrak{F}$ and defined as the multilinear map

$$D^j\mathfrak{F} : \underbrace{C^k(T\mathbb{K}) \times C^k(T\mathbb{K}) \times \cdots \times C^k(T\mathbb{K})}_{\text{j-times}} \to C^{k-j}(T\mathbb{K})$$

which is given by

$$\begin{aligned} D^j\mathfrak{F}(A_1, A_2, \ldots, A_{j-1}, A) &= \nabla_A D^{j-1}\mathfrak{F}(A_1, A_2, \ldots, A_{j-1}) \\ &\quad - \sum_{i=1}^{j-1} D^{j-1}\mathfrak{F}(A_1, A_2, \ldots, \nabla_A A_i, \ldots, A_{j-1}), \end{aligned}$$

for all $A_1, A_2, \ldots, A_{j-1} \in C^k(T\mathbb{K})$.

Definition 1. *Let $\mathbb{K}$ be a Riemannian manifold, $\Omega \subseteq \mathbb{K}$ be an open convex set and $\mathfrak{F} \in \chi(\mathbb{K})$. Then $D\mathfrak{F} = \nabla_{(.)}\mathfrak{F}$ is Lipschitz with constant $M > 0$, if for any geodesic ϱ and $x, y \in \mathbb{R}$ such that $\varrho[x, y] \subseteq \Omega$, it holds the inequality*

$$\|R_{\varrho,y,x} D\mathfrak{F}(\varrho(y))R_{\varrho,x,y} - D\mathfrak{F}(\varrho(x))\| \leq M \int_x^y \|\varrho'(p)\| dp.$$

We will write $D\mathfrak{F} \in Lip_M(\Omega)$. If $\mathbb{K} = \mathbb{R}^n$, then the above definition is same as usual Lipschitz condition for the operator $D\mathfrak{F} : \mathbb{K} \to \mathbb{K}$.

Proposition 1. *Let ϱ be a curve in $\mathbb{K}$ and $\mathfrak{F}$ be a C^1 vector field on $\mathbb{K}$, then the covariant derivative of $\mathfrak{F}$ in the direction of $\varrho'(t)$ is defined as*

$$D\mathfrak{F}(\varrho(t))\varrho'(t) = \nabla_{\varrho'(t)}\mathfrak{F}_{\varrho(t)} = \lim_{r\to 0} \frac{1}{r}\big(R_{\varrho,t+r,t}\mathfrak{F}(\varrho(t + r)) - \mathfrak{F}(\varrho(t))\big).$$

If $\mathbb{K} = \mathbb{R}^n$, then it is same as directional derivative in $\mathbb{R}^n$.

Next, we take some theorems from [1] in order to prove our convergence theorem.

Theorem 1. *Let ϱ be a geodesic in $\mathbb{K}$ and $\mathfrak{F}$ be a C^1-vector field on $\mathbb{K}$. Then*

$$R_{\varrho,t,0}\mathfrak{F}(\varrho(t)) = \mathfrak{F}(\varrho(0)) + \int_0^t R_{\varrho,\theta,0} D\mathfrak{F}(\varrho(\theta))\varrho'(\theta)d\theta.$$

Theorem 2. *Let ϱ be a geodesic in $\mathbb{K}$ and $\mathfrak{F}$ be a C^2-vector field on $\mathbb{K}$. Then*

$$R_{\varrho,t,0} D\mathfrak{F}(\varrho(t))\varrho'(t) = D\mathfrak{F}(\varrho(0))\varrho'(0) + \int_0^t R_{\varrho,\theta,0} D^2\mathfrak{F}(\varrho(\theta))(\varrho'(\theta), \varrho'(\theta))d\theta.$$

Theorem 3. *Let ϱ be a geodesic in $\mathbb{K}$ such that $[0, 1] \subseteq Dom(\varrho)$ and $\mathfrak{F}$ be a C^2-vector field on $\mathbb{K}$. Then*

$$\begin{aligned} R_{\varrho,1,0}\mathfrak{F}(\varrho(1)) &= \mathfrak{F}(\varrho(0)) + D\mathfrak{F}(\varrho(0))\varrho'(0) \\ &\quad + \int_0^1 (1 - \theta)R_{\varrho,\theta,0} D^2\mathfrak{F}(\varrho(\theta))(\varrho'(\theta), \varrho'(\theta))d\theta. \end{aligned}$$

3 Recurrence relations for a third order Newton-like method in Riemannian manifolds

In this section, we will establish the recurrence relations for third order Newton - like method in Riemannian manifolds. The third order Newton - like method

(2) in Riemannian manifolds has the form

$$\left.\begin{aligned} g_n &= -D\mathfrak{F}(a_n)^{-1}\mathfrak{F}(a_n),\\ b_n &= \exp_{a_n}(g_n),\\ \alpha_n(t) &= \exp_{a_n}(tg_n),\\ h_n &= -2\big(D\mathfrak{F}(a_n) + R_{\alpha_n,1,0}D\mathfrak{F}(b_n)R_{\alpha_n,0,1}\big)^{-1}\mathfrak{F}(a_n),\\ a_{n+1} &= \exp_{a_n}(h_n), \text{ for each } n = 0,1,2,\dots, \end{aligned}\right\} \tag{3}$$

where $D\mathfrak{F}(a_n) = \nabla_{(.)}\mathfrak{F}(a_n)$. Let $a_0 \in \Omega \subseteq \mathbb{K}$ and assume that

1. $\|D\mathfrak{F}(a_0)^{-1}\| \le \varepsilon,\ \varepsilon > 0,$
2. $\|D\mathfrak{F}(a_0)^{-1}\mathfrak{F}(a_0)\| \le \varphi,\ \varphi > 0,$
3. $\|D^2\mathfrak{F}(a)\| \le \varpi,$ for all $a \in \Omega,\ \varpi > 0,$
4. $\|R_{\varrho,b,a}D^2\mathfrak{F}(\varrho(b))R^2_{\varrho,a,b} - D^2\mathfrak{F}(\varrho(a))\| \le K\int_a^b \|\varrho'(x)\|dx, \quad K > 0,$
   where ϱ is a geodesic such that $\varrho[a,b] \subseteq \Omega$.

We take $p = \varpi\varphi\varepsilon,\ q = K\varepsilon\varphi^2,\ p_0 = 1,\ q_0 = 1,\ r_0 = p/2,\ s_0 = 2/(2-p)$, and define for $n = 0,1,2,\dots$

$$\begin{aligned} p_{n+1} &= \frac{p_n}{1 - pp_ns_n},\\ q_{n+1} &= p_{n+1}s_n^2\left[\frac{p}{2}(2r_n^2 - 7r_n + 6) + \frac{5q}{12}(1-r_n)^3s_n\right],\\ r_{n+1} &= \frac{pp_{n+1}q_{n+1}}{2},\\ s_{n+1} &= \frac{q_{n+1}}{1 - r_{n+1}}. \end{aligned}$$

In these conditions, for $n \ge 0$, we prove the following inequalities:

- $\|D\mathfrak{F}(a_n)^{-1}\| \le p_n\varepsilon,$
- $\|D\mathfrak{F}(a_n)^{-1}\mathfrak{F}(a_n)\| \le q_n\varphi,$
- $\left\|I_{a_n} - D\mathfrak{F}(a_n)^{-1}\frac{D\mathfrak{F}(a_n)+R_{\alpha_n,1,0}D\mathfrak{F}(b_n)R_{\alpha_n,0,1}}{2}\right\| \le r_n,$
- $d(a_{n+1},a_n) \le s_n\varphi,$
- $d(a_{n+1},b_n) \le (q_n + s_n)\varphi.$

To prove the above inequalities we will need some Lemmas.

Lemma 1. *Let $\mathfrak{F}$ be a C^2 vector field. Then, for all $n \ge 0$, we have*

$$R_{\alpha_n,1,0}\mathfrak{F}(b_n) = \int_0^1 (1-t)R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))(R_{\alpha_n,0,t}g_n, R_{\alpha_n,0,t}g_n)dt,$$

where α_n is a family of minimizing geodesics given as above.

Proof. By Theorem 3 and (3), we have

$$\begin{aligned}P_{\alpha_n,1,0}\mathfrak{F}(b_n) &= \mathfrak{F}(a_n) + D\mathfrak{F}(a_n)g_n + \int_0^1 (1-t)R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))(\alpha_n'(t),\alpha_n'(t))dt\\ &= \mathfrak{F}(a_n) + D\mathfrak{F}(a_n)(-D\mathfrak{F}(a_n)^{-1}\mathfrak{F}(a_n))\\ &+ \int_0^1 (1-t)R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))(\alpha_n'(t),\alpha_n'(t))dt\\ &= \int_0^1 (1-t)R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))(\alpha_n'(t),\alpha_n'(t))dt.\end{aligned}$$

Since α_n is a family of minimizing geodesics, then $\alpha_n'(t)$ is parallel and $\alpha_n'(t) = R_{\alpha_n,0,t}\alpha_n'(0),\ \alpha_n'(0) = g_n$. We have

$$R_{\alpha_n,1,0}\mathfrak{F}(b_n) = \int_0^1 (1-t)R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))(R_{\alpha_n,0,t}g_n, R_{\alpha_n,0,t}g_n)dt.$$

□

Lemma 2. *Let $\mathfrak{F}$ be a C^2 vector field on $\mathbb{K}$ and μ_n be a family of minimizing geodesics defined by $\mu_n(t) = \exp_{b_n}(tl_n)$, where $\mu_n(0) = b_n$, $\mu_n(1) = a_{n+1}$ and $l_n = R_{\alpha_n,0,1}\Big[D\mathfrak{F}(a_n)^{-1}\mathfrak{F}(a_n) - 2\big(D\mathfrak{F}(a_n) + R_{\alpha_n,1,0}D\mathfrak{F}(b_n)R_{\alpha_n,0,1}\big)^{-1}\mathfrak{F}(a_n)\Big]$. Then, for all $n \geq 0$, we have*

$$\begin{aligned}R_{\mu_n,1,0}\mathfrak{F}(a_{n+1}) &= \int_0^1 (1-t)R_{\mu_n,t,0}D^2\mathfrak{F}(\mu_n(t))R^2_{\mu_n,0,t}(l_n,l_n)dt\\ &+ \frac{1}{2}R_{\alpha_n,0,1}\int_0^1 R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))R^2_{\alpha_n,0,t}(g_n, R_{\alpha_n,1,0}l_n)dt\\ &+ R_{\alpha_n,0,1}\int_0^1 (1-t)R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))R^2_{\alpha_n,0,t}(g_n,g_n)dt\\ &- \frac{1}{2}R_{\alpha_n,0,1}\int_0^1 R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))R^2_{\alpha_n,0,t}(g_n,g_n)dt,\end{aligned}$$

where α_n is a family of minimizing geodesics given as above.

Proof. Since $\alpha_n(0) = a_n$ and $\alpha_n(1) = b_n$. Then, by Theorem 2, we have

$$\begin{aligned}
R_{\alpha_n,1,0}D\mathfrak{F}(b_n)l_n &= \frac{1}{2}[R_{\alpha_n,1,0}D\mathfrak{F}(b_n)R_{\alpha_n,0,1} - D\mathfrak{F}(a_n)]R_{\alpha_n,1,0}l_n \\
&+ \frac{1}{2}[R_{\alpha_n,1,0}D\mathfrak{F}(b_n)R_{\alpha_n,0,1} + D\mathfrak{F}(a_n)]R_{\alpha_n,1,0}l_n \\
&= \frac{1}{2}\int_0^1 R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))R^2_{\alpha_n,0,t}(g_n, R_{\alpha_n,1,0}l_n)dt \\
&+ \frac{1}{2}[R_{\alpha_n,1,0}D\mathfrak{F}(b_n)R_{\alpha_n,0,1} + D\mathfrak{F}(a_n)]R_{\alpha_n,1,0}R_{\alpha_n,0,1} \\
&\times \Big[D\mathfrak{F}(a_n)^{-1}\mathfrak{F}(a_n) - 2\big(D\mathfrak{F}(a_n) + R_{\alpha_n,1,0}D\mathfrak{F}(b_n)R_{\alpha_n,0,1}\big)^{-1}\mathfrak{F}(a_n)\Big] \\
&= \frac{1}{2}\int_0^1 R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))R^2_{\alpha_n,0,t}(g_n, R_{\alpha_n,1,0}l_n)dt \\
&+ \Big[\frac{1}{2}[R_{\alpha_n,1,0}D\mathfrak{F}(b_n)R_{\alpha_n,0,1} + D\mathfrak{F}(a_n)] - D\mathfrak{F}(a_n)\Big]D\mathfrak{F}(a_n)^{-1}\mathfrak{F}(a_n) \\
&= \frac{1}{2}\int_0^1 R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))R^2_{\alpha_n,0,t}(g_n, R_{\alpha_n,1,0}l_n)dt \\
&- \frac{1}{2}\int_0^1 R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))R^2_{\alpha_n,0,t}(g_n, g_n)dt,
\end{aligned}$$

implies that

$$\begin{aligned}
D\mathfrak{F}(b_n)l_n &= \frac{1}{2}R_{\alpha_n,0,1}\int_0^1 R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))R^2_{\alpha_n,0,t}(g_n, R_{\alpha_n,1,0}l_n)dt \\
&- \frac{1}{2}R_{\alpha_n,0,1}\int_0^1 R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))R^2_{\alpha_n,0,t}(g_n, g_n)dt.
\end{aligned}$$

By using Theorem 3 and Lemma 1, we get

$$\begin{aligned}
R_{\mu_n,1,0}\mathfrak{F}(a_{n+1}) &= R_{\mu_n,1,0}\mathfrak{F}(a_{n+1}) - \mathfrak{F}(b_n) - D\mathfrak{F}(b_n)l_n + \mathfrak{F}(b_n) + D\mathfrak{F}(b_n)l_n \\
&= \int_0^1 (1-t)R_{\mu_n,t,0}D^2\mathfrak{F}(\mu_n(t))R^2_{\mu_n,0,t}(l_n, l_n)dt \\
&+ \frac{1}{2}R_{\alpha_n,0,1}\int_0^1 R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))R^2_{\alpha_n,0,t}(g_n, R_{\alpha_n,1,0}l_n)dt \\
&+ R_{\alpha_n,0,1}\int_0^1 (1-t)R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))R^2_{\alpha_n,0,t}(g_n, g_n)dt \\
&- \frac{1}{2}R_{\alpha_n,0,1}\int_0^1 R_{\alpha_n,t,0}D^2\mathfrak{F}(\alpha_n(t))R^2_{\alpha_n,0,t}(g_n, g_n)dt.
\end{aligned}$$

□

Now, we can prove the above inequalities by the principle of mathematical induction. For $n = 0$, the first and second inequlities are trivial, but we have to prove all inequalities. We start with the third inequality. Since

$$\begin{aligned}\Big\| I_{a_0} - D\mathfrak{F}(a_0)^{-1}\frac{D\mathfrak{F}(a_0)+R_{\alpha_0,1,0}D\mathfrak{F}(b_0)R_{\alpha_0,0,1}}{2}\Big\| \\ = \Big\| D\mathfrak{F}(a_0)^{-1}\frac{D\mathfrak{F}(a_0)-R_{\alpha_0,1,0}D\mathfrak{F}(b_0)R_{\alpha_0,0,1}}{2}\Big\| \\ \le \frac{1}{2}\varpi\|D\mathfrak{F}(a_0)^{-1}\|d(a_0,b_0) \\ \le \frac{\varepsilon\varphi\varpi}{2} = \frac{p}{2} = r_0 < 1,\end{aligned}$$

which shows that the third inequality is true for $n=0$. Now, by Banach's Lemma [9],

$$D\mathfrak{F}(a_0)^{-1}\frac{D\mathfrak{F}(a_0)+R_{\alpha_0,1,0}D\mathfrak{F}(b_0)R_{\alpha_0,0,1}}{2}$$

is invertible and

$$\Big\|\Big(\frac{D\mathfrak{F}(a_0)+R_{\alpha_0,1,0}D\mathfrak{F}(b_0)R_{\alpha_0,0,1}}{2}\Big)^{-1}D\mathfrak{F}(a_0)\Big\| \le \frac{1}{1-p/2} = \frac{2}{2-p}.$$

We have

$$\begin{aligned}d(a_1,a_0) &= \Big\|\Big(\frac{D\mathfrak{F}(a_0)+R_{\alpha_0,1,0}D\mathfrak{F}(b_0)R_{\alpha_0,0,1}}{2}\Big)^{-1}\mathfrak{F}(a_0)\Big\| \\ &\le \Big\|\Big(\frac{D\mathfrak{F}(a_0)+R_{\alpha_0,1,0}D\mathfrak{F}(b_0)R_{\alpha_0,0,1}}{2}\Big)^{-1}D\mathfrak{F}(a_0)\Big\|\,\|D\mathfrak{F}(a_0)^{-1}\mathfrak{F}(a_0)\| \\ &\le \frac{2\varphi}{2-p} = s_0\varphi\end{aligned}$$

and

$$\begin{aligned}d(a_1,b_0) &\le d(a_1,a_0)+d(a_0,b_0) \\ &\le \frac{2\varphi}{2-p}+\varphi = (q_0+s_0)\varphi.\end{aligned}$$

Therefore all inequalities are true for $n=0$. Suppose that all inequalities are true for $n=0,1,2,\ldots,k$ and consider $a_k\in\Omega,\ r_{k+1}<1$ and $pp_ks_k<1$. Then, we shall prove for $n=k+1$. We have

$$\|D\mathfrak{F}(a_k)^{-1}\|\|R_{\mathfrak{Q},1,0}D\mathfrak{F}(a_{k+1})R_{\mathfrak{Q},0,1}-D\mathfrak{F}(a_k)\| \le pp_ks_k < 1.$$

By Banach's Lemma, $R_{\mathfrak{Q},1,0}D\mathfrak{F}(a_{k+1})R_{\mathfrak{Q},0,1}$ is invertible and

$$\begin{aligned}\|R_{\mathfrak{Q},1,0}D\mathfrak{F}(a_{k+1})^{-1}R_{\mathfrak{Q},0,1}\| &= \|D\mathfrak{F}(a_{k+1})^{-1}\| \\ &\le \frac{\|D\mathfrak{F}(a_k)^{-1}\|}{1-\|D\mathfrak{F}(a_k)^{-1}\|\|R_{\mathfrak{Q},1,0}D\mathfrak{F}(a_{k+1})R_{\mathfrak{Q},0,1}-D\mathfrak{F}(a_k)\|} \\ &\le \frac{p_k\varepsilon}{1-pp_ks_k} = p_{k+1}\varepsilon,\end{aligned}$$

where $\mathfrak{Q}$ is a minimizing geodesic joining the points a_k and a_{k+1} such that $\mathfrak{Q}(0) = a_k$ and $\mathfrak{Q}(1) = a_{k+1}$. Also

$$
\begin{aligned}
&\|R_{\alpha_k,0,1}\Big(\int_0^1 R_{\alpha_k,t,0}D^2\mathfrak{F}(\alpha_k(t))R^2_{\alpha_k,0,t}(1-t)dt(g_k,g_k)\\
&\quad -\frac{1}{2}\int_0^1 R_{\alpha_k,t,0}D^2\mathfrak{F}(\alpha_k(t))R^2_{\alpha_k,0,t}dt(g_k,g_k)\Big)\|\\
&\leq \left\|\int_0^1 \Big(R_{\alpha_k,t,0}D^2\mathfrak{F}(\alpha_k(t))R^2_{\alpha_k,0,t} - D^2\mathfrak{F}(a_k)\Big)(1-t)dt\right\| d(b_k,a_k)^2\\
&\quad + \left\|\frac{1}{2}\int_0^1 \Big(R_{\alpha_k,t,0}D^2\mathfrak{F}(\alpha_k(t))R^2_{\alpha_k,0,t} - D^2\mathfrak{F}(a_k)\Big)dt\right\| d(b_k,a_k)^2\\
&\leq \frac{K}{6}d(b_k,a_k)^3 + \frac{K}{4}d(b_k,a_k)^3 = \frac{5K}{12}d(b_k,a_k)^3.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|R_{\mu_k,1,0}\mathfrak{F}(a_{k+1})\| =& \|\mathfrak{F}(a_{k+1})\|\\
&\leq \frac{\varpi}{2}d(a_{k+1},b_k)^2 + \frac{\varpi}{2}d(a_{k+1},b_k)d(b_k,a_k) + \frac{5K}{12}d(b_k,a_k)^3\\
&\leq \frac{\varpi}{2}(q_k+s_k)^2\varphi^2 + \frac{\varpi}{2}q_k(q_k+s_k)\varphi^2 + \frac{5K}{12}q_k^3\varphi^3. \qquad (4)
\end{aligned}
$$

We obtain that

$$
\begin{aligned}
\|D\mathfrak{F}(a_{k+1})^{-1}\mathfrak{F}(a_{k+1})\| \leq& \|\mathfrak{F}(a_{k+1})\|\|D\mathfrak{F}(a_{k+1})^{-1}\|\\
&\leq p_{k+1}\varepsilon\Big(\frac{\varpi}{2}(q_k+s_k)^2\varphi^2 + \frac{\varpi}{2}q_k(q_k+s_k)\varphi^2 + \frac{5K}{12}q_k^3\varphi^3\Big)\\
&= p_{k+1}\Big(\frac{p}{2}(q_k+s_k)^2\varphi + \frac{p}{2}q_k(q_k+s_k)\varphi + \frac{5q}{12}q_k^3\varphi\Big)\\
&= p_{k+1}\Big(\frac{p}{2}(2q_k^2+s_k^2+3q_ks_k) + \frac{5q}{12}q_k^3\Big)\varphi\\
&= p_{k+1}\Big(\frac{p}{2}\big(2(1-r_k)^2s_k^2 + s_k^2 + 3(1-r_k)s_k^2\big)\\
&\qquad + \frac{5q}{12}(1-r_k)^3s_k^3\Big)\varphi\\
&= p_{k+1}s_k^2\Big(\frac{p}{2}(2r_k^2-7r_k+6) + \frac{5q}{12}(1-r_k)^3s_k\Big)\varphi\\
&= q_{k+1}\varphi.
\end{aligned}
$$

Next, as

$$
\begin{aligned}
&\left\|I_{a_{k+1}} - D\mathfrak{F}(a_{k+1})^{-1}\frac{D\mathfrak{F}(a_{k+1}) + R_{\alpha_{k+1},1,0}D\mathfrak{F}(b_{k+1})R_{\alpha_{k+1},0,1}}{2}\right\|\\
&= \left\|D\mathfrak{F}(a_{k+1})^{-1}\frac{D\mathfrak{F}(a_{k+1}) - R_{\alpha_{k+1},1,0}D\mathfrak{F}(b_{k+1})R_{\alpha_{k+1},0,1}}{2}\right\|\\
&\leq \frac{1}{2}\varpi\|D\mathfrak{F}(a_{k+1})^{-1}\|d(a_{k+1},b_{k+1})\\
&= \frac{pp_{k+1}q_{k+1}}{2} = r_{k+1} < 1.
\end{aligned}
$$

By Banach's Lemma, $D\mathfrak{F}(a_{k+1})^{-1}\frac{D\mathfrak{F}(a_{k+1})+R_{\alpha_{k+1},1,0}D\mathfrak{F}(b_{k+1})R_{\alpha_{k+1},0,1}}{2}$ is invertible and

$$\left\|\Big(\frac{D\mathfrak{F}(a_{k+1})+R_{\alpha_{k+1},1,0}D\mathfrak{F}(b_{k+1})R_{\alpha_{k+1},0,1}}{2}\Big)^{-1}D\mathfrak{F}(a_{k+1})\right\| \leq \frac{1}{1-r_{k+1}}.$$

This implies,

$$\begin{aligned} d(a_{k+2},a_{k+1}) &= \left\|\Big(\frac{D\mathfrak{F}(a_{k+1})+R_{\alpha_{k+1},1,0}D\mathfrak{F}(b_{k+1})R_{\alpha_{k+1},0,1}}{2}\Big)^{-1}\mathfrak{F}(a_{k+1})\right\| \\ &\leq \left\|\Big(\frac{D\mathfrak{F}(a_{k+1})+R_{\alpha_{k+1},1,0}D\mathfrak{F}(b_{k+1})R_{\alpha_{k+1},0,1}}{2}\Big)^{-1}D\mathfrak{F}(a_{k+1})\right\| \\ &\quad \times \|D\mathfrak{F}(a_{k+1})^{-1}\mathfrak{F}(a_{k+1})\| \\ &\leq \frac{q_{k+1}\varphi}{1-r_{k+1}} = s_{k+1}\varphi \end{aligned}$$

and

$$\begin{aligned} d(a_{k+2},b_{k+1}) &\leq d(a_{k+2},a_{k+1}) + d(a_{k+1},b_{k+1}) \\ &\leq s_{k+1}\varphi + q_{k+1}\varphi = (q_{k+1}+s_{k+1})\varphi. \end{aligned}$$

Thus, we conclude that all inequalities are true for $n = k+1$. Hence, by mathematical induction it holds for all n.

4 Convergence analysis

In this section, we will prove convergence and uniqueness of third order Newton - like method in Riemannian manifolds. Before that, we will prove some Lemmas.

Lemma 3. *Let* $\boldsymbol{z}_0 = 0.0952980448\ldots$ *be the smallest positive root of* $1-12a+16a^2-2a^3=0$ *and we define the functions*

$$\begin{aligned} c(a) &= \frac{3}{5}\frac{(1-12a+16a^2-2a^3)}{(1-a)^2}, \\ \mathfrak{C}(a,b) &= \frac{1}{(1-3a)^2}\Big(2a^2-7a+6+\frac{5q}{3p^2b}(1-a)^2a\Big), \\ \mathfrak{B}(a,b) &= \frac{a^2}{(1-3a)^2}\Big(2a^2-7a+6+\frac{5q}{3p^2b}(1-a)^2a\Big), \\ \mathfrak{B}_0(a) &= \mathfrak{B}(a,1). \end{aligned}$$

Then, $c(a)$ *is decreasing in* $[0,\boldsymbol{z}_0]$, $\mathfrak{C}(a,b)$ *and* $\mathfrak{B}(a,b)$ *are increasing in the variable* a *in* $[0,\boldsymbol{z}_0]$ *for* $b>1$, $\mathfrak{C}(a,b)$ *and* $\mathfrak{B}(a,b)$ *are decreasing in the variable* b, $\mathfrak{B}_0(a)$, *and* $\mathfrak{B}_0'(a)$ *are increasing in* $[0,\boldsymbol{z}_0]$.

Proof. It is easy to prove and hence omitted. □

Lemma 4. *The following recurrence relations are true for the sequences* $\{p_n\}$ *and* $\{r_n\}$.

$$p_{n+1} = \prod_{k=0}^{n}\Big(1+\frac{2r_k}{1-3r_k}\Big),$$
$$r_{n+1} = \frac{r_n^2(2r_n^2-7r_n+6)}{(1-3r_n)^2}\Big(1+\frac{5q}{3p^2p_n}\frac{(1-r_n)^2r_n}{(2r_n^2-7r_n+6)}\Big).$$

Proof. The proof follows by the use of definitions of the sequences $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ and $\{s_n\}$. □

Lemma 5. *Let* $0 < p \leq \boldsymbol{z}_0$ *and* $0 \leq q \leq 4c(\frac{p}{2})$. *Then, the sequences* $\{p_n\}$ *and* $\{r_n\}$ *are increasing and decreasing respectively. We also have* $r_n < 1,\ p_n > 1,\ pp_ns_n < 1\ \ \forall n \in \mathbb{N}$.

Proof. See [11]. □

Lemma 6. *Under the assumptions of Lemma 5, we define* $\alpha = r_2/r_1$, *then*

$$r_{n+1} \leq \alpha^{2^{n+1}} r_0 \quad \forall \in \mathbb{N}$$

and the sequence $\{r_n\}$ *converges to* 0. *Also* $\sum_{n=0}^{\infty} r_n < \infty$.

Proof. See [11]. □

Lemma 7. *The sequence* $\{p_n\}$ *is bounded above, that is, there exist a constant* M *such that* $p_n \leq M,\ \forall n \in \mathbb{N}$.

Proof. See [11]. □

Lemma 8. *The sequence* $\{s_n\}$ *is a Cauchy sequence, as* $s_n < \frac{8}{3p}s_0\alpha^{2^n}$. *Also* $\sum_{n=0}^{\infty} s_n < \infty$.

Proof. See [11]. □

Now, we can prove our main convergence theorem.

Theorem 4. *Let* $\mathbb{K}$ *be a complete Riemannian manifold,* $\Omega \subseteq \mathbb{K}$ *be an open convex set and* $\mathfrak{F} \in \chi(\mathbb{K})$ *satisfies the conditions* $(1)-(3)$ *with* $0 < p \leq 2\boldsymbol{z}_0$ *and* $0 \leq q \leq 4c(\frac{p}{2})$. *Then, the method given by* (3) *converges to a singular point* a^* *of the vector field* $\mathfrak{F}$ *with* a_n, b_n *and* a^* *belonging to* $U[a_0, r\varphi]$ *and* a^* *is the unique singular point of* $\mathfrak{F}$ *in* $U(a_0, (2/\varpi\varepsilon) - r\varphi) \cap \Omega$, *where* $r = \sum_{n=0}^{\infty} s_n$.

Proof. If $0 < p < 2\mathbf{z}_0$. Then, by Lemma 8, $\{s_n\}$ is a Cauchy sequence . Now, if $p = 2\mathbf{z}_0, q = c(p/2) = c(\mathbf{z}_0) = 0$. Then, we have $r_n = r_0 = p/2$, for $n \geq 0$. Also

$$p_{n+1} = p_n\Big(1+\frac{2r_n}{1-3r_n}\Big) = p_n\Big(1+\frac{2r_0}{1-3r_0}\Big) = \rho p_n = \rho^{n+1},$$

where $\rho = 1 + \frac{2r_0}{1-3r_0} > 1$ and

$$s_n = \frac{2r_n}{pp_n(1-r_n)} = \frac{2r_0}{pp_n(1-r_0)} = \frac{1}{\rho^n(1-r_0)} = \frac{s_0}{\rho^n},$$

this shows that $\{s_n\}$ is Cauchy sequence. Thus, the sequence $\{s_n\}$ is Cauchy in both the cases and therefore $\{a_n\}$ is a convergent sequence. If a^* is the limit of $\{a_n\}$, then we will prove that $\mathfrak{F}(a^*) = 0$. From (4), we have

$$\begin{aligned}\|\mathfrak{F}(a_{n+1})\| \leq &\frac{\varpi}{2}(q_n + s_n)^2\varphi^2 + \frac{\varpi}{2}q_n(q_n + s_n)\varphi^2 + \frac{5K}{12}q_n^3\varphi^3 \\ =&\frac{1}{\varepsilon}s_n^2\Big(\frac{p}{2}(2r_n^2 - 7r_n + 6) + \frac{5q}{12}(1-r_n)^3 s_n\Big)\varphi,\end{aligned}$$

as $\{r_n\}$ and $\{s_n\} \to 0$, when $n \to \infty$. Therefore $\mathfrak{F}(a^*) = 0$. Also

$$\begin{aligned}d(a_{n+1}, a_0) &\leq d(a_{n+1}, a_n) + d(a_n, a_{n-1}) + \cdots + d(a_1, a_0) \\ &\leq \sum_{k=0}^{n} s_k\varphi \leq r\varphi,\end{aligned}$$

this shows that $a_n \in U[a_0, r\varphi]$ and similarly we can prove that $b_n \in U[a_0, r\varphi]$. Now taking limit $n \to \infty$, we get $a^* \in U[a_0, r\varphi]$. Next, we will prove the singularity is unique. Let z^* be another singularity of $\mathfrak{F}$ in $U(a_0, (2/\varpi\varepsilon) - r\varphi)$ and ϑ be a minimizing geodesic joining the points a^* and z^* such that $\vartheta(0) = a^*$ and $\vartheta(1) = z^*$. Then, we have

$$\begin{aligned}\|R_{\vartheta,t,0}D\mathfrak{F}(\vartheta(t))R_{\vartheta,0,t} - D\mathfrak{F}(a^*)\| \leq \varpi\int_0^t \|\vartheta'(0)\|ds \\ =\varpi t d(a^*, z^*) \leq \varpi t\Big(d(a_0, a^*) + d(a_0, z^*)\Big).\end{aligned}$$

Hence

$$\begin{aligned}&\|D\mathfrak{F}(a^*)^{-1}\|\int_0^1 \|R_{\vartheta,t,0}D\mathfrak{F}(\vartheta(t))R_{\vartheta,0,t}dt - D\mathfrak{F}(a^*)\|dt \\ \leq&M\varepsilon\int_0^1 \varpi t\Big(d(a_0, a^*) + d(a_0, z^*)\Big)dt \\ <&\frac{M\varepsilon\varpi}{2}\Big(r\varphi + \frac{2}{\varpi\varepsilon} - r\varphi\Big).\end{aligned}$$

For $M = 1$, the operator $\int_0^1 R_{\vartheta,t,0}D\mathfrak{F}(\vartheta(t))R_{\vartheta,0,t}dt$ is invertible. Therefore by Banach's Lemma, we have

$$0 = R_{\vartheta,1,0}\mathfrak{F}(z^*) - \mathfrak{F}(a^*) = \int_0^1 R_{\vartheta,t,0}D\mathfrak{F}(\vartheta(t))R_{\vartheta,0,t}(\vartheta'(0))dt.$$

Therefore $\vartheta'(0) = 0$. As $0 = \|\vartheta'(0)\| = d(a^*, z^*)$, we get $a^* = z^*$. Hence the proof is complete. □

5 Numerical examples

In this section, two numerical examples are given to illustrate the effectiveness of our results.

Example 1. *Let us consider the vector field Z from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by*

$$Z(a) = Z\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -a_2 \\ a_1 - a_1 a_3^2 \\ a_1 a_2 a_3 \end{pmatrix} \tag{5}$$

with the Frobenius norm and let $\mathfrak{F} = Z|_{\boldsymbol{S}^2}$. Then, it can be easily verified that

$$Z|_{\boldsymbol{S}^2}(a) \in T_a \boldsymbol{S}^2 \quad \forall\ a \in \boldsymbol{S}^2.$$

From [6], $D\mathfrak{F}(a)$ in the basis

$$\beta_a = \left\{ \begin{pmatrix} -a_3 \\ 0 \\ a_1 \end{pmatrix}, \begin{pmatrix} 0 \\ -a_3 \\ a_2 \end{pmatrix} \right\}$$

of $T_a \boldsymbol{S}^2$ is given by

$$D\mathfrak{F}(a) = \begin{pmatrix} -\frac{1}{a_3} c_{1,1}(a) & -\frac{1}{a_3} c_{1,2}(a) \\ -\frac{1}{a_3} c_{2,1}(a) & -\frac{1}{a_3} c_{2,2}(a) \end{pmatrix},$$

where

$$c_{i,j}(a) = a_j \Big(f_{i,a_3}(a) - \sum_{m=1}^{3} a_m f_{m,a_3}(a) a_i \Big) - a_3 \Big(f_{i,a_j}(a) - \sum_{m=1}^{3} a_m f_{m,a_j}(a) a_i \Big)$$

for $i, j = 1, 2$, $f_{i,a_j} = \frac{\partial f_i}{\partial a_j}$, and $[\mathfrak{F}(a)]_{\beta_a} = \big(- f_1(a)/a_3, -f_2(a)/a_3 \big)^T$. Therefore

$$D\mathfrak{F}(a) = \begin{pmatrix} -a_1 a_2 (a_1^2 + 1) & -1 - a_1^2 (a_2^2 + a_3^2 - 1) \\ 1 - a_2^2 - a_1^2(-2 + a_2^2) - a_3^2 & -a_1 a_2 (a_2^2 + a_3^2 - 3) \end{pmatrix}.$$

Next, by using the method of Lagrange's multipliers, we get

$$\varpi = \sup\{ D\mathfrak{F}(a_1, a_2, a_3) : a_1^2 + a_2^2 + a_3^2 = 5 \} = 11$$

is a Lipschitz constant of $D\mathfrak{F}$. Initially for $a_0 = (2, -0.0013091, 1)^T$, we get

$$\begin{aligned} \|D\mathfrak{F}(a_0)^{-1}\| &= 1.00779 = \varepsilon, \\ \|D\mathfrak{F}(a_0)^{-1} \mathfrak{F}(a_0)\| &= 0.0013193 = \varphi, \\ p &= 0.0146253508 \le 2\boldsymbol{z}_0. \end{aligned}$$

Therefore we must choose q such that $0 \le q \le 4c(p/2)$. Hence the equation (5) has a unique singularity a^* in $U(a_0, (2/\varpi\varepsilon) - r\varphi) \cap \Omega$.

Example 2. *Consider the integral equation*

$$\mathfrak{F}(a)(u) = -1 + a(u) + \frac{1}{4}a(u)\int_0^1 \frac{u}{u+v}a(v)dv,\ a(u) \subset \mathbb{K} = C[0,1] \tag{6}$$

and we define the norm $\|a\| = max_{0\leq u\leq 1}|a(u)|$. *Initially for* $a_0 = a_0(u) = 1$, *we get*

$$\begin{aligned} \|D\mathfrak{F}(a_0)^{-1}\| &= 1.17718382 = \varepsilon, \\ \|D\mathfrak{F}(a_0)^{-1}\mathfrak{F}(a_0)\| &= 0.08859191 = \varphi, \\ \|D^2\mathfrak{F}(a)\| &= 0.150514997 = \varpi, \\ p &= 0.015697053 \leq 2\boldsymbol{z}_0. \end{aligned}$$

Therefore we must choose q *such that* $0 \leq q \leq 4c(p/2)$. *Hence the equation* (6) *has a unique singularity* a^* *in* $U(a_0, (2/\varpi\varepsilon) - r\varphi) \cap \Omega$.

6 Conclusions

In this article, we have extended the third order Newton-like method from Banach space to Riemannian manifolds to find the singularity of a vector field. We have presented the convergence theorem under Lipschitz continuity condition on the second order covariant derivative of a vector field and by using the recurrence relations of the method. Finally, two numerical examples are given to illustrate the effectiveness of our results.

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