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ALMOST PERIODICALLY UNITARY SOLUTIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we consider a class of stochastic differential equations with almost periodic coefficients. In the one-dimensional case, by using the unitary group of operators associated to the stationary increments of the Brownian motion, we show the unitary almost periodicity of the solution. We also prove that the tightness is a property of almost periodically unitary processes. Some examples are given.

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1 Introduction

Ever since the generalization of the almost periodicity theory to the functions with values in abstract (Polish or Banach) spaces, given by Bochner [7], several classes of almost periodic (AP) functions have been introduced, mainly: Stepanov AP, Weyl AP, Besicovitch AP, etc., see [2] for an overview and the hierarchy of these notions. This theory has played a role in various branches of mathematics, most notably in differential equations. For more details concerning this theory and its application to differential equations, see for instance [1, 17, 23, 11, 9]. We focus in this paper on the class of random functions (stochastic processes). In this case, the almost periodicity forks into different types. In [19], T. Morozan and C. Tudor introduced the almost periodicity in one-dimensional distributions (APOD), in

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[13], H. L. Hurd, A. Russek and D. Surgailis defined the almost periodicity of the finite dimensional distributions (APFD) and for continuous processes, C. Tudor introduced the almost periodicity in infinite dimensional distributions (APD), see [22]. In the case of continuous second-order stochastic processes, the almost periodicity in quadratic mean (APQM) is given by P. H. Bezandry and T. Diagana [5]. In [12], E. G. Gladyshev treated the periodic and almost periodic correlated processes (PC and APC, respectively). On the other hand, the well-known link between the unitary groups of operators and some processes, such as stationary processes and PC processes, motivated H. L. Hurd [14] to propose the definition of almost periodically unitary (APU) processes which is the central concept in this work. Some relationships between the different types of almost periodicity are shown by C. Tudor in [22], Bedouhene et al. in [4]. All the above concepts are applied to study the stochastic differential equations (SDEs) with AP coefficients except the APU (APC), to our knowledge, until now, no work has broached this issue and this is due to the fact that the set of APU processes has neither algebraic nor topological suitable properties (no stability by addition and there isn't a complete adapted metric). For instance, the study of the existence of a quadratic mean almost periodic solution to SDEs has aroused the interest of several authors, unfortunately in [18] it is shown, by counterexamples, that it is strong as a property for solutions of SDEs, which means that, generally, there is no APQM solution. However, for the almost periodicity in distribution (APOD, APFD and APD), C. Tudor and his collaborators [21, 3, 19, 20] showed the existence of solutions (of these types) to various SDEs. M. Kamenskii, O. Mellah and P. Raynaud De Fitte [16] proved, under some conditions, that the unique bounded mild solution to some semilinear SDE in a Hilbert space is APD. In this work, we show that the unique bounded solution in one dimensional case is APU. The study of the infinite dimensional case uses the notion of measure conserving transformations. This work is in progress.

This paper is organized as follows: In the second section, we recall the definitions of the different types of almost periodic processes with more details and properties on the APU processes. In section 3, we prove that an APU process satisfies the tightness property given in [4, Remark 2.2.]. We also show that the solutions to SDEs given in [18] as conterexamples to mean square almost periodicity are APU. We finish by the study of existence and uniqueness of APU solutions to some SDE. In this part, we consider the following affine stochastic differential equation

$$dX(t) = -a\,X(t)\,dt + F(t)\,dt + G(t)\,dW(t),\ \ t \in \mathbb{R}, \tag{1}$$

where a is a real positive constant, $W := \{W_t\}_{t\in\mathbb{R}}$ is a standard Brownian motion (sbm) on $\mathbb{R}$, $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ are almost periodic functions. By using the unitary group of operators connected to the stationary increments of the sbm, we show that the unique solution to (1) is APU.

2 Notations and preliminaries

In this section, we recall some important definitions.

2.1 Almost periodic functions

Let $(\mathbb{E}, ||\cdot||)$ be a Banach space. *A continuous function $f:\mathbb{R}\to(\mathbb{E},||\cdot||)$ is said to be Bohr-almost periodic if for each $\varepsilon>0$ there exists $l(\varepsilon)>0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ (called $\varepsilon-$almost period) for which*

$$\sup_{t\in\mathbb{R}}||f(t+\tau)-f(t)||<\varepsilon.$$

In this case, we say that the set of $\varepsilon-$almost periods is relatively dense in $\mathbb{R}$.

Recall that $f:\mathbb{R}\to(\mathbb{E},||\cdot||)$ is almost periodic if and anly if the set of its translations $(\tilde{f}_t)_{t\in\mathbb{R}}:=(f(\cdot+t))_{t\in\mathbb{R}}$ is relatively compact in $C_u(\mathbb{R},\mathbb{E})$, where $C_u(\mathbb{R},\mathbb{E})$ is the space of continuous functions endowed by the topology of uniform convergence on $\mathbb{R}$ (see [7]). And thus the range of f is relatively compact in E.

There is an equivalent definition of almost periodicity, due to Bochner [8], which is also used in the study of SDEs. However, in this paper, we settle for the Bohr-almost periodicity.

2.2 Almost periodic processes

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $L^2(\Omega)$ be the space of all mean square complex valued random variables defined on Ω.

2.2.1 Almost periodicity in quadratic mean

A mean square continuous second order stochastic process $X:=\{X_t\}_{t\in\mathbb{R}}$ is almost periodic in quadratic mean (APQM for short) if the function

$$\begin{cases}\mathbb{R} & \to L^2(\Omega)\\ t & \mapsto X_t\end{cases}$$

is almost periodic.

2.2.2 Periodically and almost periodically correlated processes

A mean square process X is said to be wide sens (weakly) stationary if for any $t,s\in\mathbb{R}$ its covariance function given by

$$C(t,s)=\mathrm{E}(X_t\overline{X_s})-E(X_t)E(\overline{X_s})$$

depends only on $t-s$ and its mean function $m(t)=E(X_t)$ is constant with respect to time t.

There is a connection between a wide sens stationary process $\{X_t\}_{t\in\mathbb{R}}$ and a unitary group (called also a shift group) of operators $U(\tau)$, $\tau\in\mathbb{R}$, given on the span of $\{X_t\}_{t\in\mathbb{R}}$ by

$$U(\tau)(X_t)=X_{t+\tau},\ \text{for each } t,\tau\in\mathbb{R}.$$

For each $\tau\in\mathbb{R}$, $U(\tau)$ can be extended to the closure $H(X)$ of the span of $\{X_t\}_{t\in\mathbb{R}}$ and thus to $L^2(\Omega)$. The family $\{U(\tau),\ \ \tau\in\mathbb{R}\}$ forms a one parameter shift group on $L^2(\Omega)$. And if the process is square mean continuous, the group is strongly continuous.

From this definition, we notice that for all $T\in\mathbb{R}$, $C(t+T,s+T)=C(t,s)$ and $m(t+T)=m(t)$ (constant), then Gladyshev [12] naturally gives the following definition.

A mean square process X is said to be periodically correlated with period T (PC for short) if

$$C(t+T,s+T)=C(t,s)\ \textit{and}\ m(t+T)=m(t)\ \textit{for all}\ t,s\in\mathbb{R}.$$

In the same way as the stationary process, but for only $\tau=nT$, $n\in\mathbb{Z}$, H.L. Hurd [14] shows the existence of a one parameter shift group $\{U(n),\ \ n\in\mathbb{Z}\}$ such that

$$U(n)(X_t)=(X_{t+nT}),\ \text{for each } t\in\mathbb{R},n\in\mathbb{Z}.$$

D. Gladyshev [12] generalized the definition of the periodically correlated processes to the almost periodically correlated processes, as follows:

A process X is said to be almost periodically correlated (APC for short) if its mean and covariance

$$\mathrm{m}(\cdot):\mathbb{R}\to\mathbb{C};\ C(\cdot,\cdot):\mathbb{R}\times\mathbb{R}\to\mathbb{C}$$

are uniformly continuous and the functions

$$m(\cdot):\mathbb{R}\to\mathbb{C};C(\cdot,\cdot+\tau):\mathbb{R}\to\mathbb{C},t\to C(t,t+\tau)$$

are almost periodic, for every $\tau\in\mathbb{R}$.

2.2.3 Almost periodically unitary processes

The link, given above, between stochastic processes and unitary groups motivates H.L. Hurd [14] to give the following definition, which is the central concept in this work.

A mean square continuous second order stochastic process $(X_t)_{t\in\mathbb{R}}$ is called almost periodically unitary (APU *for short), if there exists a strongly continuous unitary group (called also a shift group) $\mathcal{U}=\{U(\tau):L^2(\Omega)\to L^2(\Omega);\tau\in\mathbb{R}\}$ such that, for every $\varepsilon>0$, there exists $l(\varepsilon)>0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which*

$$\sup_{t\in\mathbb{R}}\parallel X(t+\tau)-U(\tau)\,X(t)\parallel_{L^2}<\varepsilon, \tag{2}$$

which means that the set

$$\left\{\tau; \sup_{t\in\mathbb{R}} \parallel X(t+\tau) - U(\tau)\, X(t) \parallel_{L^2} < \varepsilon\right\}$$

is relatively dense in $\mathbb{R}$*.*

In the following, we recall some properties of APU processes and give an important characterization in terms of the shift group associated to the process and some almost periodic process in quadratic mean.

Proposition 2.1. ([14, Proposition 1]) A mean square continuous second order process $(X(t))_{t\in\mathbb{R}}$ is APU with shift group $\mathcal{U}$ if and only if there exists an APQM process Y such that

$$X(t) = U(t)\, Y(t),\ \forall\, t \in \mathbb{R}. \tag{3}$$

Proposition 2.2. ([14, 15]) We have the following properties:

1. APU processes are uniformly continuous on $\mathbb{R}$ with values in $L^2(\Omega)$.
2. The range of an APU process is bounded in $L^2(\Omega)$, but it is not in general totally bounded and thus not relatively compact.
3. APQM processes are APU.
4. PC processes are APU.
5. APU processes are APC.
6. Weakly stationary processes are APU.

3 Main results

Instead of the relative compacity, which is not satisfied by the APU processes, we show in the following subsection a tightness result of APU processes, which is an important property. In the second subsection of this part, we consider two examples given in [18] and show that the solutions are APU. These examples motivate the study of the equation (1) in the third subsection.

3.1 Tightness of APU processes

Before giving the tightness result, let us recall the definition of tightness and the tightness criterion given in [4] . Let X be a continuous stochastic process and $(\widetilde{X}(r))_{r\in\mathbb{R}} := (X(\cdot + r))_{r\in\mathbb{R}}$ be the family of its translates. The process X is tight if $(\widetilde{X}(r))_{r\in\mathbb{R}}$ is tight in $\mathcal{C}_k(\mathbb{R}, \mathbb{C})$, which implies the following condition (tightness criterion):

$$\forall\, [a, b] \subset \mathbb{R},\ \forall\, \varepsilon > 0,\ \forall\, \eta > 0,\ \exists\, \delta > 0,\ \forall\, r \in \mathbb{R},$$

$$\mathrm{P}\left\{\sup_{\substack{|t-s|<\delta\\ t,s\in[a,b]}}\|X(r+t)-X(r+s)\|>\eta\right\}<\varepsilon, \tag{4}$$

where $\mathcal{C}_k(\mathbb{R},\mathbb{C})$ is the space of continuous functions endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}$.

Proposition 3.1. If the process X is APU with a strongly continuous unitary group of operators $\mathcal{U}=\{U(\tau);\,\tau\in\mathbb{R}\}$ on $L^2(\Omega)$, then X is tight.

Proof. Assume that X is APU. By Proposition 2.1, there exists an almost periodic function $Y:\mathbb{R}\to L^2(\Omega)$ such that

$$X(t)=U(t)\,Y(t),t\in\mathbb{R}.$$

Then, for all $t,s\in\mathbb{R}$ and all $r\in\mathbb{R}$

$$\Big\|X(r+t)-X(r+s)\Big\|_{L^2}=\Big\|U(r+t)\,Y(r+t)-U(r+s)\,Y(r+s)\Big\|_{L^2}.$$

By the group property of $\{U(\tau)\}_{\tau\in\mathbb{R}}$ and since $U(s+r)$ is a unitary operator on $L^2(\Omega)$, we obtain

$$\begin{aligned}\Big\|X(r+t)-X(r+s)\Big\|_{L^2}&=\Big\|U(r+s)\,\Big[U(t-s)\,Y(r+t)-Y(r+s)\Big]\Big\|_{L^2}\\&=\Big\|U(t-s)\,Y(r+t)-Y(r+s)\Big\|_{L^2}.\end{aligned}$$

It follows that

$$\begin{aligned}\Big\|X(r{+}t){-}X(r{+}s)\Big\|_{L^2}&=\Big\|\Big[U(t{-}s)\,Y(r{+}t){-}Y(r{+}t)\Big]+\Big[Y(r{+}t){-}Y(r{+}s)\Big]\Big\|_{L^2}\\&=\Big\|\Big[U(t{-}s){-}I_{L^2}\Big]Y(r{+}t)+\Big[Y(r{+}t){-}Y(r{+}s)\Big]\Big\|_{L^2}\\&=\Big\|\Big[U(t{-}s){-}U(0)\Big]Y(r{+}t)+\Big[Y(r{+}t){-}Y(r{+}s)\Big]\Big\|_{L^2}\\&\le\Big\|\Big[U(t{-}s){-}U(0)\Big]Y(r{+}t)\Big\|_{L^2}+\Big\|Y(r{+}t){-}Y(r{+}s)\Big\|_{L^2}.\end{aligned}$$

By Tchebychev inequality, we obtain that for every interval $[a,b]\subset\mathbb{R}$ and every $r\in\mathbb{R}$

$$\mathrm{P}\left\{\sup_{\substack{|t-s|<\delta\\ t,s\in[a,b]}}\Big\|X(r+t)-X(r+s)\Big\|>\eta\right\}\le\frac{1}{\eta^2}\sup_{\substack{|t-s|<\delta\\ t,s\in[a,b]}}\Big\|X(r+t)-X(r+s)\Big\|_{L^2}^2.$$

Let $\varepsilon>0$ and $\eta>0$. By the almost periodicity of $Y:\mathbb{R}\to L^2(\Omega)$ we have:

on the one hand, since $\mathcal{U} = \{U(\tau); \tau \in \mathbb{R}\}$ is a strongly continuous unitary group, and using the relative compacity of $R(Y) = \{Y(r), r \in \mathbb{R}\}$ in $L^2(\Omega)$, there exists $\delta_1 > 0$ such that for all $r \in \mathbb{R}$ we have:

$$\sup_{\substack{|t-s|<\delta_1 \\ t,s\in[a,b]}} \left\| \Big[U(t-s) - U(0)\Big] Y(r+t) \right\|_{L^2} \leq \frac{\eta\sqrt{\varepsilon}}{2}, \tag{5}$$

on the other hand, from the relative compacity of the family $(Y(\cdot + r))_{r\in\mathbb{R}}$ in $C_u(\mathbb{R}, L^2(\Omega))$, we deduce that there exists $\delta_2 > 0$, such that, for all $r \in \mathbb{R}$

$$\sup_{\substack{|t-s|<\delta_2 \\ t,s\in[a,b]}} \left\| Y(r+t) - Y(r+s) \right\|_{L^2} \leq \frac{\eta\sqrt{\varepsilon}}{2} \tag{6}$$

Therefore, X satisfies the tightness condition (4).
It remains to check (see [6, Theorem 7.3,p. 82]) that we have,
$\forall \eta > 0, \exists a > 0, \forall r \in \mathbb{R}$

$$P\Big\{ \|\widetilde{X}(r)(0)\| \geq a \Big\} = P\Big\{ \|X(r)\| \geq a \Big\} < \eta.$$

Since Y is tight, $\forall \eta > 0, \exists a > 0, \forall r \in \mathbb{R}$

$$P\Big\{ \|\widetilde{Y}(r)(0)\| \geq a \Big\} < \eta,$$

but,

$$P\Big\{ \|\widetilde{X}(r)(0)\| \geq a \Big\} = P\Big\{ \|U(r)Y(r)\| \geq a \Big\} = P\Big\{ \|Y(r)\| \geq a \Big\}.$$

□

3.2 Explicit examples of almost periodically unitary solutions

Example 3.2. (**Stationary Ornstein-Uhlenbeck process**). We consider a real-valued standard Brownian motion W. Let $\alpha, \sigma > 0$ be two real constants. The Ornstein-Uhlenbeck process X, see [18, Example 2.1], given by

$$X(t) = \sqrt{2\,\alpha\,\sigma} \int_{-\infty}^{t} e^{-\alpha(t-s)}\, dW(s) \tag{7}$$

is the unique L^2-bounded solution to the linear stochastic differential equation

$$dX(t) = -\alpha\, X(t)\, dt + \sqrt{2\,\alpha}\,\sigma\, dW(t). \tag{8}$$

The stochastic process X is Gaussian with mean 0, and for all $t \in \mathbb{R}$ and $\tau \geq 0$, its covariance function is given by

$$C(t, t+\tau) = \sigma^2\, e^{-\alpha\,\tau},$$

which means that X is a weakly stationary process, so APU (see Proposition 2.2).

Example 3.3. Let W be a real-valued standard Brownian motion. Consider the stochastic process X, defined by:

$$X(t) = e^{-t+\sin(t)} \int_{-\infty}^{t} e^{s-\sin(s)} \sqrt{1-\cos(s)}\, dW(s),\ t \in \mathbb{R}. \tag{9}$$

The process X is the unique L^2-bounded solution to the following stochastic differential equation

$$dX(t) = (-1+\cos(t))\, X(t)\, dt + \sqrt{1-\cos(t)}\, dW(t),\ t \in \mathbb{R}. \tag{10}$$

The process X is not weakly stationary. Indeed, we have, for any $t \in \mathbb{R}$, $\mathrm{E}(X(t)) = 0$ and

$$\begin{aligned} C(t+\tau, t) &= \mathrm{Cov}(X(t+\tau), X(t)) \\ &= E(X(t)\, X(t+\tau)) \\ &= e^{-t+\sin(t)} e^{-t-\tau+\sin(t+\tau)} \int_{-\infty}^{t} e^{2(s-\sin(s))}(1-\cos(s))\, ds \\ &= \frac{1}{2}\, e^{-\tau+\sin(t+\tau)-\sin(t)}. \end{aligned}$$

Let us show that X is PC. We have, for any $k, t \in \mathbb{R}$,

$$\begin{aligned} C(k+T, t+T) &= E(X(k+T)X(t+T)) \\ &= e^{-t-k-2T+\sin(t+T)+\sin(k+T)} \\ &\cdot E\Big(\Big(\int_{-\infty}^{k+T} e^{s-\sin(s)} \sqrt{1-\cos(s)}\, dW(s)\Big)\Big(\int_{-\infty}^{t+T} e^{e^{s-\sin(s)}} \sqrt{1-\cos(s)}\, dW(s)\Big)\Big). \end{aligned}$$

For $T = 2\pi$, we obtain that

$$\begin{aligned} C(k+2\pi, t+2\pi) &= e^{-t-k-4\pi+\sin(t)+\sin(k)} \\ &\cdot E\Big(\Big(\int_{-\infty}^{k+2\pi} e^{s-\sin(s)} \sqrt{1-\cos(s)}\, dW(s)\Big)\Big(\int_{-\infty}^{t+2\pi} e^{s-\sin(s)} \sqrt{1-\cos(s)}\, dW(s)\Big)\Big). \end{aligned}$$

Making the change of variable $\sigma = s - 2\pi$, we get

$$\begin{aligned} C(k+2\pi, t+2\pi) &= e^{-t-k-4\pi+\sin(t)+\sin(k)} \\ &\cdot E\Big(\Big(\int_{-\infty}^{k} e^{\sigma+2\pi-\sin(\sigma)} \sqrt{1-\cos(\sigma)}\, d\widetilde{W}(\sigma)\Big)\Big(\int_{-\infty}^{t} e^{\sigma+2\pi-\sin(\sigma)} \sqrt{1-\cos(\sigma)}\, d\widetilde{W}(\sigma)\Big)\Big), \end{aligned}$$

where $\widetilde{W}(\sigma) = W(\sigma+2\pi) - W(2\pi)$ is a Brownian motion with the same distribution as $W(\sigma)$. It follows that,

$$\begin{aligned} C(k+2\pi, t+2\pi) = e^{-t-k+\sin(t)+\sin(k)}\, E\Big(\Big(\int_{-\infty}^{k} e^{\sigma-\sin(\sigma)} \sqrt{1-\cos(\sigma)}\, dW(\sigma)\Big) \\ \Big(\int_{-\infty}^{t} e^{\sigma-\sin(\sigma)} \sqrt{1-\cos(\sigma)}\, dW(\sigma)\Big)\Big). \end{aligned}$$

Therefore, for $T = 2\pi$

$$C(k+T, t+T) = C(k,t),\ \forall\, k, t \in \mathbb{R}.$$

Hence X is PC which implies that X is APU.

3.3 Existence and uniqueness of APU solution to SDE in one dimension

We consider the following stochastic differential equation

$$dX(t) = -a\,X(t)\,dt + F(t)\,dt + G(t)\,dW(t),\ \ t \in \mathbb{R}, \tag{11}$$

where, $a > 0$ is a real fixed number, W is a real-valued standard Brownian motion and $F, G : \mathbb{R} \to \mathbb{R}$ are almost periodic functions.
In the following, we will show that the solution X to the above equation is APU: The strongly continuous unitary group $\mathcal{U}$ associated to X is inherited from stationary processes obtained by increments of the Brownian motion W.
For each $s \in \mathbb{R}$, denote by $\widetilde{W}^s = (\widetilde{W}^s_t)_{t\in\mathbb{R}}$ the stationary process obtained by the increments of the Brownien motion W:

$$\widetilde{W}^s_t = W(s+t) - W(t), \quad \forall t \in \mathbb{R}.$$

The following lemma gives explicitly the group $\mathcal{U}$.

Lemma 3.4. Let W be a real-valued standard Brownian motion. Then there exists a strongly continuous unitary group $\mathcal{U} = \{U(\tau); \tau \in \mathbb{R}\}$ on $L^2(\Omega)$ such that:

$$\widetilde{W}^s_{t+\tau} = U(\tau)(\widetilde{W}^s_t),$$

for all $s, \tau, t \in \mathbb{R}$.

Proof. To prove this lemma we can use for instance [14] and [10]. □

Remark 3.5. In particular, for $t = 0$, we have:

$$U(\tau)\left[W(s)\right] = \widetilde{W}^s_\tau := W(s+\tau) - W(\tau), \tag{12}$$

for all s and τ in $\mathbb{R}$.

Theorem 3.6. The unique L^2-bounded solution X of (11), which has the form

$$X(t) = \int_{-\infty}^{t} e^{-a(t-\sigma)}\,F(\sigma)\,d\sigma + \int_{-\infty}^{t} e^{-a(t-\sigma)}\,G(\sigma)\,dW(\sigma);\ \ \forall t \in \mathbb{R}, \tag{13}$$

is APU.

Remark 3.7. For every $r, t \in \mathbb{R}$ such that $r < t$,

$$X(t) = e^{-a(t-r)}\,X(r) + \int_r^t e^{-a(t-\sigma)}\,F(\sigma)\,d\sigma + \int_r^t e^{-a(t-\sigma)}\,G(\sigma)\,dW(\sigma).$$

Proof of Theorem 3.6

Let us show that the solution X is APU. We consider the strongly continuous unitary group $\mathcal{U} = \{U(\tau); \tau \in \mathbb{R}\}$ given by the formula (12).

Let $\varepsilon > 0$ and τ be a common $\varepsilon-$almost period of the two functions F and G. For any $t \in \mathbb{R}$, we have

$$\begin{aligned}
&\|X(t+\tau) - U(\tau)\, X(t)\|_{L^2}^2 \\
&= \mathrm{E}\Big(|\, X(t+\tau) - U(\tau)\, X(t)\, |^2 \Big) \\
&= \mathrm{E}\Bigg(\Bigg| \int_{-\infty}^{t+\tau} e^{-a(t+\tau-\sigma)}\, F(\sigma)\, d\sigma - U(\tau) \left[\int_{-\infty}^{t} e^{-a(t-\sigma)}\, F(\sigma)\, d\sigma \right] \\
&+ \int_{-\infty}^{t+\tau} e^{-a(t+\tau-\sigma)}\, G(\sigma)\, dW(\sigma) - U(\tau) \left[\int_{-\infty}^{t} e^{-a(t-\sigma)}\, G(\sigma)\, dW(\sigma) \right] \Bigg|^2 \Bigg).
\end{aligned}$$

Since $(a+b)^2 \leq 2\,a^2 + 2\,b^2$, we get

$$\begin{aligned}
&\|X(t+\tau) - U(\tau)\, X(t)\|_{L^2}^2 \\
&\leq\ 2 \left(\left| \int_{-\infty}^{t+\tau} e^{-a(t+\tau-\sigma)}\, F(\sigma)\, d\sigma - U(\tau) \left[\int_{-\infty}^{t} e^{-a(t-\sigma)}\, F(\sigma)\, d\sigma \right] \right|^2 \right) \\
&+ 2\ \mathrm{E}\left(\left| \int_{-\infty}^{t+\tau} e^{-a(t+\tau-\sigma)}\, G(\sigma)\, dW(\sigma) - U(\tau) \left[\int_{-\infty}^{t} e^{-a(t-\sigma)}\, G(\sigma)\, dW(\sigma) \right] \right|^2 \right).
\end{aligned}$$

Making the change of variable $s = \sigma - \tau$, we obtain

$$\begin{aligned}
&\|X(t+\tau) - U(\tau)\, X(t)\|_{L^2}^2 \\
&\leq\ 2 \left(\left| \int_{-\infty}^{t} e^{-a(t-s)}\, F(s+\tau)\, ds - U(\tau) \left[\int_{-\infty}^{t} e^{-a(t-s)}\, F(s)\, ds \right] \right|^2 \right) \\
&+ 2\ \mathrm{E}\left(\left| \int_{-\infty}^{t} e^{-a(t-s)}\, G(s+\tau)\, d\widetilde{W}(s) - U(\tau) \left[\int_{-\infty}^{t} e^{-a(t-s)}\, G(s)\, dW(s) \right] \right|^2 \right) \\
&:=\ I_1 + I_2,
\end{aligned}$$

where $\widetilde{W}(s) = W(s+\tau) - W(\tau)$ is a Brownian motion with the same distribution as $W(s)$.

For I_1, we have

$$\begin{aligned}
I_1 &= 2 \left| \int_{-\infty}^{t} e^{-a(t-s)}\, F(s+\tau)\, ds - U(\tau) \left[\int_{-\infty}^{t} e^{-a(t-s)}\, F(s)\, ds \right] \right|^2 \\
&= 2 \left(\left| \int_{-\infty}^{t} e^{-a(t-s)}\, F(s+\tau)\, ds - \int_{-\infty}^{t} e^{-a(t-s)}\, F(s)\, ds \right|^2 \right. \\
&= 2 \left| \int_{-\infty}^{t} e^{-a(t-s)} \left[F(s+\tau) - F(s) \right] ds \right|^2.
\end{aligned}$$

Now, using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
I_1 &\leq 2 \int_{-\infty}^{t} e^{-a(t-s)} \left| F(s+\tau) - F(s) \right| ds \\
&\leq 2 \left(\int_{-\infty}^{t} e^{-a(t-s)/2}\, e^{-a(t-s)/2} \left| F(s+\tau) - F(s) \right| ds \right)^2 \\
&\leq 2 \left(\left(\int_{-\infty}^{t} e^{-a(t-s)}\, ds \right) \left(\int_{-\infty}^{t} e^{-a(t-s)} \left| F(s+\tau) - F(s) \right|^2 ds \right) \right) \\
&\leq 2 \left(\left(\int_{-\infty}^{t} e^{-a(t-s)}\, ds \right) \left(\int_{-\infty}^{t} e^{-a(t-s)} \left| F(s+\tau) - F(s) \right|^2 ds \right) \right) \\
&\leq 2 \left(\int_{-\infty}^{t} e^{-a(t-s)}\, ds \right)^2 \sup_{s\in\mathbb{R}} \mid F(s+\tau) - F(s) \mid^2 .
\end{aligned}$$

Since F is an almost periodic function, we deduce that

$$I_1 \leq 2a^2\, \varepsilon \left(\int_{-\infty}^{t} e^{-a(t-s)}\, ds \right)^2 = 2\varepsilon.$$

For I_2, we have

$$\begin{aligned}
I_2 &= 2\, \mathrm{E}\left(\left| \int_{-\infty}^{t} e^{-a(t-s)}\, G(s+\tau)\, d\widetilde{W}(s) - U(\tau) \left[\int_{-\infty}^{t} e^{-a(t-s)}\, G(s)\, dW(s) \right] \right|^2 \right) \\
&= 2\, \mathrm{E}\left(\left| \int_{-\infty}^{t} e^{-a(t-s)}\, G(s+\tau)\, d\widetilde{W}(s) - \int_{-\infty}^{t} U(\tau) \left[e^{-a(t-s)}\, G(s)\, dW(s) \right] \right|^2 \right) \\
&= 2\, \mathrm{E}\left(\left| \int_{-\infty}^{t} e^{-a(t-s)}\, G(s+\tau)\, d\widetilde{W}(s) - \int_{-\infty}^{t} e^{-a(t-s)}\, G(s) \left[U(\tau)\, dW(s) \right] \right|^2 \right).
\end{aligned}$$

Using Lemma 3.4, we get

$$\begin{aligned}
I_2 &= 2\, \mathrm{E}\left(\left| \int_{-\infty}^{t} e^{-a(t-s)}\, G(s+\tau)\, d\widetilde{W}(s) - \int_{-\infty}^{t} e^{-a(t-s)}\, G(s)\, d(U(\tau)\, W(s)) \right|^2 \right) \\
&= 2\, \mathrm{E}\left(\left| \int_{-\infty}^{t} e^{-a(t-s)}\, G(s+\tau)\, d\widetilde{W}(s) - \int_{-\infty}^{t} e^{-a(t-s)}\, G(s)\, d\widetilde{W}(s) \right|^2 \right).
\end{aligned}$$

It follows that

$$I_2 = 2\, \mathrm{E}\left(\left| \int_{-\infty}^{t} e^{-a(t-s)} \left[G(s+\tau) - G(s) \right] d\widetilde{W}(s) \right|^2 \right).$$

Applying Itô's isometry, we get

$$I_2 = 2 \int_{-\infty}^{t} \mathrm{E}\left(\left| e^{-a(t-s)} \left[G(s+\tau) - G(s) \right] \right|^2 \right) ds.$$

By the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
I_2 &\leq 2\int_{-\infty}^{t} \mathrm{E}\left(e^{-2\,a(t-s)}\left|G(s+\tau)-G(s)\right|^2\right) ds \\
&\leq 2\int_{-\infty}^{t} e^{-2\,a(t-s)}\,\mathrm{E}\left(\left|G(s+\tau)-G(s)\right|^2\right) ds \\
&\leq 2\left(\int_{-\infty}^{t} e^{-2\,a(t-s)} ds\right) \sup_{s\in\mathbb{R}} \mathrm{E}\mid G(s+\tau)-G(s)\mid^2 .
\end{aligned}
$$

Since G is an almost periodic function, we deduce that

$$
I_2 \leq 2\,a\,\varepsilon\left(\int_{-\infty}^{t} e^{-2\,a(t-s)}\,ds\right) = \varepsilon,
$$

thus X is APU and the associated group is given by the formula (12). □

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