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GENERALIZATION OF GAUSSIAN MERSENNE NUMBERS AND THEIR NEW FAMILIES

Munesh KUMARI[1] [2] **Kalika PRASAD**[1,2,*] **Jagmohan TANTI** [3]

Abstract

In this article, we present the generalized Gaussian Mersenne numbers with arbitrary initial values and discuss two particular cases, namely, Gaussian Mersenne and Gaussian Mersenne-Lucas numbers. We present their various algebraic properties such as Binet's formula, negatively subscripted elements, Catalans's, Cassini's, and d'Ocagne's identities, partial sum, binomial sum, generating and exponential generating functions, etc. In addition, we study a new generalized sequence arising from the explicit expression made with the characteristic roots and refer to them as the k-generalized Gaussian Mersenne numbers. We present various identities of them and show their connections with the generalized Gaussian Mersenne numbers.

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1 Introduction

In 1963, Horadam [8] introduced the concept of the complex Fibonacci numbers defined as

$$C_n = F_n + iF_{n+1}, \quad \text{where } F_n \text{ is the } n\text{th Fibonacci number.}$$

Later, in 1977 Berzsenyi [1] defined the complex Fibonacci numbers by a different approach and named it the Gaussian Fibonacci numbers. Horadam [7] also defined the Gaussian Fibonacci numbers using a recurrence relation analogous to the Fibonacci numbers. Further, the concept of Gaussian numbers is extended to

[1]Department of Mathematics, Government Engineering College Bhojpur, Bihar, India, 802301, e-mail(s): muneshnasir94@gmail.com, klkaprsd@gmail.com
[2]Department of Mathematics, Central University of Jharkhand, Ranchi, India, 835205
[*]*Corresponding author*
[3]Department of Mathematics, Babasaheb Bhimrao Ambedkar University, Lucknow, India, 226025, e-mail: jagmohan.t@gmail.com

other number sequences like Lucas, Pell, Leonardo, etc., their generalizations and polynomial version, one can refer to [9, 14, 13, 15, 16, 20, 21].

In this paper, we deal with the generalization of the Mersenne numbers which is given by $2^n - 1$ and have many interesting properties. Recently, Catarino et al. [2] gave the homogeneous recurrence relation for the Mersenne numbers and studied their algebraic properties. One of the generalizations of Mersenne numbers is introduced by Soykan [18] and Kumari et al. [10] with arbitrary initial values. Dasdemir [4] also studied the generalized form of these numbers and their application in quaternion. Uysal et al. [19] studied the octonions with Mersenne numbers and obtained various interesting properties of them. Kumari et al. [11] shown the application of Mersenne numbers in r-circulant matrices. Frontczak et al. [6] shown some connections between Mersenne and generalized Fibonacci (i.e., Horadam) numbers.

The generalized Mersenne sequence $\{W_n\}_{n\geq 0}$ is given by the recurrence relation

$$W_{n+2} = 3W_{n+1} - 2W_n, \quad W_0 = c_0, \quad W_1 = c_1, \tag{1}$$

and the terms of this sequence are known as the generalized Mersenne numbers. As a special case of generalized Mersenne sequence, setting $c_0 = 0,\ \ c_1 = 1$ in (1) gives the classical Mersenne sequence $\{M_n\}_{n\geq 0}$ and for $c_0 = 2,\ \ c_1 = 3$, it gives the Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$. The characteristic equation corresponding to the above recurrence relation is

$$\lambda^2 - 3\lambda + 2 = 0. \tag{2}$$

Eqn. (2) has two roots, $\lambda_1 = 2$ and $\lambda_2 = 1$ and they satisfy $\lambda_1 + \lambda_2 = 3$, $\lambda_1\lambda_2 = 2$ and $\lambda_1 - \lambda_2 = 1$. Thus, the Binet's formula for the generalized Mersenne numbers is given by

$$W_n = (W_1 - W_0)2^n - (W_1 - 2W_0).$$

Some recent developments on Mersenne numbers and their applications, can be seen in [3, 4, 5, 12, 17].

Motivated by these works on Mersenne numbers, we generalize the Gaussian Mersenne numbers with arbitrary initial values and give a new family of the generalized Gaussian Mersenne numbers. We obtain their algebraic properties, some well-known identities like explicit formula, Catalan identity, d'Ocagne identity, summation formulas, generating functions etc.

2 Generalized Gaussian Mersenne numbers

Here, we introduce the generalized Gaussian Mersenne sequence $\{GW_n\}_{n\geq 0}$ and present their some algebraic properties, well known identities and relations with Gaussian Mersenne and Gaussian Mersenne-Lucas numbers.

Definition 1. *The generalized Gaussian Mersenne sequence* $\{GW_n\}_{n\geq 0}$ *is defined by*

$$GW_{n+2} = 3GW_{n+1} - 2GW_n, \quad \textit{with } \ GW_0 = a + i\big((3a-b)/2\big), \ GW_1 = b + ai,$$

where a *and* b *are arbitrary complex (real) numbers not all being zero.*

In terms of generalized Mersenne numbers, the generalized Gaussian Mersenne numbers can be written as

$$GW_{n+2} = W_{n+2} + iW_{n+1}.$$

For $a = 0, b = 1$ and $a = 2, b = 3$, Definition 1 yields the Gaussian Mersenne sequence $\{GM_n\}_{n\geq 0}$ and the Gaussian Mersenne-Lucas sequence $\{GH_n\}_{n\geq 0}$, respectively, i.e.

$$\begin{aligned} GM_{n+2} &= 3GM_{n+1} - 2GM_n, \quad GM_0 = \frac{-i}{2}, \quad GM_1 = 1, \\ \text{and} \quad GH_{n+2} &= 3GH_{n+1} - 2GH_n, \quad GH_0 = \left(2 + i\frac{3}{2}\right), \quad GH_1 = 3 + 2i. \end{aligned}$$

The first few generalized Gaussian Mersenne, Gaussian Mersenne and Gaussian Mersenne-Lucas numbers are:

n	GW_n	GM_n	GH_n
0	$a + i\big((3a-b)/2\big)$	$-i/2$	$2 + i\frac{3}{2}$
1	$b + ai$	1	$3 + 2i$
2	$(3b-2a) + ib$	$3 + i$	$5 + 3i$
3	$(7b-6a) + i(3b-2a)$	$7 + 3i$	$9 + 5i$
4	$(15b-14a) + i(7b-6a)$	$15 + 7i$	$17 + 9i$
5	$(31b-30a) + i(15b-14a)$	$31 + 15i$	$33 + 17i$

Table 1: Generalized Gaussian Mersenne numbers.

Theorem 1. *For* $n \geq 0$, *the Binet's formula for* GW_n *is given by*

$$GW_n = (b-a)2^n - (b-2a) + i\left((b-a)2^{n-1} - (b-2a)\right). \tag{3}$$

Proof. By the theory of difference equation, nth term of generalized Gaussian Mersenne sequence can be written as

$$GW_n = c\lambda_1^n + d\lambda_2^n, \quad \text{where } \lambda_1 = 2 \text{ and } \lambda_1 = 1. \tag{4}$$

On solving for $n = 0$ and $n = 1$, we obtain

$$c = \frac{GW_1 - GW_0\lambda_2}{\lambda_1 - \lambda_2} \quad \text{and} \quad d = \frac{GW_0\lambda_1 - GW_1}{\lambda_1 - \lambda_2}. \tag{5}$$

Now, using $GW_0 = a + i\big((3a-2b)/2\big)$, $GW_1 = b + ai$ and Eqn. (5) in Eqn. (4), we have

$$GW_n = (b-a)\lambda_1^n - (b-2a)\lambda_2^n + i\left((b-a)\lambda_1^{n-1} - (b-2a)\lambda_2^{n-1}\right).$$

This completes the proof. □

In particular, Binet's formulae for Gaussian Mersenne and Gaussian Mersenne-Lucas numbers are, respectively, given by

$$GM_n = (2^n - 1) + i(2^{n-1} - 1) \quad \text{and} \quad GH_n = (2^n + 1) + i(2^{n-1} + 1). \tag{6}$$

Theorem 2. *For $n \geq 0$, the following identities are provided.*

1. $GW_{n+1} + GW_n = 2^{n-1}3(b-a)(2+i) - 2(b-2a)(1+i)$.
2. $GW_{n+1} - GW_n = 2^{n-1}(b-a)(2+i)$.
3. $GW_{n+1} = 2GW_n + (b-2a)(1-i)$.

Proof. 1. Using the Binet's formula of the generalized Gaussian Mersenne numbers, we have

$$\begin{aligned} GW_{n+1} + GW_n &= (b-a)2^{n+1} - (b-2a) + i[(b-a)2^n - (b-2a)] \\ &\quad + (b-a)2^n - (b-2a) + i[(b-a)2^{n-1} - (b-2a)] \\ &= 2^n 3(b-a) - 2(b-2a) + i[3(b-a)2^{n-1} - 2(b-2a)] \\ &= 2^{n-1}3(b-a)(2+i) - 2(b-2a)(1+i). \end{aligned}$$

By a similar argument, the second and third identities can be proved. □

Definition 2. *The generalized Gaussian Mersenne numbers with negative subscript $\{GW_{-n}\}_{n\geq 1}$ are defined recursively as*

$$GW_0 = a + i\big((3a-2b)/2\big), \; GW_1 = b + ai, \; \textit{and } GW_{-n} = \frac{3GW_{-n+1} - GW_{-n+2}}{2}.$$

For $a = 0,\ b = 1$ and $a = 2,\ b = 3$ in Definition 2, we obtain Gaussian Mersenne and Gaussian Mersenne-Lucas numbers with negative subscript defined, respectively, as

$$\begin{aligned} GM_{-n} &= \frac{3GM_{-n+1} - GM_{-n+2}}{2}, \quad GM_0 = -i/2, \quad GM_1 = 1, \\ GH_{-n} &= \frac{3GH_{-n+1} - GH_{-n+2}}{2}, \quad GH_0 = 2 + i(3/2), \quad GH_1 = 3 + 2i. \end{aligned}$$

The first few terms of the sequences $\{GW_{-n}\}$, $\{GM_{-n}\}$ and $\{GH_{-n}\}_{n\geq 1}$ are shown in the following table:

n	GW_{-n}	GM_{-n}	GH_{-n}
1	$\frac{(6a-2b)+i\,(7a-3b)}{4}$	$\frac{(-2-3i)}{4}$	$\frac{(6+5i)}{4}$
2	$\frac{(14a-6b)+i\,(15a-7b)}{8}$	$\frac{(-6-7i)}{8}$	$\frac{(10+9i)}{8}$
3	$\frac{(30a-14b)+i\,(31a-15b)}{16}$	$\frac{(-14-15i)}{16}$	$\frac{(18+17i)}{16}$
4	$\frac{(62a-30b)+i\,(63a-31b)}{32}$	$\frac{(-30-31i)}{32}$	$\frac{(34+33i)}{32}$

Table 2: First few terms of $\{GW_{-n}\}$, $\{GM_{-n}\}$ and $\{GH_{-n}\}$.

Theorem 3. *For $n \geq 0$, the Binet's formulae for the generalized Gaussian Mersenne, Gaussian Mersenne and Gaussian Mersenne-Lucas numbers with negative subscripts are given, respectively, as*

1. $GW_{-n} = \left(\frac{(b-a)-(b-2a)2^n}{2^n}\right) + i\left(\frac{(b-a)-(b-2a)2^{n+1}}{2^{n+1}}\right).$
2. $GM_{-n} = \frac{\left(2-2^{n+1}\right)+i\left(1-2^{n+1}\right)}{2^{n+1}}.$
3. $GH_{-n} = \frac{\left(2+2^{n+1}\right)+i\left(1+2^{n+1}\right)}{2^{n+1}}.$

Proof. Replacing n by $-n$ in Binet's formulae (3) and (6), we get the required results. □

Theorem 4 (Catalan's identity)**.** *For $n, m \geq 1$, we have*

$$GW_{n+m}GW_{n-m} - GW_n^2 = (b-a)(b-2a)[(2^n - 2^{n+m-1}) + (2^{n-m-1} - 2^{n-m})] + i3(2^n - 2^{n+m-1} - 2^{n-m-1}). \tag{7}$$

Proof. Using Binet's formula (3), we have

$$\begin{aligned} GW_{n+m}GW_{n-m} - GW_n^2 &= \\ &[(b-a)2^{n+m} - (b-2a) + i\left((b-a)2^{n+m-1} - (b-2a)\right)] \\ &\quad [(b-a)2^{n-m} - (b-2a) + i\left((b-a)2^{n-m-1} - (b-2a)\right)] \\ &\quad - \left[(b-a)2^n - (b-2a) + i\left((b-a)2^{n-1} - (b-2a)\right)\right]^2 \\ &= (b-a)(b-2a)\left[(2^n - 2^{n+m-1}) + (2^{n-m-1} - 2^{n-m})\right] \\ &\quad + i3(2^n - 2^{n+m-1} - 2^{n-m-1}). \end{aligned}$$

□

Substituting $m = 1$ in Catalan's identity (7) gives the Cassini's identity for the generalized Gaussian Mersenne numbers and hence the following theorem.

Theorem 5 (Cassini's identity)**.** *For $n \geq 1$, we have*

$$GW_{n+1}GW_{n-1} - GW_n^2 = (b-a)(b-2a)\left[(2^{n-2} - 2^{n-1}) - i(2^{n-2}3)\right]. \quad (8)$$

As a special case of the above theorems, we deduce the following corollary.

Corollary 1. *For $n, m \in \mathbb{N}$, the following identities have been verified.*

1. $GM_{n+m}GM_{n-m} - GM_n^2 = \left[(2^n - 2^{n+m-1}) + (2^{n-m-1} - 2^{n-m})\right] + i3(2^n - 2^{n+m-1} - 2^{n-m-1}).$
2. $GH_{n+m}GH_{n-m} - GH_n^2 = -[[(2^n - 2^{n+m-1}) + (2^{n-m-1} - 2^{n-m})] + i3(2^n - 2^{n+m-1} - 2^{n-m-1})].$
3. $GM_{n+1}GM_{n-1} - GM_n^2 = (2^{n-2} - 2^{n-1}) - i(2^{n-2}3).$
4. $GH_{n+1}GH_{n-1} - GH_n^2 = (2^{n-1} - 2^{n-2}) + i(2^{n-2}3).$

Theorem 6 (D'Ocagne's identity)**.** *For $n, m \geq 1$, we have*

$$GW_{m+1}GW_n - GW_mGW_{n+1} = (b-a)(b-2a)\left[(2^{n-1} - 2^{m-1}) + i3(2^{n-1} - 2^{m-1})\right].$$

Proof. The argument is similar to that of Theorem 4. □

Corollary 2. *For $n, m \geq 1$, we have*

1. $GM_{m+1}GM_n - GM_mGM_{n+1} = (2^{n-1} - 2^{m-1}) + i3(2^{n-1} - 2^{m-1}).$
2. $GH_{m+1}GH_n - GH_mGH_{n+1} = (2^{m-1} - 2^{n-1}) - i3(2^{n-1} - 2^{m-1}).$

Remark 1. *The Catalan's, Cassini's and d'Ocagne's identities for the Gaussian Mersenne-Lucas numbers are same as the Gaussian Mersenne numbers but with a negative sign.*

Theorem 7 (Generating function)**.** *For the generalized Gaussian Mersenne numbers, we have*

$$GW(z) = \frac{a + (b-3a)z + i\left[\left(\frac{3}{2}a - \frac{1}{2}b\right) + \left(\frac{3}{2}b - \frac{7}{2}a\right)z\right]}{(1 - 3z + 2z^2)}.$$

Proof. Let the generating function for the sequence $\{GW_n\}_{n \geq 0}$ be given by $GW(z) = \sum_{j=0}^{\infty} GW_j z^j$. Thus, we have

$$\begin{aligned}
GW(z) - 3zGW(z) + 2z^2GW(z) &= GW_0 + z(GW_1 - 3GW_0) \\
\implies GW(z)(1 - 3z + 2z^2) &= GW_0 + z(GW_1 - 3GW_0) \\
\implies GW(z) &= \frac{GW_0(1-3z) + zGW_1}{(1 - 3z + 2z^2)} \\
\implies GW(z) &= \frac{a + (b-3a)z + i\left[\left(\frac{3}{2}a - \frac{1}{2}b\right) + \left(\frac{3}{2}b - \frac{7}{2}a\right)z\right]}{(1 - 3z + 2z^2)}.
\end{aligned}$$
□

The following corollary gives the generating functions for Gaussian Mersenne and Gaussian Mersenne-Lucas numbers.

Corollary 3. $GM(z) = \dfrac{z + i\left(\frac{3}{2}z - \frac{1}{2}\right)}{(1 - 3z + 2z^2)}$ *and* $GH(z) = \dfrac{(2 - 3z) + i\left(\frac{3}{2} - \frac{5}{2}z\right)}{(1 - 3z + 2z^2)}$.

Theorem 8. *The exponential generating function for generalized Gaussian Mersenne numbers is*

$$E(z) = (b - a)(1 + \frac{i}{2})e^{2z} - (b - 2a)e^{z}(1 + i).$$

Proof. Let $E(z) = \sum_{n=0}^{\infty} GW_n \dfrac{z^n}{n!}$ be the exponential generating function for the sequence $\{GW_n\}_{n\geq 0}$. Then using Binet's formula (3), the result can be easily proved. □

Theorem 9. *The exponential generating functions for even and odd-indexed sequences* $\{GW_{2n}\}$ *and* $\{GW_{2n+1}\}$ *are given as*

$$E_{GW_{2n}}(z) = (b - a)(1 + \frac{i}{2})\cosh 2\sqrt{z} - (b - 2a)(1 + i)\cosh\sqrt{z}$$

$$\text{and} \quad E_{GW_{2n+1}}(z) = \frac{1}{\sqrt{z}}\Big[(b - a)(1 + \frac{i}{2})\sinh 2\sqrt{z} - (b - 2a)(1 + i)\sinh\sqrt{z}\Big].$$

Proof. The proof follows using the fact that the exponential generating functions for even and odd-indexed sub-sequences $\{GW_{2n}\}_{n\geq 0}$ and $\{GW_{2n+1}\}_{n\geq 0}$ are given by

$$E_{GW_{2n}}(z) = \frac{E(\sqrt{z}) + E(-\sqrt{z})}{2} \quad \text{and} \quad E_{GW_{2n+1}}(z) = \frac{E(\sqrt{z}) - E(-\sqrt{z})}{2\sqrt{z}},$$

where $E(z)$ is the exponential generating function of the sequence $\{GW_n\}_{n\geq 0}$. □

Corollary 4. *The exponential generating functions for the Gaussian Mersenne and Gaussian Mersenne-Lucas numbers are given as follows:*

1. $E_{GM_n}(z) = (1 + \frac{i}{2})e^{2z} - (1 + i)e^{z}$.
2. $E_{GH_n}(z) = (1 + \frac{i}{2})e^{2z} + (1 + i)e^{z}$.
3. $E_{GM_{2n}}(z) = (1 + \frac{i}{2})\cosh 2\sqrt{z} - (1 + i)\cosh\sqrt{z}$.
4. $E_{GH_{2n}}(z) = (1 + \frac{i}{2})\cosh 2\sqrt{z} + (1 + i)\cosh\sqrt{z}$.
5. $E_{GM_{2n+1}}(z) = \frac{1}{\sqrt{z}}\Big[(1 + \frac{i}{2})\sinh 2\sqrt{z} - (1 + i)\sinh\sqrt{z}\Big]$.
6. $E_{GH_{2n+1}}(z) = \frac{1}{\sqrt{z}}\Big[(1 + \frac{i}{2})\sinh 2\sqrt{z} + (1 + i)\sinh\sqrt{z}\Big]$.

The next theorem deals with the finite sum of the generalized Gaussian Mersenne numbers. Hence, the result for Gaussian Mersenne and Gaussian Mersenne-Lucas numbers are given in the subsequent corollary.

Theorem 10. *For all positive integers n, the following sum formulas have been verified.*

1. $\sum_{k=0}^{n} GW_k = GW_{n+1} - (n+1)\, GW_1 + (2n+1)\, GW_0$.
2. $\sum_{k=0}^{n} GW_{2k} = \dfrac{4GW_{2n} - 3nGW_1 + (6n-1)\, GW_0}{3}$.
3. $\sum_{k=0}^{n} GW_{2k+1} = \dfrac{4GW_{2n+1} - (3n+1)\, GW_1 + 6nGW_0}{3}$.

Proof (1). Using the Binet's formula $GW_k = c2^k + d$, where $c = GW_1 - GW_0$ and $d = 2GW_0 - GW_1$, we get

$$\begin{aligned}\sum_{k=0}^{n} GW_k &= c\sum_{k=0}^{n} 2^k + d\sum_{k=0}^{n}(1)^k = c\left(2^{n+1}-1\right) + d\,(n+1)\\ &= GW_{n+1} - (n+1)\, GW_1 + (2n+1)\, GW_0.\end{aligned}$$

Similarly, the second and third identities can be proved. □

Corollary 5. *For all positive integer n, the following sum formulas have been verified.*

1. $\sum_{k=0}^{n} GM_k = GM_{n+1} - (n+1) - i\left(\frac{2n+1}{2}\right)$.
2. $\sum_{k=0}^{n} GM_{2k} = \dfrac{4GM_{2n} - 3n - i\left(\frac{6n-1}{2}\right)}{3}$.
3. $\sum_{k=0}^{n} GM_{2k+1} = \dfrac{4GM_{2n+1} - (3n+1) - i3}{3}$.
4. $\sum_{k=0}^{n} GH_k = GH_{n+1} + (n-1) + i\left(\frac{2n-1}{2}\right)$.
5. $\sum_{k=0}^{n} GH_{2k} = \dfrac{4GH_{2n} + (3n-2) + i\left(3n - \frac{3}{2}\right)}{3}$.
6. $\sum_{k=0}^{n} GH_{2k+1} = \dfrac{4GH_{2n+1} + (3n-3) + i\,(3n-2)}{3}$.

3 k-Generalized Gaussian Mersenne numbers

In this section, we give a new family of the Gaussian Mersenne numbers in generalized form, refered as k-generalized Gaussian Mersenne numbers and investigate their properties.

Definition 3. *Let $k \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$ then $\exists!\ s, r \in \mathbb{N} \cup \{0\}$ such that $n = sk + r$, $0 \le r < k$. Then the k-generalized Gaussian Mersenne numbers $\{GW_n^{(k)}\}_{n\ge 0}$ are defined as*

$$\begin{aligned}GW_n^{(k)} = &\left[(b-a)\lambda_1^s - (b-2a)\lambda_2^s + i\left((b-a)\lambda_1^{s-1} - (b-2a)\lambda_2^{s-1}\right)\right]^{k-r}\\ &\left[(b-a)\lambda_1^{s+1} - (b-2a)\lambda_2^{s+1} + i\left((b-a)((b-a)\lambda_1^{s} - (b-2a)\lambda_2^{s})\right)\right]^{r},\end{aligned}$$

where λ_1 and λ_2 are the roots of the characteristic Eqn. (2).

From Eqn. (3) and Definition 3, the relation between k-generalized Gaussian Mersenne and generalized Gaussian Mersenne numbers is described as

$$GW_n^{(k)} = GW_s^{k-r} GW_{s+1}^r, \quad n = sk + r. \tag{9}$$

If $k = 1$ then $r = 0$ and hence $m = n$. So, from Eqn. (9), we have $GW_n^{(1)} = GW_n$.
The next remark informs us of some special relations between k-generalized Gaussian Mersenne and generalized Gaussian Mersenne numbers for $k = 2, 3$.

Remark 2. *For $s \in \mathbb{N}$, the following identities are verified.*

1. $GW_{2s}^{(2)} = GW_s^2$.
2. $GW_{2s+1}^{(2)} = GW_s GW_{s+1}$.
3. $GW_{2s+1}^{(2)} = 3GW_{2s}^{(2)} - 2GW_{2s-1}^{(2)}$.
4. $GW_{3s}^{(3)} = GW_s^3$.
5. $GW_{3s+1}^{(3)} = GW_s^2 GW_{s+1}$.
6. $GW_{3s+1}^{(3)} = 3GW_{3s}^{(3)} - 2GW_{3s-1}^{(3)}$.
7. $GW_{3s+2}^{(3)} = GW_s GW_{s+1}^2$.

A list of first few k-generalized Gaussian Mersenne numbers is shown in the following table.

$GW_n^{(k)}$	$k=1$	$k=2$
$GW_0^{(k)}$	$a + i\left(\frac{3}{2}a - \frac{b}{2}\right)$	$\left(\frac{-5}{4}a^2 - \frac{b^2}{4} + \frac{3}{2}ab\right) + i(3a^2 - ab)$
$GW_1^{(k)}$	$b + ai$	$a + i\left(\frac{3}{2}a - \frac{b}{2}\right)$
$GW_2^{(k)}$	$(3b - 2a) + ib$	$b + ai$
$GW_3^{(k)}$	$(7b - 6a) + i(3b - 2a)$	$(3b - 2a) + ib$
$GW_4^{(k)}$	$(15b - 14a) + i(7b - 6a)$	$(4b^2 + 8a^2 - 12ab) + i(6b^2 - 4ab)$
$GW_5^{(k)}$	$(31b - 30a) + i(15b - 14a)$	$(18b^2 + 12a^2 - 3ab) + i(16b^2 + 4a^2 - 18ab)$

$GW_n^{(k)}$	$k=3$	$k=4$
$GW_0^{(k)}$	$\left(\frac{-23}{4}a^3 - \frac{3}{4}ab^2 + \frac{9}{2}a^2b\right) +$ $i\left(\frac{9}{8}a^3 + \frac{b^3}{8} + \frac{15}{8}a^2b - \frac{9}{8}ab^2\right)$	$\left(\frac{-119}{16}a^4 + \frac{36}{16}a^3b + \frac{30}{16}a^2b^2 - \frac{12}{16}ab^3 + \frac{16}{16}b^4\right)$ $+i\left(\frac{-60}{8}a^4 + \frac{92}{8}a^3b - 9a^2b^2 - \frac{ab^3}{2}\right)$
$GW_1^{(k)}$	$\left(\frac{-5}{4}a^2 - \frac{b^2}{4} + \frac{3}{2}ab\right)$ $+i(3a^2 - ab)$	$\left(\frac{-23}{4}a^3 - \frac{3}{4}ab^2 + \frac{9}{2}a^2b\right) +$ $i\left(\frac{9}{8}a^3 + \frac{b^3}{8} + \frac{15}{8}a^2b - \frac{9}{8}ab^2\right)$
$GW_2^{(k)}$	$a + i\left(\frac{3}{2}a - \frac{1}{2}b\right)$	$\left(\frac{-5}{4}a^2 - \frac{b^2}{4} + \frac{3}{2}ab\right) + i(3a^2 - ab)$
$GW_3^{(k)}$	$b + ai$	$a + i\left(\frac{3}{2}a - \frac{1}{2}b\right)$
$GW_4^{(k)}$	$(3b - 2a) + ib$	$b + ai$
$GW_5^{(k)}$	$(4b^2 + 8a^2 - 12ab)$ $+i(6b^2 - 4ab)$	$(3b - 2a) + ib$

Table 3: The k-generalized Gaussian Mersenne numbers for $k = 1, 2, 3, 4$.

Theorem 11. *For $k, s \in \mathbb{N}$, we have the following results,*

1. $GW_{sk}^{(k)} = GW_s^k$.

2. $GW^{(s)}_{sk+1} = 3GW^{(s)}_{sk} - 2GW^{(s)}_{sk-1}$.

3. $GW^{(k)}_{sk+k} - GW^{(k)}_{sk} = GW^{k}_{s+1} - GW^{k}_{s}$.

Proof. 1. Let $n = sk$, then $r = 0$ and hence from Eqn. (9), we have the desired result.

2. From Eqn. (9) and Definition 1, we have

$$3GW^{(s)}_{sk} - 2GW^{(s)}_{sk-1} = 3GW^{s}_{k} - 2GW_{k-1}GM^{s-1}_{k} = GW^{s-1}_{k}GW_{k+1} = GW^{(s)}_{sk+1}.$$

3. It can be easily established using the fist identity of Theorem 11.

□

Theorem 12. *Let $n, m \geq 0$ such that $n + m > 1$, then we have*

$$GW^{(2)}_{2(n+m-1)} - GW_{n+m}GW_{n+m-2} = (b-a)(b-2a)\left[(2^{n+m-2} - 2^{n+m-1}) - i(3.2^{n+m-2})\right].$$

Proof. From Eqn. (8) and the fist identity of Theorem 11, we have

$$\begin{aligned} GW^{(2)}_{2(n+m-1)} - GW_{n+m}GW_{n+m-2} &= GW^{2}_{(n+m-1)} - GW_{n+m}GW_{n+m-2} \\ &= (b-a)(b-2a)\left[(2^{n+m-2} - 2^{n+m-1}) - i(3.2^{n+m-2})\right]. \end{aligned}$$

□

Theorem 13. *Let $s, k \in \mathbb{N}$, then for fixed k, s, the following results hold.*

1. $$\sum_{m=0}^{k-1} \binom{k-1}{m} GW^{(k)}_{sk+m} = [2^{n-1}3(b-a)(2+i) - 2(b-2a)(1+i)]^{k-1}GW_s.$$

2. $$\sum_{m=0}^{k-1} (-1)^m \binom{k-1}{m} GW^{(k)}_{sk+m} = [2^{n-1}(a-b)(2+i)]^{k-1}GW_s.$$

3. $$\sum_{m=0}^{k-1} GW^{(k)}_{sk+m} = \frac{GW_s(GW^{(k)}_{(s+1)k} - GW^{(k)}_{sk})}{2^{n-1}(b-a)(2+i)}.$$

4. $$\sum_{m=0}^{k} mGW^{(k)}_{sk+m} = \frac{GW^{(k+2)}_{s(k+2)+1} - kGW^{(k+2)}_{s(k+2)+k} + (k-1)GW^{(k+2)}_{s(k+2)+k+1}}{(2^{n-1}(b-a)(2+i))^2}.$$

Proof. 1. From relation (9), we have

$$\begin{aligned} \sum_{m=0}^{k-1} \binom{k-1}{m} GW^{(k)}_{sk+m} &= \sum_{m=0}^{k-1} \binom{k-1}{m} GW^{k-m}_{s} GW^{m}_{s+1} \\ &= GW_s \sum_{m=0}^{k-1} \binom{k-1}{m} GW^{m}_{s+1} GW^{k-1-m}_{s} \\ &= GW_s(GW_s + GW_{s+1})^{k-1} \quad \text{(using Binomial theorem)} \\ &= [2^{n-1}3(b-a)(2+i) - 2(b-2a)(1+i)]^{k-1}GW_s \quad \text{(using Theorem 2).} \end{aligned}$$

2. Proceeding as first identity, we write

$$\begin{aligned}\sum_{m=0}^{k-1}(-1)^m\binom{k-1}{m}GW_{sk+m}^{(k)} &= (-1)^{k-1}\sum_{m=0}^{k-1}(-1)^{k-1-m}\binom{k-1}{m}GW_s^{k-m}GW_{s+1}^m\\ &= (-1)^{k-1}GW_s\sum_{m=0}^{k-1}\binom{k-1}{m}GW_{s+1}^m(-GW_s)^{k-1-m}\\ &= (-1)^{k-1}GW_s(GW_{s+1}-GW_s)^{k-1} \quad \text{(using Binomial theorem)}\\ &= (-1)^{k-1}[2^{n-1}(b-a)(2+i)]^{k-1}GW_s \quad \text{(using Theorem 2 (2))}\\ &= [2^{n-1}(a-b)(2+i)]^{k-1}GW_s.\end{aligned}$$

3. From (9), we have $GW_{sk+m}^{(k)} = GW_s^{k-m}GW_{s+1}^m = GW_s^k(GW_{s+1}/GW_s)^m$. So,

$$\begin{aligned}\sum_{m=0}^{k-1}GW_{sk+m}^{(k)} &= GW_s^k\sum_{m=0}^{k-1}\Big(\frac{GW_{s+1}}{GW_s}\Big)^m\\ &= GW_s^k\frac{(GW_{s+1}/GW_s)^k-1}{GW_{s+1}/GW_s-1}\\ &= GW_s\Big(\frac{GW_{s+1}^k-GW_s^k}{GW_{s+1}-GW_s}\Big)\\ &= \frac{GW_s(GW_{(s+1)k}^{(k)}-GW_{sk}^{(k)})}{GW_{s+1}-GW_s}\\ &= \frac{GW_s(GW_{(s+1)k}^{(k)}-GW_{sk}^{(k)})}{2^{n-1}(b-a)(2+i)}.\end{aligned}$$

4. We should note that $\sum_{m=1}^{k}mx^{m-1} = (1-kx^{k-1}+(k-1)x^k)/(1-x)^2$. Hence,

$$\begin{aligned}\sum_{m=0}^{k}mGW_{sk+m}^{(k)} &= GW_s^{k-1}GW_{s+1}\sum_{m=1}^{k}m\Big(\frac{GW_{s+1}}{GW_s}\Big)^{m-1}\\ &= GW_s^{k-1}GW_{s+1}\Big(\frac{1-k(GW_{s+1}/GW_s)^{k-1}+(k-1)(GW_{s+1}/GW_s)^k}{(1-GW_{s+1}/GW_s)^2}\Big)\\ &= \frac{GW_s^{k-1}GW_{s+1}-kGW_{s+1}^k+(k-1)GW_{s+1}^{k+1}/GW_s}{(1-GW_{s+1}/GW_s)^2}\\ &= \frac{GW_s^{k+1}GW_{s+1}-kGW_s^2GW_{s+1}^k+(k-1)GW_sGW_{s+1}^{k+1}}{(GW_s-GW_{s+1})^2}\\ &= \frac{GW_{s(k+2)+1}^{(k+2)}-kGW_{s(k+2)+k}^{(k+2)}+(k-1)GW_{s(k+2)+k+1}^{(k+2)}}{(2^{n-1}(b-a)(2+i))^2} \quad \text{(using (9))}.\end{aligned}$$

□

Theorem 14. *For $n,k\geq 2$, then Cassini's identity for k-generalized Gaussian*

Mersenne numbers is

$$(GW^{(k)}_{nk+a-1})^2 - GW^{(k)}_{nk+a}GW^{(k)}_{nk+a-2} = \begin{cases} GW_n^{2k-2}\,(b-a)\,(b-2a)\left[(2^{n-2}-2^{n-1}) - i(3.2^{n-2})\right], & a=1 \\ 0, & a\neq 1. \end{cases}$$

Proof. Let $a=1$, then from Eqn. (9) and Theorem 11, we have

$$\begin{aligned} GW^{(k)}_{nk+1}GW^{(k)}_{nk-1} - (GW^{(k)}_{nk})^2 &= (GW_n^{k-1}GW_{n+1})(GW_{n-1}GW_n^{k-1}) - (GW_n^k)^2 \\ &= GW_n^{2k-2}[GW_{n+1}GW_{n-1} - (GW_n)^2] \\ &= GW_n^{2k-2}\,(b-a)\,(b-2a)\left[(2^{n-2}-2^{n-1}) - i(3.2^{n-2})\right] \text{(using Eqn. (8))}. \end{aligned}$$

Moreover, if $a\neq 1$, then by using Eqn. (9)

$$\begin{aligned} GW^{(k)}_{nk+a}GW^{(k)}_{nk+a-2} - (GW^{(k)}_{nk+a-1})^2 &= (GW_n^{k-a}GW_{n+1}^{a})(GW_n^{k-a+2}GW_{n+1}^{a-2}) - (GW_n^{k-a+1}GW_{n+1}^{a-1})^2 \\ &= GW_n^{2k-2a+2}[GW_{n+1}^{2a-2} - GW_{n+1}^{2a-2}] \\ &= 0. \end{aligned}$$

□

Theorem 15. *For $k,s\in\mathbb{N}$, the following relations between k-generalized Gaussian Mersenne and generalized Gaussian Mersenne numbers are obtained.*

1. $\sum_{j=0}^{k-1}(3)^{-j}GW^{(k)}_{sk+j} = \frac{GW_s}{2GW_{s-1}}\Big(\frac{3^kGW^{(k)}_{sk} - GW^{(k)}_{(s+1)k}}{3^{k-1}}\Big)$.
2. $\sum_{j=0}^{k-1}(-2)^{k-1-j}(3)^j\binom{k-1}{j}GW^{(k)}_{sk+j} = GW_sGW^{(k-1)}_{(s+2)(k-1)}$.
3. $\sum_{j=0}^{k-1}(-1)^j(3)^{k-1-j}\binom{k-1}{j}GW^{(k)}_{sk+j} = (-2)^{k-1}GW_sGW^{(k-1)}_{(s-1)(k-1)}$.

Proof. 1. By using Eqn. (9), we write

$$\begin{aligned} \sum_{j=0}^{k-1}(3)^{-j}GW^{(k)}_{sk+j} &= \sum_{j=0}^{k-1}\Big(\frac{GW_{s+1}}{3GW_s}\Big)^jGW_s^k \\ &= GW_s^k\left(\frac{\Big(\frac{GW_{s+1}}{3GW_s}\Big)^k - 1}{\Big(\frac{GW_{s+1}}{3GW_s}\Big) - 1}\right) \\ &= GW_s\Big(\frac{3^kGW_s^k - GW_{s+1}^k}{3^{k-1}2GW_{s-1}}\Big) \\ &= \frac{GW_s}{2GW_{s-1}}\Big(\frac{3^kGW^{(k)}_{sk} - GW^{(k)}_{(s+1)k}}{3^{k-1}}\Big). \end{aligned}$$

2. Using Eqn. (9) and binomial theorem, we get

$$\sum_{j=0}^{k-1}(-2)^{k-1-j}(3)^j\binom{k-1}{j}GW_{sk+j}^{(k)} = GW_s(3GW_{s+1}-2GW_s)^{k-1}$$
$$= GW_sGW_{(s+2)}^{k-1} = GW_sGW_{(s+2)(k-1)}^{(k-1)}.$$

3. The argument is similar to 2. □

4 Conclusion

In summary, we introduced the generalized Gaussian Mersenne numbers and investigated their algebraic properties. In addition, we introduced a new family of k-generalized Gaussian Mersenne numbers in closed form and shown some relations with generalized Gaussian Mersenne numbers. Here, we examined Binet's formula, Cassini's and Catalan's identity, generating and exponential generating functions, various partial and binomial sums, etc. of these numbers.

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