Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 5(67), No. 1 - 2025, 133-146
https://doi.org/10.31926/but.mif.2025.5.67.1.10

WARPED PRODUCT POINTWISE $\mathcal{PR}$- SEMI-SLANT IMMERSIONS IN PARA-KÄHLER MANIFOLDS

Anuj KUMAR[1] and Anil SHARMA [*,2]

Abstract

This study investigates the geometry of warped product pointwise $\mathcal{PR}$-semi-slant immersions M in para-Kähler manifolds $\overline{M}^{2m}$ that naturally englobes the warped product $\mathcal{PR}$ and semi-slant immersions. We first derive non-existence of warped product $M_\lambda \times_f M_T$, and then by presenting a numerical example examine the presence of warped product M such that $M = M_T \times_f M_\lambda$ in $\overline{M}^{2m}$. Finally, we derive some characterizations for pointwise $\mathcal{PR}$-semi-slant submanifolds to be locally warped product $M_T \times_f M_\lambda$ in terms of shape operator and endomorphisms. The mixed totally geodesic case is also discussed.

2020 *Mathematics Subject Classification:* 53B25, 53B30, 53C15, 53C42
Key words: warped product, para-Kähler manifold, pointwise slant submanifold, pseudo-Riemannian manifold

1 Introduction

The geometry of the warped bundle in different spaces has always been a topic of great interest in differential geometry. In particular, the warped product of (pseudo)-Riemannian manifolds has a long and fascinating history and is currently extensively studied because of its numerous application to mathematics and physics especially in, harmonic map, Ricci soliton, general relativity theory and black holes. For examples warped product $C \times_f \mathbb{S}^1$ called the surface of revolution is influential in the construction of different models in some relativistic theories and warped product $\mathbb{S}_+^{m-1} \times_f \mathbb{S}^1$ involved open upper hemisphere and circle for some warping function f on $\mathbb{S}_+^{m-1}$ play an important role in the research of harmonic maps, Ricci solitons and Einstein manifolds [5, 22, 14, 1, 17, 15].

[1]Department of Mathematics and Statistics, Central University of Punjab, Bathinda-151401, Punjab, India, e-mail: kumaranujcup@gmail.com

[2*] *Corresponding author*, Department of Mathematics, AIT-CSE, University Institute of Science, Chandigarh University, Mohali, Punjab-140413, India, e-mail: anilsharma3091991@gmail.com

Bishop-O'Neill [6] originated the geometry of warped product manifolds carrying non-positive curvature as, consider two pseudo-Riemannian manifolds B and F. The *warped product* $B \times_f F$ of B and F is the manifold $B \times F$ endowed with pseudo-Riemannian structure such that $\|X\|^2 = \|\pi^*(X)\|^2 + (f \circ \pi)^2 \|\sigma^*(X)\|^2$ $\forall$ tangent vector $X \in \mathfrak{X}(B \times_f F)$ where π and σ denotes the natural projections on $B \times F$ to B and F, respectively. Hence in $B \times_f F$, B is labeled the *base manifold* while F is named the *fiber* and f a positive C^∞ wraping function on B. However, the theory gained popularity after Chen started investigating the Cauchy-Riemann warped structure in even-dimensional manifold endowed Riemann metric i.e., Kählerian $\overline{N}$ and proved that the proper warped product Cauchy-Riemann submanifolds in the form $N_\perp \times_f N_T$ didn't exist [7]. Thereafter, Sahin [18] extended the geometry of Cauchy-Riemann warped product to semi-slant warped products and presented nonexistence theorems for such warped products in Kähler manifold. Later on, Sahin [19] continued by introducing a new generalised class called warped product pointwise semi-slant submanifolds in Kählerian manifolds. These geometric setups may not be often found appropriate in physics, specifically in the relativity theory where the metric isn't naturally positive definite. Meanwhile, Chen-Munteanu [10] started the geometry of $\mathcal{PR}$ submanifold along with its warped aspects in para-Kähler manifolds and presented some analogies and differences between Kähler and para-Kähler manifolds. Analogous to this, Srivastava-Sharma [24] continued the study for paracosymplectic manifolds. Thenceforth, many differential geometer's studied (pseudo)-Riemannian warped geometry as pointwise slant, semi (pseudo) slant submanifolds with several viewpoints in different ambients (cf. [25, 4, 3, 26, 20, 21, 11, 23, 16]). Motivation to present research is due to considerably two reasons, one its numerous applications. Second, to extend the study for M that naturally contains slant, pointwise slant, $\mathcal{PR}$ and semi-slant submanifolds in para-Kähler manifolds $\overline{M}^{2m}$.

The following is a brief description of the manuscript. In Sect. 2, we review few fundamentals of the para-Kähler manifolds and submanifolds. Sect. 3, contains the definition of pointwise $\mathcal{PR}$-semi-slant submanifolds, some important results, and relations of integrability and totally geodesic foliation. In Sect. 4, we first prove the nonexistence of warped immersion as pointwise $\mathcal{PR}$-semi-slant of the type $M_\lambda \times_f M_T$ and then give a numerical illustration for the existence of warped product immersion $M_T \times_f M_\lambda \to \overline{M}^{2m}$. Finally, we obtain some conditions that are both necessary and sufficient in terms of shape operator and canonical structures t, n for pointwise $\mathcal{PR}$-semi-slant submanifolds to be locally warped products in $\overline{M}^{2m}$.

2 Preliminaries

An almost paracomplex structure on a $2m$ dimentional manifold $\overline{M}^{2m}$ is $\mathcal{P}$ such that

$$\mathcal{P}^2 = id, \tag{2.1}$$

where $\mathcal{P}$ is an endomorphism and id the identity map with ± 1 eigenbundles $\mathfrak{X}^{\pm}(\overline{M})$ of dimension m. For this, the combination $(\overline{M}^{2m}, \mathcal{P})$ is referred to be an almost para-complex manifold. An almost para-Hermitian manifold $(\overline{M}^{2m}, \mathcal{P}, \overline{g})$ is a manifold associated with an almost paracomplex structure $\mathcal{P}$ and a pseudo-Riemannian metric $\overline{g}$ satisfying

$$\overline{g}(\mathcal{P}X, \mathcal{P}Y) = -\overline{g}(X, Y). \tag{2.2}$$

Then from above equatons, we can easily obtain

$$\overline{g}(\mathcal{P}X, Y) + \overline{g}(X, \mathcal{P}Y) = 0 \ \forall X, Y \in \mathfrak{X}(\overline{M}^{2m}), \tag{2.3}$$

where $\mathfrak{X}(\overline{M}^{2m})$ being Lie algebra of vector fields on $\overline{M}^{2m}$.

Definition 2.1. An almost para-Hermitian manifold $\overline{M}^{2m}$ is called a *para-Kähler manifold* [8] if $\mathcal{P}$ is parallel with respect to $\overline{\nabla}$, *i*.e.,

$$(\overline{\nabla}_X \mathcal{P})Y = 0, \quad \forall X, Y \in \mathfrak{X}(\overline{M}^{2m}). \tag{2.4}$$

Here, $\overline{\nabla}$ represents the Levi-Civita connection on $\overline{M}^{2m}$ w.r.t. $\overline{g}$.

2.1 Geometry of submanifolds

Let M be a differentiable manifold immersed in a $2m$-dimensional para-Kähler manifold $\overline{M}^{2m}$. We use the notation g for the induced metric tensor on M such that $g = \overline{g}|_M$ of constant signature and rank [12]. Thus, $\forall\ p \in M$, tangent space $\mathfrak{X}_p(M)$ is a non-degenerated subspace of $\mathfrak{X}_p(\overline{M})$ with $\mathfrak{X}_p(\overline{M}) = \mathfrak{X}_p(M) \oplus \mathfrak{X}_p(M)^{\perp}$, where $\mathfrak{X}_p(M)^{\perp}$ indicates the normal space of M. If $\mathfrak{X}(M^{\perp})$ indicates a normal bundle to M and $\mathfrak{X}(M)$ the tangent bundle to M, then the Gauss-Weingarten formulae are thus defined by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\overline{\nabla}_X \zeta = -A_\zeta X + \nabla^{\perp}_X \zeta, \tag{2.6}$$

for any $X, Y \in \mathfrak{X}(M)$ and $\zeta \in \mathfrak{X}(M^{\perp})$, where ∇ (resp., $\nabla^{\perp}$) is the induced tangent (resp., normal) connection on $\mathfrak{X}(M)$ (resp., $\mathfrak{X}(M^{\perp})$), A_ζ at ζ is the Weingarten map and the second fundamental form h, is given by

$$g(A_\zeta X, Y) = \overline{g}(h(X, Y), \zeta). \tag{2.7}$$

A submanifold M is totally geodesic (resp., umbilical) if its $h = 0$ (resp., $h(X, Y) = g(X, Y)\mathcal{H}$), where $\mathcal{H}$ indicates the mean curvature vector. If $\mathcal{H} = 0$ then submanifold M is called *minimal*. If we write, for all $\xi \in \mathfrak{X}(M)$ and $\zeta \in \mathfrak{X}(M^{\perp})$ that

$$\mathcal{P}\xi = t\xi + n\xi, \tag{2.8}$$

$$\mathcal{P}\zeta = t'\zeta + n'\zeta, \tag{2.9}$$

where $t\xi$ ($t'\zeta$) and $n\xi$ ($n'\zeta$) are tangential (normal) components of $\mathcal{P}\xi$ ($\mathcal{P}\zeta$), then using Eqs. (2.1), (2.2) and (2.8), it is observe that

$$\overline{g}(X, tY) = -\overline{g}(tX, Y) \quad \forall X, Y \in \mathfrak{X}(M). \tag{2.10}$$

Now by the consequence of (2.4), (2.8) and (2.9), we obtain

$$(\nabla_X t)Y = A_{nY}X + t'h(X, Y), \tag{2.11}$$
$$(\nabla_X n)Y = n'h(X, Y) - h(X, tY), \tag{2.12}$$

$\forall\ X, Y \in \mathfrak{X}(M)$.

3 Pointwise $\mathcal{PR}$-semi-slant submanifolds

Chen-Garay [9] introduced the concept of pointwise slant submanifolds for Hermittian manifolds. Motivated to this Sharma [20, 21] continued the study by defining pointwise slant submanifolds in para-Hermitian manifolds as;

Definition 3.1. Let ϕ be an immersion $\phi : M \to \overline{M}^{2m}$ into an almost para-Hermitian manifold and D_λ be the non-degenerate distribution on M. Then D_λ is said to be *pointwise slant* distribution on M, accordingly M *pointwise slant submanifold*, if there exists a real valued function λ such that

$$t^2 = \lambda id, \quad g(tX, Y) = -g(X, tY)$$

$\forall$ non-null tangent vectors $X, Y \in D_\lambda$ at each point $p \in M$. Here, λ is called *slant function*, independent the choice of $X, Y \in M$.

Now analogous to [19, 21], we define pointwise $\mathcal{PR}$-semi-slant submanifolds M in $\overline{M}^{2m}$;

Definition 3.2. Let $M \to \overline{M}^{2m}$ be an immersion of non-degenerate submanifold M in a para-Hermitian manifold $\overline{M}^{2m}$. Then M is a *pointwise $\mathcal{PR}$-semi-slant submanifold* if it admits a pair of orthogonal distributions i.e., totally holomorphic $\mathfrak{D}_T$ and pointwise slant $\mathfrak{D}_\lambda$ with slant function λ satisfying $\mathfrak{X}(M) = \mathfrak{D}_T \oplus \mathfrak{D}_\lambda$ such that $\mathcal{P}(\mathfrak{D}_T) \subseteq \mathfrak{D}_T$.

Let us denote by d_1 and d_2 the dimension of $\mathfrak{D}_T$ and $\mathfrak{D}_\lambda$, respectively then we deduce that pointwise $\mathcal{PR}$-semi-slant submanifold M of $\overline{M}^{2m}$ is

- pointwise slant (resp., pointwise semi-slant) submanfold, if $d_1 = 0$ and $\mathfrak{D}_\lambda$ (resp., the pair $\mathfrak{D}_T$, $\mathfrak{D}_\lambda$) indicates on M with slant function $\lambda = \cos^2(\theta)$ [19].

- $\mathcal{PR}$-semi slant submanifold, if $d_1.d_2 \neq 0$ and slant function λ is globally constant [20, 2], in particular if $d_1 \neq 0$ and $\lambda = 0$ then M is a $\mathcal{PR}$-submanifold [10].

Finally, we call M *proper*, if $d_1.d_2 \neq 0$ and λ is non-constant function, *mixed totally geodesic*, if $h(\mathfrak{D}_T, \mathfrak{D}_\lambda)$ vanishes.

Next, we present example to validate proper M of $\overline{M}^{2m}$.

Example 3.3. Let $\overline{M}^{2m} = \mathbb{R}^8$ be a 8-dimensional manifold with the coordinate system $(\overline{x}_1, \cdots, \overline{x}_8)$. Define a structure $(\mathcal{P}, \overline{g})$ on $\overline{M}^{2m}$ by

$$\mathcal{P}e_1 = e_2,\ \mathcal{P}e_2 = e_1, \cdots,\ \mathcal{P}e_7 = e_8, \mathcal{P}e_8 = e_7. \tag{3.1}$$

$$\overline{g} = \sum_{i=1}^{4}(d\overline{x}_i)^2 - \sum_{j=5}^{8}(d\overline{x}_j)^2, \tag{3.2}$$

such that $e_1 = \dfrac{\partial}{\partial \overline{x}_1}, \cdots, e_8 = \dfrac{\partial}{\partial \overline{x}_8}$. Now by direct computations, we achieve an almost para-Hermitian manifold. For $\overline{\nabla}$ w.r.t. $\overline{g}$, we readily conclude that the manifold $(\overline{M}^{2m}, \mathcal{P}, \overline{g})$ is a para-Kähler manifold. Assume the immersion $\phi : (M, g) \to (\overline{M}^{2m}, \overline{g})$ defined by

$$\phi(u, v, r, \alpha) = (\cos(\alpha), r, \sin(\alpha) + v, 2u, v + \sin(\alpha), 3u, \cos(\alpha), \alpha), \tag{3.3}$$

where $r, \alpha \in \mathbb{R}/\{0\}$. Then $\mathfrak{X}_p(M)$ of M is spanned by

$$\begin{aligned} &Z_u = 2e_4 + 3e_6,\ Z_v = e_3 + e_5, \\ &Z_r = e_2,\ Z_\alpha = -\sin(\alpha)e_1 + cos(\alpha)e_3 + \cos(\alpha)e_5 - \sin(\alpha)e_7 + e_8, \end{aligned} \tag{3.4}$$

where $Z_u, Z_v, Z_r, Z_\alpha \in \mathfrak{X}(M)$. Consequently from Eqs. (3.1), we obtain

$$\begin{aligned} &\mathcal{P}(Z_u) = 2e_3 + 3e_5, \mathcal{P}(Z_v) = e_4 + e_6, \\ &\mathcal{P}(Z_r) = e_1, \mathcal{P}(Z_\alpha) = -\sin(\alpha)e_2 + cos(\alpha)e_4 + \cos(\alpha)e_6 - \sin(\alpha)e_8 + e_7. \end{aligned} \tag{3.5}$$

From Eqs. (3.2), (3.4) and (3.5), we analyze that holomorphic distribution $\mathfrak{D}_T = span\{Z_u, Z_v\}$ and pointwise slant distribution $\mathfrak{D}_\lambda = span\{Z_r, Z_\alpha\}$ with slant function $\lambda = \sin^2(\alpha)$. Hence we call M a proper M in $\overline{M}^{2m}$.

Furthermnore, on $\mathfrak{D}_T$ and $\mathfrak{D}_\lambda$ if the projections are denoted by P_T and P_λ, respectively. Then $\forall\ X \in \mathfrak{X}(M)$ with effect of projections can be represented as $X = P_T X + P_\lambda X$. Now, by applying Eq. (2.8) and $\mathcal{P}$ to previous expression,we derive the following

$$\begin{aligned} &\mathcal{P}X = tP_T X + tP_\lambda X + nP_\lambda X,\ tP_T X \in \mathfrak{X}(\mathfrak{D}),\ nP_T X = 0, \\ &\implies tP_\lambda X \in \mathfrak{X}(\mathfrak{D}_\lambda), nP_\lambda X \in \mathfrak{X}(M^\perp), \\ &and\ tX = tP_T X + tP_\lambda X,\ nX = nP_\lambda X. \end{aligned}$$

Since, $\mathfrak{D}_\lambda$ is pointwise slant distribution, hence from Definition 3.1 and above expressions, we can deduce that

$$t^2 X = \lambda X, \tag{3.6}$$

for some real-valued non-constant function λ on M.

Now, we present the characterizations for pointwise $\mathcal{PR}$-semi-slant submanifold M of $\overline{M}^{2m}$:

Theorem 3.4. *In order for a submanifold of a $\overline{M}^{2m}$ to be a pointwise $\mathcal{PR}$-semi-slant M, it is necessary and sufficient that $\exists$ a distribution D on M and a λ such that for $X \in \mathfrak{X}(M)$, (a) $D = \{X : (t_D)^2 X = \lambda X\}$, (b) $nX = 0$, if $X \perp D$. where λ denotes the slant function of M.*

Proof. Analogous to Theorem 3.4 of [19], the proof of this thoerem can be achieved easily. □

Straightforward from Theorem 3.4, we draw corollary for future use;

Corollary 3.5. *For a pointwise $\mathcal{PR}$-semi-slant submanifold M in $\overline{M}^{2m}$, we have*

$$g(tX, tY) = -\lambda\, g(X, Y), \tag{3.7}$$

$$g(nX, nY) = (\lambda - 1)\, g(X, Y), \tag{3.8}$$

$\forall\ X, Y \in \mathfrak{X}(\mathfrak{D}_\lambda)$.

Next, we derive some important results;

Theorem 3.6. *Let $M \to \overline{M}^{2m}$ be a proper pointwise $\mathcal{PR}$-semi-slant immersion in a para-Kähler manifold. In order for a holomorphic distribution $\mathfrak{D}_T$ on M*

(i) being integrable, it is necessary and sufficient that $h(\mathcal{P}Y, X) = h(\mathcal{P}X, Y)$,

(ii) define a totally geodesic foliation, it is necessary and sufficient that $g(A_{ntZ}X, Y) = g(A_{nZ}X, \mathcal{P}Y)$

$\forall\ X, Y \in \mathfrak{D}_T$ and $Z \in \mathfrak{D}_\lambda$.

Proof. From Eqs. (2.1), (2.4) and (2.8), we derive that $g([X, Y], Z) = -g(\overline{\nabla}_Y X, t^2 Z + ntZ) + g(h(Y, \mathcal{P}X), nZ)$. Using Eq. (3.6) in above equality, we attain $(\lambda - 1)g([X, Y], Z) = g(h(\mathcal{P}X, Y), nZ) - g(h(X, \mathcal{P}Y), nZ)$. Hence (i) achieved. Now, for (ii), we know $g(\nabla_X Y, Z) = g(\overline{\nabla}_X Y, Z)$ by virtue of definition 3.2 and Eq. (2.5). Employing Eqs. (2.1)–(2.7), in above expression, we recieve that $g(\nabla_X Y, Z) = g(\nabla_X Y, t^2 Z) + g(A_{ntZ}X, Y) - g(A_{nZ}X, \mathcal{P}Y)$. Now, applying Eq. (3.6) in previous equation, we deduce $(\lambda - 1)g(\nabla_X Y, Z) = g(A_{ntZ}X, Y) - g(A_{nZ}X, \mathcal{P}Y)$. This derived (ii) and thus, finishes the derivation of the theorem. □

Theorem 3.7. *Let $M \to \overline{M}^{2m}$ be a proper pointwise $\mathcal{PR}$-semi-slant immersion in a para-Kähler manifold. In order for a pointwise slant distribution $\mathfrak{D}_\lambda$ on M,*

(i) being integrable, it is necessary and sufficient that$g(A_{nW}Z - A_{nZ}W, \mathcal{P}X) = g(A_{ntZ}W - A_{ntW}Z, X)$,

(ii) defines a totally geodesic foliation, it is necessary and sufficient that $g(A_{nW}Z, \mathcal{P}X) = g(A_{ntW}Z, X)$,

$\forall\ X \in \mathfrak{D}_T$ and $Z, W \in \mathfrak{D}_\lambda$.

Proof. The proof is identical to that of Theorem 3.6 and hence omitted. □

4 Warped product pointwise $\mathcal{PR}$-semi-slant immersions

In this part, we establish some characterizations for warped product pointwise $\mathcal{PR}$-semi-slant immersions in para Kähler manifolds $\overline{M}^{2m}$.

Firstly, we revive few fundamentals from [6] for further use;

If we denote by ∇ the Levi-Civita connection on $B \times_f F$, then for any $X, Y \in \mathfrak{X}(B)$ and $Z, W \in \mathfrak{X}(F)$, we see that

$$\nabla_X Y \in \mathfrak{X}(TB), \quad \nabla_X Z = \nabla_Z X = (\frac{Xf}{f})Z, \quad \nabla_Z W = \frac{-g(Z,W)}{f}\nabla f, \tag{4.1}$$

where $g(\nabla f, X) = Xf$ specifies the gradient ∇f of f.

Remark 4.1. Since *lift* is of the utmost use in computation on product manifold, therefore, for simplicity, we will examine on $M = B \times_f F$ the $X \in \mathfrak{X}(B)$ with the *lift* $\tilde{X}$ and $Z \in \mathfrak{X}(F)$ with the *lift* $\tilde{Z}$. Moreover, it is indeed worth remembering that for the warped product $B \times_f F$ such that B (resp., F) is totally geodesic (resp., umbilical) in M.

Next we say that,

Definition 4.2. A proper pointwise $\mathcal{PR}$-semi-slant submanifolds M in para Kähler manifolds $\overline{M}^{2m}$ is called *warped product pointwise $\mathcal{PR}$-semi-slant submanifolds* if it is a warped product of the form: $M_T \times_f M_\lambda$ such that M_λ and M_T are pointwise proper slant and holomorphic integral submanifolds of $\mathfrak{D}_\lambda$ and $\mathfrak{D}_T$ on M respectively, and f a non-constant positive smooth function on M_T. If the warping function f is constant then it is called *pointwise $\mathcal{PR}$-semi-slant product.*

Here we derive the non-existence result as a proposotion;

Proposition 4.3. *Let $M \to \overline{M}^{2m}$ be an immersion. Then for any $X, Y \in \mathfrak{X}(M_T)$ and $Z \in \mathfrak{X}(M_\lambda)$ there are no warped product pointwise $\mathcal{PR}$-semi-slant submanifolds of the form $M = M_\lambda \times_f M_T$ in para-Kähler manifolds $\overline{M}^{2m}$.*

Proof. We have from Eqs. (2.1)-(2.9), that $g(\nabla_X Z, Y) = g(\overline{\nabla}_X(t^2 Z + ntZ), Y) - g(\overline{\nabla}_X nZ, \mathcal{P}Y)$. Applying Eq. (3.6) and $g(Y, Z) = 0$, we arrive at

$$g(\nabla_X Z, Y) = \lambda g(\overline{\nabla}_X Z, Y) - g(A_{ntZ}X, Y) + g(A_{nZ}X, \mathcal{P}Y). \tag{4.2}$$

Thus from Eq. (4.2), we conclude that

$$(\lambda - 1)g(\nabla_X Z, Y) = -\overline{g}(h(X, Y), ntZ) + \overline{g}(h(X, \mathcal{P}Y), nZ). \tag{4.3}$$

We get by swapping X and Y in Eq. (4.3) and then subtracting from (4.3), that

$$\overline{g}(h(\mathcal{P}X, Y), nZ) = \overline{g}(h(X, \mathcal{P}Y), nZ). \tag{4.4}$$

Furthermore, from Eqs. (2.1), (2.4), (2.8) and Gauss-Weingarten formulas, we deduce that

$$g(A_{nZ}X, \mathcal{P}Y) = g(\overline{\nabla}_X Z, Y) + g(\overline{\nabla}_X tZ, \mathcal{P}Y). \tag{4.5}$$

Now, From Eqs. (4.1), (4.4) and (4.5), we observe that $tZ(\ln f)g(X, \mathcal{P}Y) = 0$. Replacing Z by tZ and X by $\mathcal{P}X$ in above expression we derive $\lambda Z(\ln f)g(X,Y) = 0$. Thus f is constant. Since, M_λ is pointwise proper slant submanifold and X, Y, Z are non-null vector fields. The proof is now complete. □

Next, we first present an illustration that signifies the presence of warped product M of the form $M = M_T \times_f M_\lambda$ in $\overline{M}^{2m}$ and then proves some important lemmas for further use.

Example 4.4. Let $\overline{M}^{2m} = \mathbb{R}^{12}$ be a 12-dimensional manifold with the coordinate system $(\overline{x}_1, \cdots, \overline{x}_{12})$. Define a structure $(\mathcal{P}, \overline{g})$ on $\overline{M}^{2m}$ by

$$\mathcal{P}e_1 = e_2,\ \mathcal{P}e_2 = e_1, \cdots, \mathcal{P}e_{11} = e_{12}, \mathcal{P}e_{12} = e_{11}. \tag{4.6}$$

$$\overline{g} = \sum_{i=1}^{6}(d\overline{x}_i)^2 - \sum_{j=7}^{12}(d\overline{x}_j)^2, \tag{4.7}$$

where $e_1 = \dfrac{\partial}{\partial \overline{x}_1}, \cdots, e_{12} = \dfrac{\partial}{\partial \overline{x}_{12}}$. By straightforward computations, we get an almost para-Hermitian manifold. For $\overline{\nabla}$ w. r. t. $\overline{g}$, We may easily deduce that the manifold $(\overline{M}^{2m}, \mathcal{P}, \overline{g})$ is a para-Kähler manifold. Consider the immersion $\phi : (M, g) \to (\overline{M}^{2m}, \overline{g})$ defined by

$$\begin{aligned} \overline{x}_1 = u\sinh(\theta), \overline{x}_2 = v\sinh(\theta), \overline{x}_3 = u\sinh(\alpha), \overline{x}_4 = v\sinh(\alpha), \\ \overline{x}_5 = k_1, \overline{x}_6 = k_2, \overline{x}_7 = u\cosh(\theta), \overline{x}_8 = v\cosh(\theta), \\ \overline{x}_9 = u\cosh(\alpha), \overline{x}_{10} = v\cosh(\alpha), \overline{x}_{11} = \theta, \overline{x}_{12} = \alpha, \end{aligned} \tag{4.8}$$

where $u, v \in \mathbb{R} - \{0\}$ and $\alpha, \theta \in (0, \pi/2)$. Then, from direct computations for tangent and normal bundles of M and Eqs. (3.2) (4.8), we conclude that holomorphic distribution $\mathfrak{D}_T = span\{Z_u, Z_v\}$ and pointwise slant distribution $\mathfrak{D}_\lambda = span\{Z_\theta, Z_\alpha\}$ with slant function $\lambda = \frac{1}{u^2+v^2-1}$. Thus M evolve into a proper M in $\overline{M}^{2m}$. Now the induced g on M is deduced as $g = -2du^2 - 2dv^2 + (u^2 + v^2 - 1)\{d\alpha^2 + d\theta^2\} = g_{M_T} +_{f^2} g_{M_\lambda}$. Thus, M is a 4-dimensional warped product M in $\mathbb{R}^{12}$ with wraping function $f = \sqrt{u^2 + v^2 - 1}$.

Lemma 4.5. *If $M = M_T \times_f M_\lambda$ be a warped product pointwise $\mathcal{PR}$-semi-slant submanifolds of a para-Kähler manifold $\overline{M}^{2m}$, then*

(*a*) $g(A_{nZ}X, Y) = 0$,

(*b*) $A_{nZ}W = A_{nW}Z$,

(*c*) $g(A_{ntW}X, Z) = -\mathcal{P}X(\ln f)g(tW, Z) + X(\ln f)\lambda g(Z, W)$,

(*d*) $g(A_{nW}X, Z) = -\mathcal{P}X(\ln f)g(W, Z) + X(\ln f)g(Z, tW)$,

(*e*) $g(A_{nW}\mathcal{P}X, Z) = -\frac{Xf}{f}g(W, Z) + \mathcal{P}X(\ln f)g(Z, tW)$, *for all $X, Y \in \mathfrak{X}(M_T)$ and $Z, W \in \mathfrak{X}(M_\lambda)$.*

Proof. Let M be a submanifold of $\overline{M}^{2m}$ then using Eqs. (2.1), (2.4) and (2.8) we receive

$$g(A_{nZ}X, Y) = -g(\overline{\nabla}_X \mathcal{P}Y, Z) - g(\overline{\nabla}_X Y, tZ). \tag{4.9}$$

Formula-(a) can be achieved from the fact that the pair $(\mathfrak{D}_T, \mathfrak{D}_\lambda)$ is orthogonal and Eqs. (2.10), (4.1) in equation (4.9). Again from Eqs. (2.1), (2.4), (2.8) and Gauss-Weingarten formulas, we attain that

$$g(A_{nZ}X, W) = -g(\nabla_X tW, Z) + g(\overline{\nabla}_X Z, nW) - g(\nabla_X W, tZ). \tag{4.10}$$

Putting Eq. (4.1) in (4.10), we arrive at

$$g(A_{nZ}X, W) = -\frac{Xf}{f}g(tW, Z) + g(\overline{\nabla}_X Z, nW) - \frac{Xf}{f}g(W, tZ). \tag{4.11}$$

Now, using Eq. (2.10) in Eq. (4.11) we prove the Formula-(b). Further, replacing W by tW in Formula-(b) results in $A_{nZ}tW = A_{ntW}Z$, then applying Eqs. (2.5)–(2.8) as well as the fact $\overline{M}^{2m}$ is para-Kähler in previous expression, we acquire that

$$g(A_{ntW}X, Z) = -g(\nabla_{tW} tX, Z) - g(\nabla_{tW} X, tZ). \tag{4.12}$$

Formula-(c) can be achieved by using Eqs. (3.7)and (4.1) in Eq. (4.12). Next, Formula-(d) can be easily derived by replacing W by tW in Formula-(c) and using Eq. (3.6). Finally by virtue of Eq. (2.1) and replacement of X by $\mathcal{P}X$ in Formula-(d) we get Formula-(e). This proves the lemma. □

Lemma 4.6. *If $M = M_T \times_f M_\lambda$ be a warped product pointwise $\mathcal{PR}$-semi-slant submanifold of a para-Kähler manifold $\overline{M}^{2m}$, then*

(a) $(\nabla_U t)X = \mathcal{P}X(\ln f)P_\lambda U - \frac{Xf}{f}tP_\lambda U,$

(b) $(\nabla_U t)Z = \nabla(\ln f)g(tP_\lambda U, Z) - \mathcal{P}\nabla(\ln f)g(P_\lambda U, Z),$

for all $X \in \mathfrak{X}(M_T)$, $U \in \mathfrak{X}(M)$ and $Z \in \mathfrak{X}(M_\lambda)$.

Proof. Employing the projections P_T and P_λ of distributions endowed with the definition of M l .h. s. that is, left hand side of Formula-a, we acquire $(\nabla_U t)X = (\nabla_{P_T U} t)X + (\nabla_{P_\lambda U} t)X$. Now using Eq. (2.11) in previous relation, we get $(\nabla_U t)X = A_{nX}P_T U + t' h(X, P_T U) + (\nabla_{P_\lambda U} t)X$. Since M_T is totally geodesic, therefore first two terms in r. h. s. of previous equality vanishes. Using Eq. (4.1), we achieve the r. h. s. of the Formula-a. Again, by using the concept of projections and Eq. (4.1) in left hand side of Formula-2b, we obtain $(\nabla_U t)Z = P_T U(\ln f)tZ - P_T U(\ln f)tZ + (\nabla_{P_\lambda U} t)Z$. Now taking inner product with X, we have $g((\nabla_U t)Z, X) = g(\nabla_{P_\lambda U} tZ, X) + g(\nabla_{P_\lambda U} Z, tX)$. By the property of pseudo-Riemannian connection and Eq. (4.1), we acquire that $g((\nabla_U t)Z, X) = = -\frac{Xf}{f}g(tZ, P_\lambda U) - \mathcal{P}X(\ln f)g(Z, P_\lambda U)$. Finally, applying property of gradient, we get the r.h.s. of the Formula-b. Thus proves the lemma. □

Lemma 4.7. *If $M = M_T \times_f M_\lambda$ be a warped product pointwise $\mathcal{PR}$-semi-slant submanifold of a para-Kähler manifold $\overline{M}^{2m}$ then*

(a) $(\nabla_U n)X = -\frac{Xf}{f} nP_\lambda U$,

(b) $(\nabla_U n)Z = n'h(U,Z) - h(U,tZ)$,

for all $X \in \mathfrak{X}(M_T)$, $U \in \mathfrak{X}(M)$ and $Z \in \mathfrak{X}(M_\lambda)$.

Proof. Using the notion of projections for distributions, Eqs. (2.11), (2.12) as well as the fact M_T is totally geodesic, we prove the lemma. □

In the sequel, we derive some important characterizations as our main results.

Theorem 4.8. *Suppose $\overline{M}^{2m}$ is a para-Kähler manifold and let $M \to \overline{M}^{2m}$ be an immersion of a pointwise $\mathcal{PR}$-semi-slant submanifold M into $\overline{M}^{2m}$. Then a necessary and sufficient condition for M to be locally warped product of the form $M_T \times_f M_\lambda$ is that the shape operator of M satisfies*

$$A_{ntW}X + A_{nW}\mathcal{P}X = (\lambda - 1)X(\mu)W,\ X \in \mathfrak{X}(\mathfrak{D}), W \in \mathfrak{X}(\mathfrak{D}_\lambda), \tag{4.13}$$

for some function μ on M such that $Z(\mu) = 0$, $Z \in \mathfrak{X}(\mathfrak{D}_\lambda)$.

Proof. Let M be a warped product in $\overline{M}^{2m}$. Then clearly from Formulas (c) and (e) of Lemma 4.5, we obtain Eq. (4.13). Placing $\mu = \ln f$ suggests that $Z(\mu) = 0$ when f is a function on M_T. Conversely, consider that M is a pointwise $\mathcal{PR}$-semi-slant submanifold of $\overline{M}^{2m}$ with Eq. (4.13) satisfied. From Formula (a) of Lemma 4.5, we can say that $A_{nZ}X \in \mathfrak{X}(\mathfrak{D}_\lambda)$. Then by virtue of inner product with Eq. (4.13) of Y and Theorem 3.6-(ii), we deduce that the integral manifolds M_T of $\mathfrak{D}_T$ defines a totally geodesic foliation in M. Now using integrablity condition from Theorem 3.7-(i) and Eq. (2.7), we arrive at

$$g(A_{ntZ}X + A_{nZ}\mathcal{P}X, W) = g(A_{ntW}X + A_{nW}\mathcal{P}X, Z) \tag{4.14}$$

Employing Eqs. (2.1)-(2.6), (2.8), (3.6) in l. h. s. of Eq. (4.14), we obtain that

$$g(A_{ntZ}X + A_{nZ}\mathcal{P}X, W) = (\lambda - 1)g(\nabla_W Z, X). \tag{4.15}$$

Now, taking product of Eq. (4.13) with Z, we find that

$$g(A_{ntW}X + A_{nW}\mathcal{P}X, Z) = g((\lambda - 1)X(\mu)W, Z). \tag{4.16}$$

From Eqs. (4.14), (4.15) and (4.16), we arrive at $h_\lambda(Z,W) = \nabla(\mu)g(W,Z)$ where h_λ denote the second fundamental form of $\mathfrak{D}_\lambda$ in M and $\nabla\mu$ the gradient vector of function μ. Since $Z(\mu) = 0\ \forall\ Z \in \mathfrak{X}(\mathfrak{D}_\lambda)$. Therefore, form previous expression it is not hard to see that the integrable manifold M_λ of $\mathfrak{D}_\lambda$ is parallel with non-vanishing mean curvature. Hence totally umbilical in M. Then from a Theorem 1.2 of [13], we reach to the following conclusion that M is a warped product in $\overline{M}^{2m}$. Thus the theorem proof is now complete. □

Theorem 4.9. *Let $M \to \overline{M}^{2m}$ be an immersion of a pointwise $\mathcal{PR}$-semi-slant submanifold M into a para-Kähler manifold $\overline{M}^{2m}$. Then M is locally a warped product $M_T \times_f M_\lambda$ if and only if the endomorphism t of the tangent bundle on M satisfies*

$$\begin{aligned}(\nabla_U t)V =& \mathcal{P} P_T V(\mu) P_\lambda U - P_T V(\mu) t P_\lambda U + \nabla\mu g(tP_\lambda U, P_\lambda V \\ &+ \mathcal{P}\nabla\mu g(P_\lambda U, P_\lambda V),\end{aligned} \tag{4.17}$$

for some function μ on M such that $Z(\mu) = 0$, $Z \in \mathfrak{X}(\mathfrak{D}_\lambda)$ and $V \in \mathfrak{X}(M)$.

Proof. Consider a warped product M in $\overline{M}^{2m}$. Now, we get $(\nabla_U t)V = (\nabla_U t)P_T V + (\nabla_U t)P_\lambda V$ by virtue of projections P_T and P_λ on distributions $\mathfrak{D}_T$ and $\mathfrak{D}_\lambda$ respectively in M. Then, using lemma 4.6, we obtain Eq. (4.17). Since f is a function on M_T, setting $\mu = \ln f$ implies that $Z(\mu) = 0$. Conversely, assume that M is a pointwise $\mathcal{PR}$-semi-slant submanifold of $\overline{M}^{2m}$ such that Eq. (4.17) satisfied. Replace U by X and V by Y into (4.17), we have $(\nabla_X t)Y = 0$, for all $X, Y \in \mathfrak{X}(M_T)$. By taking inner product of above relation with Z and from equation (2.1), (2.8) and (2.10), we obtain that $g(h(X,Y), nZ) = 0$. This shows that the integral manifold M_T of $\mathfrak{D}_T$ is integrable and totally geodesic in M. On the other hand, interchange U and V with Z and W in Eq. (4.17), we obtain that

$$(\nabla_Z t)W = \mathcal{P}\nabla\mu g(Z,W) - \nabla\mu g(Z, tW). \tag{4.18}$$

By taking product of Eq. (4.18) with X, we recieve

$$g(h_\lambda(Z,tW), X) + g(h_\lambda(Z,W), \mathcal{P}X) = -X(\mu)g(Z,tW) - \mathcal{P}X(\mu)g(Z,W). \tag{4.19}$$

Replace W by tW and X by $\mathcal{P}X$ into (4.19), we obtain that

$$\lambda g(h_\lambda(Z,W), \mathcal{P}X) + g(h_\lambda(Z,tW), X) = -\lambda\mathcal{P}X(\mu)g(Z,W) - X(\mu)g(Z,tW). \tag{4.20}$$

In view of (4.18) and (4.20), we have

$$g(h_\lambda(Z,W), \mathcal{P}X) = -g(\mathcal{P}X, \nabla\mu)g(Z,W). \tag{4.21}$$

Again replace X by $\mathcal{P}X$ into above expression, we have

$$g(h_\lambda(Z,W), X) = -g(X, \nabla\mu)g(Z,W).$$

Now using property of gradient in above relation, we derive $h_\lambda(Z,W) = -\nabla(\mu)g(Z,W)$ where h_λ denotes the second fundamental form of $\mathfrak{D}_\lambda$ in M and $\nabla\mu$ the gradient vector of function μ. Since $Z(\mu) = 0 \ \forall \ Z \in \mathfrak{X}(\mathfrak{D}_\lambda)$. Therefore form previous expression it is not hard to see that the integrable manifold M_λ of $\mathfrak{D}_\lambda$ is parallel with non-vanishing mean curvature. Hence totally umbilical in M. Then from the Theorem 1.2 of [13], we arrive at M is a warped product in $\overline{M}^{2m}$. Thus the theorem proof is now complete. □

Theorem 4.10. *Let $M \to \overline{M}^{2m}$ be an immersion of a pointwise $\mathcal{PR}$-semi-slant submanifold M into a para-Kähler manifold $\overline{M}^{2m}$. Then M is locally a warped product $M_T \times_f M_\lambda$ if and only if the endomorphism n of a normal-bundle on M satisfies*

$$(\nabla_U n)V = -P_T V(\mu) n P_\lambda U + n' h(U, P_\lambda V) - h(U, t P_\lambda V), \tag{4.22}$$

for some function μ on M such that $Z(\mu) = 0$, $Z \in \mathfrak{X}(\mathfrak{D}_\lambda)$ and $U, V \in \mathfrak{X}(M)$.

Proof. This theorem proof can be accomplished by employing Lemma 4.7 and using the same steps as in the proof of Theorem 4.9. □

Finally, we derive the condition for the non-existence of warped product $M_T \times_f M_\lambda$ in $\overline{M}^{2m}$.

Proposition 4.11. *A warped product pointwise $\mathcal{PR}$-semi-slant submanifold of the form $M = M_T \times_f M_\lambda$ immersed into a para-Kähler manifold $\overline{M}^{2m}$ is pointwise $\mathcal{PR}$-semi-slant product if M is mixed totally geodesic $\forall\, X, Y \in \mathfrak{X}(M_T)$ and $Z, W \in \mathfrak{X}(M_\lambda)$.*

Proof. Assume that M is a warped product pointwise $\mathcal{PR}$-semi-slant submanifold and mixed totally geodesic in $\overline{M}^{2m}$. Then by the (2.5) and (2.8), we obtain $g(h(X,Z), nW) = g(\overline{\nabla}_Z X, \mathcal{P}W) - g(\overline{\nabla}_Z X, tW) = 0$. By using the fact that structure is para-Kähler and Eq. (4.1), we obtain

$$\mathcal{P}X(\ln f) g(Z,W) - \frac{Xf}{f} g(Z, tW) = 0. \tag{4.23}$$

Replace Z by tZ in above expression and applying Eq. (3.7), we have

$$\mathcal{P}X(\ln f) g(tZ,W) + \lambda \frac{Xf}{f} g(Z,W) = 0. \tag{4.24}$$

Again, replace X by $\mathcal{P}X$ in (4.23), we arrive at

$$\frac{Xf}{f} g(Z,W) - \mathcal{P}X(\ln f) g(Z, tW) = 0. \tag{4.25}$$

By the virtue of Eqs. (4.24) and (4.25), we deduce that

$$(1-\lambda)\frac{Xf}{f} g(Z,W) = 0. \tag{4.26}$$

Therefore from Eq. (4.26), we conclude that the warping function f is constant. Hence contradiction to our assumption that M is a warped product in $\overline{M}^{2m}$. Thus proved the proposition. □

Conflict of Interest: The authors declare no competing interests.
Funding: Not Applicable.
Data Availability: Not Applicable.
Ethical Conduct: The manuscript is not currently being submitted for publication elsewhere and has not been previously published.

References

[1] Afanasev, D. E. and Katanaev, M. O., *Global properties of warped solutions in general relativity with an electromagnetic field and a cosmological constant*, Phys. Rev. D., **100** (2019), 16.

[2] Alegre, P. and Carriazo, A., *Bi-slant submanifolds of para-Hermitian manifolds*, Mathematics, **7** (2019), no 7, Article ID 618.

[3] Ali, A. and Pişcoran, L-I., *Geometric classification of warped products isometrically immersed into Sasakian space forms*, Math. Nachr. **292** (2019), no 2, 234–251.

[4] Al-Solamy, F. R., Khan, K. A. and Uddin, S., *Geometry of warped product semi-slant submanifolds of nearly Kaehler manifolds*, Res. Math. **71** (2017), no 3, 783–799.

[5] Beem, J. K. and Ehrlich, P. E., *Global Lorentzian geometry*, Marcel Dekker, 1981.

[6] Bishop, R. L. and O'Neill, B., *Manifolds of negative curvature*, Trans. Amer. Math. Soc., **145** (1969), 1–49.

[7] Chen, B. Y., *Geometry of warped product $\mathcal{CR}$-submanifolds in Kaehler manifolds i, ii*, Monatsh. Math., **133** and **134**, (2001), no 3 and 2, 177–195 and 103–119.

[8] Chen, B. Y., *Pseudo-Riemannian geometry, δ-invariants and applications*, Word Scientific, 2011.

[9] Chen B. Y. and Garay, O. J., *Pointwise slant submanifolds in almost Hermitian manifolds*, Turkish J. Math., **36** (2012), 630–640.

[10] Chen, B. Y. and Munteanu, M. I., *Geometry of $\mathcal{PR}$-warped products in para-Kahler manifolds*, Taiwanese J. Mathematics, **16** (2012), no 4, 1293–1327.

[11] Dhiman, M., Kumar A. and Srivastava, S. K., *$\mathcal{PR}$-semi slant warped product submanifold of para-Kenmotsu manifolds*, Res. Math. **77** (2022), no 4, Article ID 142.

[12] Etayo, F., Fioravanti, M. and Trias, U. R., *On the submanifolds of an almost para-Hermitian manifold*, Acta. Math. Hungar., **85** (1999), no 4, 277–286.

[13] Hiepko, S., *Eine innere kennzeichnung der verzerrten produkte*, Math. Ann., **241** (1979), 209–215.

[14] Hong, S. T., *Warped products and black holes*, Nuovo Cimento Soc. Ital. Fis. B., **120** (2005), 1227–1234.

[15] Li, Y., Srivastava, S. K., Mofarreh, F., Kumar, A., and Ali, A., *Ricci soliton of $\mathcal{CR}$-warped product manifolds and their classifications*, Symmetry, **15** (2023), no 5, Article ID 976.

[16] Mofarreh, F., Srivastava, S. K., Kumar, A. and Ali, A., *Geometric inequalities of $\mathcal{PR}$-warped product submanifold in para-Kenmotsu manifold*, AIMS Mathematics, **7** (2022), no 10, 19481-19509.

[17] O'Neill, B., *Semi-Riemannian geometry with applications to relativity*, Academic Press, 1983.

[18] Sahin, B., *Non-existence of warped product semi-slant submanifolds of Kaehler manifolds*, Geometriae Dedicata, **117** (2006), no 1, 195–202.

[19] Sahin, B., *Warped product pointwise semi-slant submanifolds of Kahler manifolds*, Portugal. Math.(N.S.), **70** (2013), 251–268.

[20] Sharma, A., *Non-existence of $\mathcal{PR}$-semi slant warped product submanifold in a para-Kaehler manifold*, Kyungpook Mathematical Journal, **60** (2020), no 1, 197–210.

[21] Sharma, A., *Pointwise $\mathcal{PR}$-pseudo slant submanifolds in para-Kaehler manifolds*, Bull. Transilvania Univ. Brasov Ser. III: Math. Comput. Sci., **63** (2021), no 1, 231–240.

[22] Solomon, B., *Harmonic maps to sphere*, J. Differential geometry, **21** (1985), 151–162.

[23] Srivastava, S. K., Mofarreh, F., Kumar, A. and Ali, A., *Characterizations of $\mathcal{PR}$-pseudo-slant warped product submanifold of para-Kenmotsu manifold with slant base*, Symmetry, **14** (2022), no 5, Article ID 1001.

[24] Srivastava, S. K. and Sharma, A., *Geometry of $\mathcal{PR}$-semi-invariant warped product submanifolds in paracosymplectic manifold*, J. Geom. **108** (2017), 61–74.

[25] Srivastava, S. K. and Sharma, A., *Pointwise pseudo-slant warped product submanifolds in a Kaehler manifold*, Mediterr. J. Math. **14** (2017), Article ID 20.

[26] Uddin, S., Alghamdi, F. and Al-Solamy, F. R., *Geometry of warped product pointwise semi-slant submanifolds of locally product Riemannian manifolds*, J. Geom. Phys., **152** (2020), Article ID 2020103658.