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MAPS ON MATRICES PRESERVING IDEMPOTENCY OF A TRIADIC RELATION

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Abstract

Let $\mathbf{M}_n$ be the algebra of all $n \times n$ real or complex matrices. In this paper we give a full description of continuous maps on $\mathbf{M}_n$ such that $\Phi(A)(\Phi(B) - \Phi(C))$ is idempotent if and only if $A(B - C)$ is idempotent for all $A, B, C \in \mathbf{M}_n$.

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1 Introduction and statement of the result

Linear preserver problems deal with maps on subsets of algebras that preserve certain sets, functions, relations, etc. Compared to the linear preserver problems, a more general task would be to consider the maps being non-linear only. In many cases, there exists a gap between linear maps and non-linear maps, and it is much more difficult to deal with the non-linear ones. To fill this gap, sometimes we need stronger assumptions to reach a regular form between "wild" characters. For all these topics we refer to the interesting book [4] and the references therein as well. Throughout this paper, the following notations will be used:

- $\mathbb{F}$, field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$;
- $\mathbb{F}^n$, vectors with n real or complex components;
- I, identity matrix;
- I_r, identity matrix of size $r \times r$;
- $\mathbf{M}_n = \mathbf{M}_n(\mathbb{F})$, algebra of all $n \times n$ matrices over field $\mathbb{F}$;
- $\mathbf{I}_n = \mathbf{I}_n(\mathbb{F})$, the set of all idempotents in $\mathbf{M}_n$; i.e., $A \in \mathbf{I}_n$ iff $A^2 = A$;

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- $\overline{A}$, matrix obtained from $A \in \mathbf{M}_n$ by applying the complex-conjugation entrywise, $\overline{A} = \overline{(a_{ij})} = (\overline{a_{ij}})$;
- $\operatorname{Ker}(A)$, kernel or null space of $A \in \mathbf{M}_n$; i.e., $\operatorname{Ker}(A) = \{x \in \mathbb{F}^n \mid Ax = 0\}$;
- $\operatorname{Im}(A)$, image or range of $A \in \mathbf{M}_n$; i.e., $\{Ax \mid x \in \mathbb{F}^n\}$;
- $\dim(V)$, dimension of subspace V of $\mathbb{F}^n$;
- E_{ij}, $n \times n$ matrix with (i, j)-entry being 1 and other entries 0;
- $\sigma(A)$, set of all eigenvalues of $A \in \mathbf{M}_n$.

For every non-zero $n \times 1$ vector x and non-zero $1 \times n$ vector y^t, the xy^t is rank-1 matrix and every rank-1 matrix can be written in this way. The rank-1 matrix xy^t is idempotent if and only if $y^t x = 1$ and xy^t is nilpotent if and only if $y^t x = 0$.

We begin by discussing two familiar preserver problems which will serve as motivation for what will follow. For a given nonempty subset S of $\mathbf{M}_n$, often, we are interested in describing the form of mappings $\Phi : \mathbf{M}_n \to \mathbf{M}_n$ satisfying $A*B \in S$ iff $\Phi(A)*\Phi(B) \in S$, where $*$ is an operation such as $A-B, AB, AB-BA, AB+BA, ABA, \ldots$ and S is any of sets $\{0\}$, rank-1 matrices, rank-1 idempotents, rank-1 nilpotents, full rank matrices, $\mathbf{I}_n, \ldots$. This paper was inspired by two following theorems.

Theorem 1. *[5, Theorem 3.4] Let $n \geq 3$ and let $\Phi : \mathbf{M}_n \to \mathbf{M}_n$ be a bijective continuous map. Assume that*

$$A - \lambda B \in \mathbf{I}_n \Longleftrightarrow \Phi(A) - \lambda\Phi(B) \in \mathbf{I}_n,$$

for all $A, B \in \mathbf{M}_n, \lambda \in \mathbb{C}$. Then there exists an invertible $T \in \mathbf{M}_n$ such that Φ has the following forms

$$A \mapsto TAT^{-1} \text{ or } A \mapsto TA^tT^{-1} \qquad (A \in \mathbf{M}_n). \tag{1}$$

Peter Šemrl showed that assuming $\lambda = 1$ in Theorem 1, it is not possible to reach a regular form of the map Φ such as the form (1). Therefore, we need a slightly stronger assumption. The natural question is what other assumptions can reach us with reasonable results? Can other assumptions be mapped as substitutes for surjectivity's condition? The next theorem, which is the starting point of our research, is as follows:

Theorem 2. *[3, Theorem 1.2] Let $n \geq 3$. Then a unital surjective map $\Phi : \mathbf{M}_n \to \mathbf{M}_n$ satisfies*

$$AB \in \mathbf{I}_n \setminus \{0\} \Longleftrightarrow \Phi(A)\Phi(B) \in \mathbf{I}_n \setminus \{0\} \qquad (A, B \in \mathbf{M}_n),$$

if and only if there exist a field automorphism $f : \mathbb{C} \to \mathbb{C}$ and an invertible matrix $S \in \mathbf{M}_n$ such that $\Phi(A) = Sf(A)S^{-1}$ for all $A \in \mathbf{M}_n$, where $f(A) = (f(a_{ij}))$ if $A = (a_{ij})$.

This was a summary of what led us to the following theorem. Our main theorem reads as follows.

Theorem 3. *Let $\mathbf{M}_n$ be the algebra of all $n \times n$ real or complex matrices and $\mathbf{I}_n$ the set of all idempotents in $\mathbf{M}_n$. Let $\Phi : \mathbf{M}_n \to \mathbf{M}_n$ be a continuous map such that*

$$A(B-C) \in \mathbf{I}_n \Longleftrightarrow \Phi(A)(\Phi(B)-\Phi(C)) \in \mathbf{I}_n \qquad (A,B,C \in \mathbf{M}_n). \tag{2}$$

Then there exists an invertible matrix $T \in \mathbf{M}_n$ and a constant λ with $\lambda^2 = 1$ such that Φ has the following forms

$$A \mapsto \lambda TAT^{-1} \text{ or } A \mapsto \lambda T\overline{A}T^{-1} \qquad (A \in \mathbf{M}_n). \tag{3}$$

Let $n = 1$. Since only idempotents in $\mathbb{F}$ are 0 and 1, the Theorem 3 reads as follows.

Theorem 4. *Let $f : \mathbb{F} \to \mathbb{F}$ be a continuous map such that*

$$a(b-c) \in \{0,1\} \Longleftrightarrow f(a)(f(b)-f(c)) \in \{0,1\} \qquad (a,b,c \in \mathbb{F}). \tag{4}$$

Then there exists a scalar $\varepsilon \in \{-1,1\}$ such that $f(z) = \varepsilon z$ or $\varepsilon\bar{z}$ for all $z \in \mathbb{F}$.

Proof. For the sake of readability, we divided the proof into four steps.

Step 1. *f is injective.*

Let $f(a) = f(b)$ for some $a, b \in \mathbb{F}$. Therefore $f(1)(f(a)-f(b))$ and $f(1)(f(b)-f(a))$ both are in $\{0,1\}$. It follows that $a-b = (a-b)^2 = (b-a)^2 = b-a$ which implies $a = b$, as desired.

Step 2. $f(0) = 0$.

It is easy to check that both $a(b-c)$ and $a(c-b)$ are in $\{0,1\}$ whenever $a = 0$ or $b = c$. Both of $f(0)(f(1)-f(0))$ and $f(0)(f(0)-f(1))$ are in $\{0,1\}$ since $0(1-0) = 0(0-1) = 0$. Hence $f(0) = 0$ or $f(1) = f(0)$. But injectivity of f follows $f(0) = 0$.

Step 3. *$f(r) = \pm r$ for all $r \in \mathbb{R}$.*

Clearly, $f(1) = \pm 1$. Without loss of generality we can assume that $f(1) = 1$. This implies that for every $a, b \in \mathbb{F}$, $f(a) - f(b) = 1$ whenever $a - b = 1$. Hence by a recursive process we get $f(n) = n$ and $f(kp) = kf(p)$ for all $k, n \in \mathbb{Z}$ and $p \in \mathbb{Q}$. This implies that $m = f(n\frac{m}{n}) = nf(\frac{m}{n})$ which implies $f(\frac{m}{n}) = \frac{m}{n}$ for all $m, n \in \mathbb{Z}$ with $n \neq 0$. Now the continuity of f implies that $f(r) = r$ for all $r \in \mathbb{R}$, as desired.

Step 4. *$f(ci) = \pm ci$ for all $c \in \mathbb{R}$.*

Clearly, $f(i) = \pm i$ because $i(0-i) = 1$. Without loss of generality we can assume that $f(i) = i$. Let $p \in \mathbb{Q}$. For every $k \in \mathbb{Z}$ we have $f((k+1)pi) = f(kpi) + f(pi)$ because $(pi)^{-1}((k+1)pi - kpi) = 1$. Now by a recursive process we conclude that $f(kpi) = kf(pi)$ which gives $f(pi) = pi$ for all $p \in \mathbb{Q}$. Now the continuity of f implies the desired result.

Now if $f(r) = r$ for each $r \in \mathbb{R}$ and $f(i) = i$, then

$$\frac{1}{a}(a + ib - ib) = 1 \Longrightarrow f(a + ib) = a + ib \qquad (a, b \in \mathbb{R}).$$

The other cases can be proved similarly. This completes the proof. □

2 Proof of Theorem 3

Let $\Phi : \mathbf{M}_n \to \mathbf{M}_n$ be a continuous map satisfying (2).

Lemma 1. Φ *is injective.*

Proof. Suppose $\Phi(A) = \Phi(B)$ for some $A, B \in \mathbf{M}_n$. Then $\Phi(I)(\Phi(A) - \Phi(B))$ and $\Phi(I)(\Phi(B) - \Phi(A))$ are idempotent. It follows that $A - B = (A - B)^2 = B - A$, so $A = B$ and Φ is injective. □

Lemma 2. *For any three operators $A, B, C \in \mathbf{M}_n$ we have*

$$A(B - C) = 0 \Longleftrightarrow \Phi(A)(\Phi(B) - \Phi(C)) = 0.$$

Proof. Let $A, B, C \in \mathbf{M}_n$. Then

$$\begin{aligned} A(B - C) = 0 &\Longleftrightarrow A(B - C) \in \mathbf{I}_n \text{ and } A(C - B) \in \mathbf{I}_n \\ &\Longleftrightarrow \Phi(A)(\Phi(B) - \Phi(C)) \in \mathbf{I}_n \text{ and } \Phi(A)(\Phi(C) - \Phi(B)) \in \mathbf{I}_n \\ &\Longleftrightarrow \Phi(A)(\Phi(B) - \Phi(C)) = 0. \end{aligned}$$

□

Lemma 3. $\Phi(0)\Phi(A) = \Phi(0)\Phi(B)$ *for every $A, B \in \mathbf{M}_n$.*

Proof. For every $A, B \in \mathbf{M}_n$ both of $\Phi(0)(\Phi(A) - \Phi(B))$ and $\Phi(0)(\Phi(B) - \Phi(A))$ are idempotent. This implies that $\Phi(0)(\Phi(A) - \Phi(B)) = \Phi(0)(\Phi(B) - \Phi(A))$. This completes the proof of the lemma. □

Let $R = \Phi(0)$ and consider transformation

$$\psi : A \mapsto \Phi(A) - R.$$

Now ψ satisfies the hypothesis of the original theorem with $\psi(0) = 0$.

Corollary 1. ψ *preserve zero product in both directions; i.e.,*

$$AB = 0 \Longleftrightarrow \psi(A)\psi(B) = 0.$$

Proof. It follows from Lemma 2 and that $\psi(0) = 0$. □

It is worth to mention that Corollary 1 with $\psi(0) = 0$ follow that $AB \in \mathbf{I}_n \setminus \{0\}$ if and only if $\psi(A)\psi(B) \in \mathbf{I}_n \setminus \{0\}$. In particular, $\psi(E)^2 \in \mathbf{I}_n \setminus \{0\}$ for all $E \in \mathbf{I}_n \setminus \{0\}$.

Lemma 4. *For any idempotent E we have $\psi(E)^2 = \psi(E)\psi(I)$.*

Proof. Let $E \in \mathbf{I}_n$. We have $E(I - E) = 0$. Now Lemma 2 leads up to the conclusion. □

Lemma 5. *$\psi(I) = I$ or $\psi(I) = -I$.*

Proof. We complete the proof by two steps.

Step 1. $\sigma(\psi(I)) \subseteq \{-1, 1\}$.

We have $\psi(I)^4 = \psi(I)^2$. Hence by the spectral mapping theorem we can get $\sigma(\psi(I)) \subseteq \{-1, 0, 1\}$. It is enough to show that $\psi(I)$ is invertible. For doing this we show $\operatorname{Ker}\psi(I) = \{0\}$. Let $x \in \operatorname{Ker}\psi(I)$. By Lemma 4, $\psi(E)^2x = \psi(E)\psi(I)x = 0$ for each $E \in \mathbf{I}_n$. This implies that $x \in \operatorname{Ker}\psi(E)^2$ and so $\operatorname{Ker}\psi(I) \subseteq \operatorname{Ker}\psi(E)^2$ for every $E \in \mathbf{I}_n$. It follows that $\operatorname{Ker}\psi(I) \subseteq \bigcap_{E\in\mathbf{I}_n} \operatorname{Ker}\psi(E)^2$. Clearly,

$$EF = 0 \text{ or } FE = 0 \Longrightarrow \operatorname{Im} E \cap \operatorname{Im} F = \{0\} \qquad (E, F \in \mathbf{I}_n). \tag{5}$$

For every i, j, $1 \leq i \neq j \leq n$, we have $E_{ii}E_{jj} = 0$. By (5) and using zero product preserving property of ψ we get $\operatorname{Im}\psi(E_{ii})^2 \cap \operatorname{Im}\psi(E_{jj})^2 = \{0\}$ since $\psi(E)^2 \in \mathbf{I}_n$ for all $E \in \mathbf{I}_n$. Now with a similar discussion as [1, Lemma 2.3] we show that $\psi(E_{ii})^2$ is rank-1 matrix for all $i = 1, 2, \ldots, n$. Set $A_i = \psi(E_{ii})^2$. Clearly, $A_i^2 = A_i, A_i \neq 0$ for $i = 1, 2, \ldots, n$ and $A_iA_j = 0$ whenever $i \neq j$. It follows that $\{0\} \neq \operatorname{Im} A_i \subset \operatorname{Ker} A_j$ whenever $i \neq j$. Thus

$$\sum_{i\neq j} \operatorname{Im} A_i \subset \operatorname{Ker} A_j,$$

and since $\operatorname{Im} A_j \nsubseteq \operatorname{Ker} A_j$ we have

$$\operatorname{Im} A_j \nsubseteq \sum_{i\neq j} \operatorname{Im} A_i \qquad (j = 1, 2, \ldots, n).$$

Hence, we can easily get that $\dim \sum_{i\neq j} \operatorname{Im} A_i \geq n-1$ for every $j \in \{1, 2, \ldots, n\}$ and since $\operatorname{Ker} A_j \neq \mathbb{F}^n$ we infer that $\sum_{i\neq j} \operatorname{Im} A_i = \operatorname{Ker} A_j$ is of dimension $n - 1$. This implies that all the images of matrices A_i are one-dimensional subspaces and they are linearly independent. Therefore, $\mathbb{F}^n = \bigoplus_{i=1}^n \operatorname{Span}\{x_i\}$ for some basis $\{x_i\}$ for $\operatorname{Im}\psi(E_{ii})^2$. Now we show that $\bigcap_{i=1}^n \operatorname{Ker}\psi(E_{ii})^2 = \{0\}$. Let $x \in \mathbb{F}^n$ such that $x \in \bigcap_{i=1}^n \operatorname{Ker}\psi(E_{ii})^2$. We can find scalars $c_1, c_2, \ldots, c_n$ such that $x = \sum_{i=1}^n c_ix_i$. Accordingly, we have $\Phi(E_{jj})^2x = c_jx_j = 0$ for each $1 \leq j \leq n$ because for each i with $i \neq j, x_i \in$ Ker $\psi(E_{jj})^2$ for $1 \leq j \leq n$. It follows that $c_i = 0$ for every $i \in \{1, 2, \ldots, n\}$ and so $x = 0$. Now since $\operatorname{Ker}\psi(I) \subseteq \bigcap_{i=1}^n \operatorname{Ker}\psi(E_{ii})^2$ we get the result, as desired.

Step 2. $\sigma(\psi(I)) = \{-1\}$ *or* $\{1\}$.

Assume, with contrary, that $\sigma(\psi(I)) = \{-1, 1\}$. Then by applying a similarity transformation we can suppose that $\psi(I) = I_r \oplus (-I_{n-r})$. Let $E \in \mathbf{I}_n$. Set

$$\psi(E) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A \in \mathbf{M}_r$, $D \in \mathbf{M}_{(n-r)}$, and B is $r \times (n-r)$ and C is $(n-r) \times r$. We first show that $A^2 = A, CA = C$ and $B = D = 0$. We have

$$\begin{aligned} \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} &= \psi(I)\psi(E) \\ &= (\psi(I)\psi(E))^2 = \begin{pmatrix} A^2 - BC & AB - BD \\ -CA + DC & -CB + D^2 \end{pmatrix} \end{aligned} \tag{6}$$

and

$$\begin{aligned} \begin{pmatrix} A & -B \\ C & -D \end{pmatrix} &= \psi(E)\psi(I) \\ &= \psi(E)^2 = \begin{pmatrix} A^2 + BC & AB + BD \\ CA + DC & CB + D^2 \end{pmatrix}. \end{aligned} \tag{7}$$

With compare Eq. (6) and Eq. (7) we obtain

$$A^2 = A,\ BD = -B,\ CA = C,\ D^2 = -D \text{ and } AB = BC = CB = DC = 0.$$

On the other hand, we have

$$\psi(E) = \psi(E)\psi(I)^2 = \psi(E)^3.$$

This implies that

$$\begin{pmatrix} A & -B \\ C & -D \end{pmatrix} = \psi(E)^3 = \psi(E) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

which implies $B = D = 0$, as claimed. Moreover, by a direct calculation we obtain $\psi(E) \in \mathbf{I}_n$. Now for all i, j with $1 \le i \ne j \le n$ we have $\psi(E_{ii})\psi(E_{jj}) = 0$ because $E_{ii}E_{jj} = 0$. Therefore we can find an invertible matrix S such that $\psi(E_{ii}) = SE_{ii}S^{-1}$ for each $1 \le i \le n$, see [7, Lemma 3.1 (i)]. Hence, we have $E_{ii} = S^{-1}\psi(E_{ii})S = S^{-1}\psi(E_{ii})SS^{-1}\psi(I)S = E_{ii}S^{-1}\psi(I)S$ for all $1 \le i \le n$. This implies that $S^{-1}\psi(I)S = I$, a contradiction. This contradiction ensures that $\sigma(\psi(I)) = \{1\}$ or $\{-1\}$. □

Let $\psi(I) = -I$ and consider transformation $\Lambda : \mathbf{M}_n \to \mathbf{M}_n$ defined by $\Lambda(X) = -\psi(X)$. Then Λ satisfies the hypothesis of the original theorem with $\Lambda(I) = I$. Hence without loss of generality we may and do suppose that $\psi(I) = I$.

Corollary 2. Φ *coincides with* ψ.

Proof. It is enough to show that $\Phi(0) = 0$. We have $0(I - 0) = 0$. Hence $0 = \Phi(0)(\Phi(I) - \Phi(0)) = \Phi(0)(\psi(I)) = \Phi(0)$, as asserted. □

We will prove the theorem separately for $n = 2$, and $n \geq 3$. Let $n = 2$. Since $E_{11}E_{22} = E_{22}E_{11} = 0$ and Φ preserve zero product we can find invertible matrix T such that $\Phi(E_{11}) = TE_{11}T^{-1}$ and $\Phi(E_{22}) = TE_{22}T^{-1}$, see [7, Lemma 3.1 (i)]. Without loss of generality we can therefore assume that

$$\Phi(E_{11}) = E_{11} \quad \text{and} \quad \Phi(E_{22}) = E_{22}. \tag{8}$$

The proof is divided into the following four steps.

Step 1. *$\Phi(\alpha E_{11}) = \alpha E_{11}$ or $\overline{\alpha} E_{11}$ for all $\alpha \in \mathbb{F}$.*

Since $E_{11}E_{22} = E_{22}E_{11} = 0$ and Φ is an injective continuous map such that preserve zero product we have $\Phi(\alpha E_{11}) = f(\alpha)E_{11}$ for some injective continuous function $f : \mathbb{F} \to \mathbb{F}$. By preserving property of map Φ we have

$$\alpha(\beta - \gamma) \in \{0, 1\} \Longleftrightarrow f(\alpha)(f(\beta) - f(\gamma)) \in \{0, 1\} \qquad (\alpha, \beta, \gamma \in \mathbb{F}).$$

Now Theorem 4 with $f(1) = 1$ leads up to the conclution. In a similar way we can get the same conclusion for $\Phi(\alpha E_{22})$. Therefore for every $\alpha \in \mathbb{F}$ we have the following four cases:

(i) $\Phi(\alpha E_{11}) = \alpha E_{11}$ and $\Phi(\alpha E_{22}) = \alpha E_{22}$

(ii) $\Phi(\alpha E_{11}) = \alpha E_{11}$ and $\Phi(\alpha E_{22}) = \overline{\alpha} E_{22}$

(ii) $\Phi(\alpha E_{11}) = \overline{\alpha} E_{11}$ and $\Phi(\alpha E_{22}) = \alpha E_{22}$

(iv) $\Phi(\alpha E_{11}) = \overline{\alpha} E_{11}$ and $\Phi(\alpha E_{22}) = \overline{\alpha} E_{22}$

We consider only the first case. In all other cases one can argue in a quite similar way.

Step 2. *For every complex number $\alpha, \beta \in \mathbb{F}$ there exist injective continuous functions $f, g : \mathbb{F} \to \mathbb{F}$ such that*

$$\Phi(\alpha E_{11} + \beta E_{12}) = \alpha E_{11} + f(\beta)E_{12} \tag{9}$$
$$\Phi(\alpha E_{11} + \beta E_{21}) = \alpha E_{11} + g(\beta)E_{21} \tag{10}$$
$$\Phi(\alpha E_{22} + \beta E_{12}) = \alpha E_{22} + f(\beta)E_{12} \tag{11}$$
$$\Phi(\alpha E_{22} + \beta E_{21}) = \alpha E_{22} + g(\beta)E_{21} \tag{12}$$

First we prove Eq. (9). By Corollary 1, $E_{22}\Phi(\alpha E_{11} + \beta E_{12}) = 0$ because $E_{22}(\alpha E_{11} + \beta E_{12}) = 0$. This implies that $\Phi(\alpha E_{11} + \beta E_{12}) = \alpha' E_{11} + \beta' E_{12}$ for some scalar $\alpha', \beta' \in \mathbb{F}$. We have $\alpha^{-1}E_{11}(\alpha E_{11} + \beta E_{12}) = E_{11} + \alpha^{-1}\beta E_{12} \in \mathbf{I}_2$. This implies that $\alpha^{-1}E_{11}(\alpha' E_{11} + \beta' E_{12}) = \alpha^{-1}\alpha' E_{11} + \alpha^{-1}\beta' E_{12} \in \mathbf{I}_2$ which implies that $\alpha=\alpha'$, as desired. In a similar manner we can show that there exist injective continuous functions $g,h,k:\mathbb{F}\to\mathbb{F}$ such that

$$\Phi(\alpha E_{11}+\beta E_{21})=\alpha E_{11}+g(\beta)E_{21} \tag{13}$$
$$\Phi(\alpha E_{22}+\beta E_{12})=\alpha E_{22}+h(\beta)E_{12} \tag{14}$$
$$\Phi(\alpha E_{22}+\beta E_{21})=\alpha E_{22}+k(\beta)E_{21}. \tag{15}$$

Set $\alpha=0$ in formulas (9), (13), (14) and (15). Injectivity of Φ implies that $f=h$ and $g=k$, as claimed.

Step 3. *Let* $A=\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$. *Then* $\Phi(A)=\begin{pmatrix} a_{11} & f(a_{12})\\ g(a_{21}) & a_{22}\end{pmatrix}$ *for some injective continuous functions* $f,g:\mathbb{F}\to\mathbb{F}$ *with* $f(0)=g(0)=0$.

Let $\Phi(A)=\begin{pmatrix} a'_{11} & a'_{12}\\ a'_{21} & a'_{22}\end{pmatrix}$. We have

$$E_{11}\left(\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}-\begin{pmatrix} a_{11} & a_{12}\\ 0 & 0\end{pmatrix}\right)=0$$

and

$$E_{22}\left(\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}-\begin{pmatrix} 0 & 0\\ a_{21} & a_{22}\end{pmatrix}\right)=0.$$

Consequently, by Corollary 1 and step 2 we get

$$E_{11}\left(\begin{pmatrix} a'_{11} & a'_{12}\\ a'_{21} & a'_{22}\end{pmatrix}-\begin{pmatrix} a_{11} & f(a_{12})\\ 0 & 0\end{pmatrix}\right)=0$$

and

$$E_{22}\left(\begin{pmatrix} a'_{11} & a'_{12}\\ a'_{21} & a'_{22}\end{pmatrix}-\begin{pmatrix} 0 & 0\\ g(a_{21}) & a_{22}\end{pmatrix}\right)=0.$$

These imply that $a'_{11}=a_{11}, a'_{12}=f(a_{12}), a'_{21}=g(a_{21}), a'_{22}=a_{22}$, as desired.

Step 4. *There exists a non-zero constant* c *such that* $f(a)=ca$ *and* $g(a)=c^{-1}a$ *for all* $a\in\mathbb{F}$.

Every non-scalar 2×2 idempotent matrix has trace 1 and determinant 0, which can be written as $D=\begin{pmatrix} d & d_{12}\\ d_{21} & 1-d\end{pmatrix}$ where $d_{12}d_{21}=d(1-d)$. Let $a,b\in\mathbb{F}$ be non-zero scalars and set $E=\begin{pmatrix} e & a\\ b & 1-e\end{pmatrix}\in\mathbf{I}_2$. Then by step 3, $\Phi(E)=\begin{pmatrix} e & f(a)\\ g(b) & 1-e\end{pmatrix}$. Since Φ preserve idempotents we have $f(a)g(b)=e(1-e)=ab$. We have $f(1)g(1)=1$. With $a=1$ and $b=1$ we get $g(b)=(f(1))^{-1}b$ and $f(a)=f(1)a$, respectively. Put $c=f(1)$. This completes the proof.

Now if $\Phi(\alpha E_{11})=\alpha E_{11}$ and $\Phi(\alpha E_{22})=\overline{\alpha}E_{22}$ for all $\alpha\in\mathbb{C}$ then with a similar argument as before we can show that $\Phi(A)=\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & \overline{a_{22}}\end{pmatrix}$ for all $A=$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbf{M}_2$. But this leads to a contradiction because suppose $E = \begin{pmatrix} e & a \\ b & 1-e \end{pmatrix} \in \mathbf{I}_2$ for some non-zero purely imaginary number $e \in \mathbb{C}$. Then $\Phi(E) = \begin{pmatrix} e & a \\ b & \overline{1-e} \end{pmatrix} \in \mathbf{I}_2$ which implies $e + 1 - \overline{e} = 1$, a contradiction. By a similar discussion the third case can not happen. Now set $S = \begin{pmatrix} 1 & 0 \\ 0 & c^{-1} \end{pmatrix}$. Then $\Phi(A) = SAS^{-1}$. Again by applying a similarity transformation we get $\Phi(A) = A$. This completes the proof of Theorem 3 for $n = 2$.

It remains to prove the theorem for $n \geq 3$. Clearly, Φ preserve rank-1 idempotents. Hence, by [6, Theorem 1.2] there exists an invertible matrix $T \in \mathbf{M}_n$ and a non-zero endomorphism $h : \mathbb{F} \to \mathbb{F}$ such that

$$\Phi(P) = Th(P)T^{-1},$$

for all rank-1 idempotent $P \in \mathbf{I}_n$, where $h(P) = (h(p_{ij}))$ if $P = (p_{ij})$. Continuity of Φ implies that h is a non-zero continuous endomorphism on $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. It is not difficult to show that continuous endomorphism on the field of real numbers is identity and on the field of complex numbers is identity or conjugate identity; i.e., $h(z) = z$, for all $z \in \mathbb{C}$, or $h(z) = \bar{z}$ for all $z \in \mathbb{C}$, for more details see [2]. Therefore for real case we have $\Phi(P) = TPT^{-1}$ and for complex case we have $\Phi(P) = TPT^{-1}$ or $\Phi(P) = T\overline{P}T^{-1}$. Since the transformations $A \mapsto TAT^{-1}$ for all invertible matrix T and $A \mapsto \overline{A}$ satisfy the hypothesis of the original theorem so, without loss of generality we can assume that $\Phi(P) = P$ for all rank-1 idempotent $P \in \mathbf{I}_n$. In order to complete the proof of Theorem 3 it is enough to show that $\Phi(A) = A$ for every $A \in \mathbf{M}_n$ and we do this by completing the following steps:

Step 1. $\Phi(P) = P$ *for all* $P \in \mathbf{I}_n$.

By [3, proposition 2.3], for each $P \in \mathbf{I}_n$ we have $\Phi(P) = \lambda I + (1-\lambda)P$ for some $\lambda \in \mathbb{F} \setminus \{1\}$. Since $\Phi(P)^2 = \Phi(P)$. We get $\lambda = 0$, as desired.

Step 2. $\Phi(N) = N$ *for all* rank-1 *nilpotent* $N \in \mathbf{M}_n$.

By [3, proposition 2.3], for each rank-1 nilpotent $N \in \mathbf{M}_n$ we have $\Phi(N) = \lambda I + (1-\lambda)N$ for some $\lambda \in \mathbb{F} \setminus \{1\}$. Since $N^2 = 0$, so $\Phi(N)^2 = 0$. This yields that $\lambda = 0$, as desired.

Step 3. $\Phi(A) = A$ *for all* $A \notin \mathbb{F}I$.

Since $A \notin \mathbb{F}I$ there exists an x such that x and Ax are linear independent. We can find y^t such that $y^t x = 0$ and $y^t Ax = 1$. On the other hand by [3, Proposition 2.3] $\Phi(A) = \lambda I + (1-\lambda)A$ for some $\lambda \in \mathbb{F} \setminus \{1\}$. Since $Axy^t \in \mathbf{I}_n$ and $\Phi(xy^t) = xy^t$ we have $\Phi(A)xy^t \in \mathbf{I}_n$. Therefore $1 = y^t\Phi(A)x = \lambda y^t x + (1-\lambda)y^t Ax = 1 - \lambda$. It follows that $\lambda = 0$ and the proof is completed.

Step 4. $\Phi(A) = A$ *for all* $A \in \mathbb{F}I$.

We first show that there exists a continuous injective function f on $\mathbb{F}$ with $f(0) = 0$ and $f(1) = 1$ such that $\Phi(\lambda I) = f(\lambda)I$. Clearly, this follows from continuity and injectivity of Φ, the identity of the map Φ on $\mathbf{I}_n$ and easy fact that $A \in \mathbb{F}^*I$ if and only if $AP \notin \mathbf{I}_n \setminus \{0\}$ for every $P \in \mathbf{I}_n \setminus \{0\}$, where $\mathbb{F}^* = \mathbb{F} \setminus \{0, 1\}$. Now by preserving property of map Φ we have

$$\alpha(\beta - \gamma) \in \{0, 1\} \Longleftrightarrow f(\alpha)(f(\beta) - f(\gamma)) \in \{0, 1\}, \qquad (\alpha, \beta, \gamma \in \mathbb{F}).$$

Consequently, by Theorem 4 there exists a scalar $\epsilon \in \{-1, 1\}$ such that $f(\lambda) = \epsilon\lambda$ or $\epsilon\overline{\lambda}$ for all $\lambda \in \mathbb{F}$. The cases $f : \lambda \mapsto \overline{\lambda}, -\overline{\lambda}$, or $-\lambda$ can not be happen, because for example if $f(\lambda) = \overline{\lambda}$ then $iE_{11}(-iI) = E_{11} \in \mathbf{I}_n$ but $iE_{11}(iI) = -E_{11} \notin \mathbf{I}_n$, a contradiction. It is easy to find counterexamples for the other cases. These contradictions complete the proof.

The last step completes the proof of the Theorem 3.

3 Concluding remark

It would be expecting to get the same result for an infinite dimensional case. Although this expectation turns out to be false as we show with the following example. Let H be an infinite dimensional Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. It is known that H is isomorphic with $H \oplus H$. We define $\Phi : B(H) \to B(H)$ by

$$\Phi : T \mapsto \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}.$$

Then Φ is injective, non-surjective, continuous and satisfiable in the original assumption of our theorem but does not have a standard form as we expected. Therefore for extend Theorem 3 to the infinite-dimensional case we need a stronger assumption. We leave it for an interested reader.

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