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BOUNDS OF CURVATURES FOR SUBMANIFOLDS OF PRODUCT SPACES

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Abstract

The paper is devoted to study of bounds for the normalized scalar curvature and the generalized normalized δ-Casorati curvatures for submanifolds of product spaces.

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Key words: Casorati curvature, DDVV conjucture, locally product spaces, scalar curvature, Wintgen inequality.

1 Introduction

The theory of product manifolds and their submanifolds is of fundamental importance in geometry due to its various geometric and physical aspects. The notion of locally product manifolds was introduced by Tachibana[26] in 1960. Adati[1] studied invariant, anti-invariant and non-invariant submanifolds of a locally product manifold. Bejancu[4] investigated semi-invariant submanifolds of locally product manifolds. Şahin[24] studied slant and semi slant submanifolds of locally product manifolds. Kiliç et al.[13] established the Chen-Ricci inequalities for different submanifolds of locally product manifolds. On the other hand, Yano and Kon[33] studied the submanifolds of Kaehlerian product manifolds. Shahid[25] studied CR-submanifolds of Kaehlerian product manifolds.

In 1890, the notion of Casorati curvature, an extrinsic invariant was introduced by Casorati[5] for surfaces in Euclidean spaces. In general, the Casorati curvature C of a submanifold in a Riemannian manifold is defined to be normalized square of

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the second fundamental form [6]. In particular, for hypersurfaces of a Riemannian manifold M^{n+1}, the Casorati curvature is given by

$$C = \frac{1}{n}(k_1^2 + k_2^2 + ... + k_n^2)$$

where $k_1^2, ..., k_n^2$ denote the principal curvatures of the hypersurface. It extends the concept of the principal direction of a hypersurface of a Riemannian manifold [11]. This notion gives a better intuition of curvature compared to the Gaussian curvature. Gaussian curvature may vanish for surfaces that look intuitively curved, while the Casorati curvature vanishes only at planar points. The Casorati curvature has been extended to arbitrary submanifolds in Riemannian geometry. In the last decade geometers rigorously worked in this direction to obtain some optimal inequalities for submanifolds of different ambient spaces [15, 16, 17, 27, 28, 31].

Wintgen[32] obtained an inequality

$$\mathcal{K} \leq \|H\|^2 - |\mathcal{K}^{\perp}|, \tag{1}$$

where $\mathcal{K}$, $\|H\|^2$ and $\mathcal{K}^{\perp}$ is Gauss curavture, squared mean curvature and normal curvature respectively of any surface $\mathcal{M}^2$ in an Euclidean Space E^4. The equality holds if and only if the ellipse of the curvature of $\mathcal{M}^2$ in E^4 is a circle.The above inequality is called Wintgen inequality.

De Smet et.al.[8] developed the generalized Wintgen inequality and named as DDVV conjecture for the submanifolds in real space forms as follows:

Conjucture 1. *Let $f : \mathcal{M}^n \to \bar{\mathcal{M}}^m(c)$ be an isometric immersion of n-dimensional submanifolds of a real space form $\overline{\mathcal{M}}^m(c)$ of constant sectional curvature c, then*

$$\rho \leq \|H\|^2 - \rho^{\perp} + c$$

where ρ and $\rho^{\perp}$ are the normalized scalar curvature and the normalized normal scalar curvature respectively.

The normalized scalar curvature is defined by

$$\rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j), \tag{2}$$

where τ is scalar curvature.

The normalized normal scalar curvature are defined as

$$\rho^{\perp} = \frac{2\tau^{\perp}}{n(n-1)} = \frac{2}{n(n-1)} \sqrt{\sum_{1 \leq i < j \leq n} \sum_{1 \leq \alpha < \beta \leq m} (R^{\perp}(e_i, e_j, \xi_\alpha, \xi_\beta))}, \tag{3}$$

where $\tau^{\perp}$ is normal scalar curvature.

Lu[18] and Ge and Tang[9] finally settled the general case of DDVV conjecture independently. The Wintgen inequality holds good for every submanifold $\mathcal{M}^n$ in any real space form $\mathcal{M}^{n+m}$ $, n \geq 2, m \geq 2$. The geometers prove the conjecture for different submanifolds of different ambient spaces [2, 3, 18, 19, 20, 21, 22].

In this paper, we derive inequalities in terms of Casorati curvatures and Wintgen inequality for submanifolds of product spaces. In Section 2, we give a brief introduction about product spaces and their submanifolds . In Section 3, we obtain some inequalities for δ-Casorati curvatures for submanifolds of product spaces and in the last section we obtain Wintgen inequalities for product spaces.

2 Preliminaries

Let $(\mathcal{M}^n, G)$ be an n-dimensional Riemannnian manifold isometrically immersed into product space $\overline{\mathcal{M}}^m(c) = (\overline{\mathcal{M}}^{m_1}(c_1) \times \overline{\mathcal{M}}^{m_2}(c_2), \overline{G})$, where $\overline{\mathcal{M}}^{m_1}(c_1)$ and $\overline{\mathcal{M}}^{m_2}(c_2)$ are m_1-dimensional and m_2-dimensional space forms of curvatures c_1 and c_2 respectively. We define a non-trivial tensor field F of type $(1,1)$ a product structure of $\overline{\mathcal{M}}^m(c)$ such that $F^2 = I$ and $F \neq \pm I$, where I is the identity transformation and obviously F satifies

$$\overline{G}(FX, FY) = \overline{G}(X, Y), \qquad \overline{\nabla} F = 0, \tag{4}$$

for all vector fields X and Y on $\overline{\mathcal{M}}^m(c)$.

The curvature tensor $\overline{R}$ of $\overline{\mathcal{M}}^m(c)$ is expressed as[13]

$$\begin{aligned}
\overline{R}(X,Y,Z,W) &= a\Big\{\overline{G}(X,W)\overline{G}(Y,Z) - \overline{G}(X,Z)\overline{G}(Y,W) \\
&\quad + \overline{G}(X,FW)\overline{G}(Y,FZ) - \overline{G}(X,FZ)\overline{G}(Y,FW)\Big\} \\
&\quad + b\Big\{\overline{G}(X,FW)\overline{G}(Y,Z) - \overline{G}(X,FZ)\overline{G}(Y,W) \qquad (5) \\
&\quad + \overline{G}(X,W)\overline{G}(Y,FZ) - \overline{G}(X,Z)\overline{G}(Y,FW)\Big\},
\end{aligned} \tag{6}$$

where $a = \frac{c_1+c_2}{2}$ and $b = \frac{c_1-c_2}{2}$. For any vector field X tangent to $\mathcal{M}$, we can write

$$FX = fX + tX, \tag{7}$$

where fX and tX represents the tangential and normal parts of FX respectively.

From (4) and (7), we can easily see that

$$G(fX, Y) = G(X, fY), \tag{8}$$

for all vector fields in $\mathcal{M}$. The squared norm of f at any point $p \in \mathcal{M}$ is given by

$$\|f\|^2 = \sum_{i,j=1}^{n} G(fE_i, E_j)^2, \tag{9}$$

where $\{E_1, E_2, \ldots, E_n\}$ be an orthonormal basis of the tangent space $T_p\mathcal{M}$.

Let $(\mathcal{M}^n, G)$ be a submanifold of a Riemannian manifold $(\overline{\mathcal{M}}^m, \overline{G})$. The Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y), \tag{10}$$

$$\overline{\nabla}_X V = -A_V X + D_X Y, \tag{11}$$

for all X, Y tangent to $\mathcal{M}^n$ and vector field V normal to $\mathcal{M}^n$, where $\overline{\nabla}$, ∇ and D be the Riemannian connection, the induced Riemannian conection and induced normal conection in $\overline{\mathcal{M}}^m(c)$, $\mathcal{M}^n$ and $T^{\perp}\mathcal{M}^n$ respectively. The second fundamental form B and the shape operator A_V are related by

$$G(B(X, Y), V) = G(A_V X, Y). \tag{12}$$

Let $p \in \mathcal{M}^n$ and $\{E_1, ..., E_n\}$ be an orthonormal basis of the tangent space $T_p\mathcal{M}^n$ and $\{E_{n+1}, ..., E_m\}$ be an orthonormal basis of $T^{\perp}\mathcal{M}^n$. The mean curvature vector, denoted by $H(p)$, is defined by

$$H(p) = \frac{1}{n} \sum_{i=1}^{n} B(E_i, E_i). \tag{13}$$

Also, we set

$$B_{ij}^r = G(B(E_i, E_j), E_r), \qquad i, j \in \{1, ..., n\}, \;\; r \in \{n+1, ..., m\}$$

and

$$\|B\|^2 = \sum_{i,j=1}^{n} (B(E_i, E_j), B(E_i, E_j)). \tag{14}$$

For $(\mathcal{M}^n, G)$ submanifold of a Riemannian manifold $(\overline{\mathcal{M}}^m, \overline{G})$. we denote by $K(\pi)$ the sectional curvature of $\mathcal{M}^n$ associated with a plane section $\pi \subset T_p\mathcal{M}^n, p \in \mathcal{M}^n$. For an orthonormal basis $\{E_1, E_2, ..., E_n\}$ of the tangent space $T_p\mathcal{M}^n$, the scalar curvature ρ is defined by

$$\rho = \sum_{i<j} K_{ij},$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by E_i and E_j.

Let R be the curvature tensor of $\mathcal{M}^n$, then the Gauss and Ricci equations are

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) + G(B(X, W), B(Y, Z)) - G(B(X, Z), B(Y, W)),$$

$$\overline{R}^{\perp}(X, Y, \eta, \zeta) = c[G(JX, \eta)G(JY, \zeta) - G(JX, \zeta)G(JY, \eta)] - G([A_\eta, A_\zeta]X, Y),$$

for any vector fields X, Y, Z, W tangent to $\mathcal{M}^n$ and η, ζ normal to $\mathcal{M}^n$ respectively.

The norm of the squared mean curvature of the submanifold is defined by

$$\|H\|^2 = \frac{1}{n^2} \sum_{\gamma=n+1}^{m} (\sum_{i=1}^{n} B_{ii}^{\gamma})^2,$$

and the squared norm of second fundamental form h is denoted by $\mathcal{C}$ defined as

$$\mathcal{C} = \frac{1}{n} \sum_{\gamma=n+1}^{m} \sum_{i,j=1}^{n} (B_{ij}^{\gamma})^2,$$

known as Casorati curvature of the submanifold.

If we suppose that Γ is an s-dimensional subspace of $T\mathcal{M}$ with $s \geq 2$, and $\{E_1, E_2, \ldots, E_s\}$ is an orthonormal basis of Γ, then the scalar curvature of the s-plane section Γ is given as

$$\tau(\Gamma) = \sum_{1 \leq \gamma < \beta \leq s} K(E_\gamma \wedge E_\beta),$$

and the Casorati curvature $\mathcal{C}$ of the subspace Γ is as follows

$$\mathcal{C}(\Gamma) = \frac{1}{s} \sum_{\gamma=n+1}^{m} \sum_{i,j=1}^{s} (B_{ij}^{\gamma})^2.$$

The normalized δ-Casorati curvature $\delta_{\mathcal{C}}(n-1)$ and $\widetilde{\delta}_{\mathcal{C}}(n-1)$ are defined as

$$[\delta_{\mathcal{C}}(n-1)]_p = \frac{1}{2}\mathcal{C}_p + \frac{n+1}{2n} inf\{\mathcal{C}(\Gamma)|\Gamma : \text{a hyperplane of } T_p\mathcal{M}\} \quad (15)$$

and

$$[\widetilde{\delta}_{\mathcal{C}}(n-1)]_p = 2\mathcal{C}_p + \frac{2n-1}{2n} sup\{\mathcal{C}(\Gamma)|\Gamma : \text{a hyperplane of } T_p\mathcal{M}\}. \quad (16)$$

For a positive real number $t \neq n(n-1)$, the generalized normalized δ-Casorati curvatures $\delta_{\mathcal{C}}(t; n-1)$ and $\widetilde{\delta}_{\mathcal{C}}(t; n-1)$ are given as

$$[\delta_{\mathcal{C}}(t; n-1)]_p = t\mathcal{C}_p + \frac{(n-1)(n+t)(n^2-n-t)}{nt} inf\{\mathcal{C}(\Gamma)|\Gamma : \text{a hyperplane of } T_p\mathcal{M}\}$$

if $0 < t < n^2 - n$, and

$$[\widetilde{\delta}_{\mathcal{C}}(t; n-1)]_p = r\mathcal{C}_p + \frac{(n-1)(n+t)(n^2-n-t)}{nt} sup\{\mathcal{C}(\Gamma)|\Gamma : \text{a hyperplane of } T_p\mathcal{M}\},$$

if $t > n^2 - n$.

Oprea[23] gives new direction to prove the Chen inequalities using optimization techniques. For a submanifold $(\mathcal{M}, G)$ of a Riemannian manifold $(\overline{\mathcal{M}}, G)$ and $h : \mathcal{M} \to \mathbf{R}$ be a differentiable function. If we have a constrained problem

$$min_{x \in M} h(x) \quad (17)$$

then the following result holds.

Lemma 1. *[23] Let $x_\circ \in \mathcal{M}$ is the solution of the problem (17), then*
(i) $(grad\ \ h)(x_\circ) \in T^{\perp}_{x_\circ}\mathcal{M}$
(ii) the bilinear form
$\mathcal{B} : T_{x_\circ}\mathcal{M} \times T_{x_\circ}\mathcal{M} \to \mathbf{R}$
$\mathcal{B}(X,Y) = Hess_h(X,Y) + G(B(X,Y),(grad\ \ h)(x_\circ))$
is positive semi-definite, where B is the second fundamental form of $\mathcal{M}$ in $\overline{\mathcal{M}}$ and grad h is the gradient of h.

Example 1. *Consider a submanifold M in E^9 given by*

$$M = \{(t,-t,0,t,-t,\cos u\cos v\cos w,\cos u\cos v\sin w,\cos u\sin v,\sin u\}$$

for any $t \in R$ and $u,v,w \in [0,\pi/2)$. Let F be an almost product structure on E^9 defined by

$$FX = (x^2,x^1,x^3,x^5,x^4,x^6,x^7,x^8,x^9)$$

where $X = (x^1,x^2,x^3,x^4,x^5,x^6,x^7,x^8,x^9)$. Then we have

$$PX = \frac{1}{2}(x^1+x^2,x^1+x^2,2x^3,x^4+x^5,x^4+x^5,2x^6,2x^7,2x^8,2x^9)$$

and

$$QX = \frac{1}{2}(x^1-x^2,x^1-x^2,0,x^4-x^5,x^5-x^4,0,0,0,0)$$

which shows that M is a locally product of the unit 3-sphere given by the spherical coordinates in E^9 given as

$$(\cos u\cos v\cos w,\cos u\cos v\sin w,\cos u\sin v,\sin u,0,0,0,0,0)$$

for $u \in [0,\pi/2)$ and the other coordinates in $[0,\pi/2]$ and a plane section M_1 in E^9 given by

$$M_1 = \{(t,-t,0,t,t,0,0,0,0 : t \in R)\}.$$

Thus M is an almost constant curvature manifold with $a = b = \frac{1}{4}$.

3 Inequalities for generalized normalized δ-Casorati curvatures

Theorem 2. *Let $\mathcal{M}^n$ be a n-dimensional submanifold of a manifold $\overline{\mathcal{M}}^m(c)$. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t;n-1)$ satisfies

$$\begin{aligned}\rho \;\leq\; & \frac{[\delta_{\mathcal{C}}(t;n-1)]_p}{n(n-1)} + a\{1+\frac{(trf)^2}{n(n-1)} - \frac{1}{n(n-1)}\|f\|^2\} \\ & + \frac{2b}{n}\{trf\}, \end{aligned} \tag{18}$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta}_c(t;n-1)$ satisfies

$$\rho \leq \frac{[\widetilde{\delta}_{\mathcal{C}}(n-1)]_p}{n(n-1)} + a\big\{1 + \frac{(trf)^2}{n(n-1)} - \frac{1}{n(n-1)}\|f\|^2\big\} + \frac{2b}{n}\big\{trf\big\}, \tag{19}$$

for any real number $t > n(n-1)$. Moreover , the equality holds in (18) and (19) iff $\mathcal{M}$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{\mathcal{M}}$, such that with respect to suitable tangent orthonormal frame $\{E_1, \ldots, E_n\}$ and normal orthonormal frame $\{E_{n+1}, \ldots, E_m\}$, the shape operator $A_r \equiv A_{E_\gamma}$, $\gamma \in \{n+1, \ldots, m\}$, take the following form

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 & 0 \\ 0 & a & 0 & \ldots & 0 & 0 \\ 0 & 0 & a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{n(n-1)}{t}a \end{pmatrix}, \tag{20}$$

$$A_{n+2} = \cdots = A_m = 0.$$

Proof. Let $\{E_1, E_2, \ldots, E_n\}$ and $\{E_{n+1}, E_{n+2}, \ldots, E_m\}$ be an orthonormal bases of $T_p\mathcal{M}$ and $T_p^{\perp}\mathcal{M}$ respectively at a point $p \in \mathcal{M}$. Using (5), we have

$$2\tau = a\big\{n(n-1) + (trf)^2 - \|f\|^2\big\} + 2b\big\{(n-1)trf\big\} + n^2\|H\|^2 - n\mathcal{C}. \tag{21}$$

Consider a polynomial $\mathcal{Q}$ in the components of second fundamental form B defined as

$$\mathcal{Q} = t\mathcal{C} + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\mathcal{C}(L) - 2\tau(p) + a\big\{n(n-1) + (trf)^2 - \|f\|^2\big\} + 2b\big\{(n-1)trf\big\},$$

where Γ is hyperplane of tangent space at a point p. We assume that Γ is spanned by $\{E_1, E_2, \ldots, E_{n-1}\}$ and $\mathcal{Q}$ has an expression of the form

$$\mathcal{Q} = \frac{t}{n}\sum_{\gamma=n+1}^{m}\sum_{i,j=1}^{n}(B_{ij}^{\gamma})^2 + \frac{(n+t)(n^2-n-t)}{nt}\sum_{\gamma=n+1}^{m}\sum_{i,j=1}^{n-1}(B_{ij}^{\gamma})^2 \tag{22}$$

$$-2\tau(p) + a\big\{n(n-1) + (trf)^2 - \|f\|^2\big\} + 2b\big\{(n-1)trf\big\}. \tag{23}$$

From (21) and (22), we arrive at

$$\begin{aligned} \mathcal{Q} &= \sum_{\gamma=n+1}^{m} \sum_{i=1}^{n-1} \Big[\big(\frac{(n^2+nr-n-2t)}{t}\big)(B_{ii}^{\gamma})^2 + \frac{2(n+t)}{n}(B_{in}^{\gamma})^2 \Big] \\ &\quad + \sum_{\gamma=n+1}^{m} \Big[2\Big(\frac{2(n+t)(n-1)}{t}\Big) \sum_{(i<j)=1}^{n} (B_{ij}^{\gamma})^2 - 2 \sum_{(i<j)=1}^{n} B_{ii}^{\gamma} B_{jj}^{\gamma} + \frac{t}{n}(B_{nn}^{\gamma})^2 \Big] \\ &\geq \sum_{\gamma=n+1}^{m} \sum_{i=1}^{n-1} \Big[\Big(\frac{(n^2+n(r-1)-2t)}{t}\Big)(B_{ii}^{\gamma})^2 - 2 \sum_{(i<j)=1}^{n} B_{ii}^{\gamma} B_{jj}^{\gamma} + \frac{t}{n}(B_{nn}^{\gamma})^2 \Big]. \end{aligned} \tag{24}$$

For $t = n+1, \ldots, m$, suppose we have a quadratic form $h_\gamma : \mathbf{R}^n \to \mathbf{R}$ defined as

$$h_\gamma(B_{11}^{\gamma}, \ldots, B_{nn}^{\gamma}) = \sum_{i=1}^{n-1} \frac{n^2+n(r-1)-2r}{r}(B_{ii}^{\gamma})^2 - 2 \sum_{(i<j)=1}^{n} B_{ii}^{\gamma} B_{jj}^{\gamma} + \frac{t}{n}(B_{nn}^{\gamma})^2$$

and the optimization problem

$$\begin{aligned} &\min \quad h_\gamma \\ &\text{subject to} \quad G : B_{11}^{\gamma} + \cdots + B_{nn}^{\gamma} = c^{\gamma}, \end{aligned}$$

where c is a real constant. The partial derivatives of g_γ are

$$\begin{cases} \frac{\partial h_\gamma}{\partial B_{ii}^{\gamma}} = \frac{2(n+t)(n-1)}{t} B_{ii}^{\gamma} - 2\sum_{l=1}^{n} B_{ll}^{\gamma}, \\ \frac{\partial h_\gamma}{\partial B_{nn}^{\gamma}} = \frac{2t}{n} B_{nn}^{\gamma} - 2\sum_{l=1}^{n-1} B_{ll}^{\gamma}, \end{cases} \tag{25}$$

where $i = \{1, 2, \ldots, n-1\}, i \neq j$, and $\gamma \in \{n+1, \ldots, m\}$.

The vector $gradh_\gamma$ is normal at G for the optimal $(B_{11}^{\gamma}, \ldots, B_{nn}^{\gamma})$ of the problem. Thus, it is collinear with the vector $(1, 1, \ldots, 1)$. Using (25), the critical point of the corresponding problem has the form

$$\begin{cases} B_{ii}^{\gamma} = \frac{t}{n(n-1)} B_{ii}^{\gamma} v^{\gamma}, i \in \{1, \ldots, n-1\} \\ B_{ii}^{\gamma} = v^{\gamma}. \end{cases} \tag{26}$$

By use of (26) and $\sum_{i=1}^{\gamma} B_{ii}^{\gamma} = c^{\gamma}$, we arrive at

$$\begin{cases} B_{ii}^{\gamma} = \frac{t}{(n+t)(n-1)} c^{\gamma}, i \in \{1, \ldots, n-1\} \\ B_{ii}^{\gamma} = \frac{n}{(n+t)} c^{\gamma}. \end{cases} \tag{27}$$

For an arbitrary fixed point $p \in D$, the 2-form $\mathcal{B} : T_pD \times T_pD \to$ has the following form

$$\mathcal{B}(X, Y) = Hess(h_\gamma(X, Y)) + \langle B(X, Y), (grad(h))(x_\circ) \rangle \tag{28}$$

where B and $\langle , \rangle$ are the second fundamental form of D in $\mathbf{R}^n$ and standard inner product on $\mathbf{R}^n$ respectively. The Hessian matrix of g_γ is of the form

$$Hess(h_\gamma) = \begin{pmatrix} 2\frac{(n+t)(n-1)}{t} - 2 & -2 & \dots & -2 & -2 \\ -2 & 2\frac{(n+t)(n-1)}{t} - 2 & \dots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \dots & 2\frac{(n+t)(n-1)}{t} - 2 & -2 \\ -2 & -2 & \dots & -2 & \frac{2t}{n} \end{pmatrix}.$$

Though F is totally geodesic in $\mathbf{R}^n$, take a tangent vector $X = (X_1, \dots, X_n)$ at any arbitrary point p on D, verifying the relation $\sum_{i=1}^{n} X_i = 0$, we have the following

$$\begin{aligned} \mathcal{B}(X, X) &= \frac{2(n^2 - n + tn - 2t)}{t} \sum_{i=1}^{n-1} X_i^2 + \frac{2t}{n} X_n^2 - 2\Big(\sum_{i=1}^{n} X_i\Big)^2 \qquad (29) \\ &= \frac{2(n^2 - n + tn - 2t)}{t} \sum_{i=1}^{n-1} X_i^2 + \frac{2t}{n} X_n^2 \\ &\geq 0. \end{aligned}$$

Hence the point $(B^\gamma_{11}, \dots, B^\gamma_{nn})$ is the global minimum point by Lemma 1 and $h_\gamma(B^\gamma_{11}, \dots, B^\gamma_{nn}) = 0$. Thus, we have $\mathcal{Q} \geq 0$ and hence

$$\begin{aligned} 2\tau \leq\ & t\mathcal{C} + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\mathcal{C}(L) + a\big\{n(n-1) + (trf)^2 - \|f\|^2\big\} \\ & +2b\big\{(n-1)trf\big\}, \end{aligned}$$

whereby, we obtain

$$\begin{aligned} \rho \leq\ & \frac{t}{n(n-1)}\mathcal{C} + \frac{(n+t)(n^2-n-t)}{n^2 t}\mathcal{C}(L) + a\big\{1 + \frac{(trf)^2}{n(n-1)} - \frac{1}{n(n-1)}\|f\|^2\big\} \\ & + \frac{2b}{n}\{trf\}, \end{aligned}$$

for every tangent hyperplane Γ of $\mathcal{M}$. If we take the infimum over all tangent hyperplanes L, the result trivially follows. Moreover the equality sign holds iff

$$B^\gamma_{ij} = 0,\ \forall\ i, j \in \{1, \dots, n\},\ i \neq j \text{ and } \gamma \in \{n+1, \dots, m\} \qquad (30)$$

and

$$B^\gamma_{nn} = \frac{n(n-1)}{t} B^\gamma_{11} = \dots = \frac{n(n-1)}{t} B^\gamma_{n-1n-1}, \quad \forall \gamma \in \{n+1, \dots, m\}. \qquad (31)$$

From (30) and (31), we obtain that the equality holds if and only if the submanifold is invariantly quasi-umbilical with normal connections in $\overline{\mathcal{M}}$, such that the shape operator takes the form (20) with respect to an orthonormal tangent and orthonormal normal frames.

In the same way, we can prove (ii).

Remark 1. *The special cases of the theorem can be proved by taking the submanifolds as F-invariant, F-anti-invariant, slant et al.*

□

4 DDVV inequality for Riemannian product spaces

In this section we prove DDVV inequality for submanifolds of product space forms.

Theorem 3. *Let M^n be a n-dimensional submanifold of a manifold $\overline{\mathcal{M}}^m(c)$. Then, we have*

$$\rho + \rho^{\perp} \leq \|H\|^2 + a + \frac{2}{n(n-1)}\Big((trf)^2 + \|f\|^2\Big) - \frac{b}{n}(trf). \tag{32}$$

The equality holds if and only if, the shape operator with respect to the suitable orthonormal frame $\{E_1, E_2, \dots, E_m\}$ takes the following form

$$A_{E_{n+1}} = \begin{pmatrix} \xi_1+\psi & 0 & 0 & \dots & 0 & 0 \\ 0 & \xi_1-\psi & 0 & \dots & 0 & 0 \\ 0 & 0 & \xi_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \xi_1 & 0 \\ 0 & 0 & 0 & \dots & 0 & \xi_1 \end{pmatrix}, A_{E_{n+2}} = \begin{pmatrix} \xi_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & \xi_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & \xi_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \xi_2 & 0 \\ 0 & 0 & 0 & \dots & 0 & \xi_2 \end{pmatrix},$$

$$A_{E_{n+3}} = \begin{pmatrix} \xi_3 & 0 & 0 & \dots & 0 & 0 \\ 0 & \xi_3 & 0 & \dots & 0 & 0 \\ 0 & 0 & \xi_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \xi_3 & 0 \\ 0 & 0 & 0 & \dots & 0 & \xi_3 \end{pmatrix}, A_{E_{n+4}} = \cdots = A_{E_m} = 0, \tag{33}$$

where ξ_1, ξ_2, ξ_3 and ψ are real functions on submanifold.

Proof. Using (21), we have

$$2\tau = a\{n(n-1) + (trf)^2 - \|f\|^2\} + 2b\{(n-1)trf\} + \sum_{\gamma=n+1}^{m} \sum_{1\leq<j\leq n} \Big[B^{\gamma}_{ii}B^{\gamma}_{jj} - (B^{\gamma}_{ij})\Big]. \tag{34}$$

By taking into consideration (4) and Ricci equation, we get

$$\begin{aligned} \rho^{\perp} &= \frac{2\tau^{\perp}}{n(n-1)} \\ &= \frac{2}{n(n-1)}\sqrt{\sum_{n+1\leq\gamma<\beta\leq m}\ \sum_{1\leq i<j\leq n}\left[\sum_{k=1}^{n}\Big(B^{\gamma}_{ik}B^{\beta}_{jk} - B^{\gamma}_{jk}B^{\beta}_{ik}\Big)\right]^2}. \end{aligned} \tag{35}$$

On the other hand, we know that

$$\begin{aligned} n^2H^2 &= \sum_{\gamma=n+1}^{m}\Big(\sum_{i=1}^{n} B_{ii}^{\gamma}\Big)^2 \\ &= \frac{1}{n-1}\sum_{\gamma=n+1}^{m}\sum_{1\leq i<j\leq n}^{n}\Big(B_{ii}^{\gamma}-B_{jj}^{\gamma}\Big)^2 + \frac{2n}{n-1}\sum_{\gamma=n+1}^{m}\sum_{1\leq i<j\leq n}^{n} B_{ii}^{\gamma}B_{jj}^{\gamma}. \quad (36) \end{aligned}$$

By using the equation of Gauss and Ricci, we have

$$\begin{aligned} &\sum_{\gamma=n+1}^{m}\sum_{1\leq i<j\leq n}^{n}\Big(B_{ii}^{\gamma}-B_{jj}^{\gamma}\Big)^2 + 2n\sum_{\gamma=n+1}^{m}\sum_{1\leq i<j\leq n}^{n}(B_{ii}^{\gamma})^2 \\ \geq\ & 2n\Big[\sum_{n+1\leq\gamma<\beta\leq m}\ \sum_{1\leq i<j\leq n}\Big[\sum_{k=1}^{n}\Big(B_{ik}^{\gamma}B_{jk}^{\beta}-B_{jk}^{\gamma}B_{ik}^{\beta}\Big)\Big]^2\Big]^{\frac{1}{2}}. \quad (37) \end{aligned}$$

Substituting the value of (35) and (36) in (37), we have the following equation

$$nH^2 - n\rho^{\perp} = \frac{2}{n-1}\sum_{\gamma=n+1}^{m}\sum_{1\leq<j\leq n}\Big[B_{ii}^{\gamma}B_{jj}^{\gamma}-\big(B_{ij}^{\gamma}\big)\Big]. \quad (38)$$

Using (34) and (38), we arrive at

$$nH^2 - n\rho^{\perp} = \frac{2}{n-1}\Big[\tau - \frac{a}{2}\{n(n-1)-(trf)^2+\|f\|^2\} - b\big\{(n-1)trf\big\}\Big]. \quad (39)$$

The desired result holds easily from (39). The equality case of (32) at some point $p \in \mathcal{M}$ holds if and only if shape operator takes the form (33).

□

Remark 2. *The special cases of the theorem can be proved by taking the submanifolds as F-invariant, F-anti-invariant, slant et al.*

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