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A NOTE ON GEOMETRIC CONSTRUCTION OF SPECTRALLY ARBITRARY ZERO-NONZERO PATTERN MATRICES

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Abstract

In this paper, we give a geometric construction for spectrally arbitrary zero-nonzero pattern matrices. The geometric construction also deals with computation of a matrix realization for a given characteristic polynomial.

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1 Introduction

In this paper, we are developing the geometric construction for spectrally arbitrary zero-nonzero pattern matrices.

We begin with some basic definitions and terminologies. A *zero-nonzero pattern matrix* of order n is an $n \times n$ matrix whose entries belong to the set $\{*, 0\}$, where $*$ represents a nonzero entry. A *Qualitative class* or simply a *zero-nonzero pattern class* of a zero-nonzero pattern matrix S is denoted by $Q(S)$ and is defined as

$$Q(S) := \{A = (a_{ij}) \in M_n(\mathbb{R}) : \ a_{ij} \neq 0 \text{ if } s_{ij} = * \text{ otherwise } a_{ij} = 0\},$$

where s_{ij} is the $(i, j)^{th}$ entry of the zero-nonzero pattern S.

A given property P of an $n \times n$ matrix is said to be an *allowable* property for a given zero-nonzero pattern matrix S if there exists an $n \times n$ matrix $A \in Q(S)$ such that A satisfies P. We say that a zero-nonzero pattern matrix S *requires* a property P if every matrix in $Q(S)$ satisfies P.

The basic definitions are as given in [4].

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2 Main results

Definition 1. *[1] A zero-nonzero pattern matrix S of order $n \times n$ is said to be spectrally arbitrary if every monic polynomial of degree n is the characteristic polynomial of some matrix A in the qualitative class of S.*

We assign a one to one correspondence between vectors in $\mathbb{R}^n$ and coefficients of a characteristic polynomial for an $n \times n$ matrix in $M_n(\mathbb{R})$. For a vector $v = (a_1, a_2, \ldots, a_{n-1}, a_n)$ there is a characteristic polynomial $x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^{n-1}a_{n-1}x + (-1)^n a_n$ for some square matrix A of order n.

Definition 2. *Let S be a given zero-nonzero pattern matrix of order n. For a given vector $v = (a_1, a_2, \ldots, a_{n-1}, a_n)$ in $\mathbb{R}^n$ if there exists a matrix $A \in Q(S)$ whose characteristic polynomial is $x^n - a_1x^{n-1} + a_2x^{n-2} + \cdots + (-1)^{n-1}a_{n-1}x + (-1)^n a_n$, then a matrix A is called as a realization of the vector v.*

In this paper we are mainly interested in answering the following two open questions from the literature [2] to zero-nonzero pattern matrices.
Question 1. Given a zero-nonzero pattern, is it spectrally arbitrary?
Question 2. If a zero-nonzero pattern is spectrally arbitrary, then for any given monic polynomial with appropriate order how do we find a matrix in its qualitative class whose characteristic polynomial is the given polynomial?

Example 1. *Let*

$$S = \begin{pmatrix} * & * & 0 & * \\ 0 & 0 & * & * \\ * & 0 & 0 & * \\ * & 0 & 0 & * \end{pmatrix}.$$

be a zero nonzero pattern of order 4.

Let e_1, e_2, e_3, e_4 be unit vectors along the coordinate axes surrounding a hyperoctant. We want to find realizations of these vectors in $Q(S)$. Consider a matrix realization $A \in Q(S)$ given by $A = \begin{pmatrix} 2 & 2 & 0 & a \\ 0 & 0 & 1 & b \\ 1 & 0 & 0 & c \\ 1 & 0 & 0 & d \end{pmatrix}$, where a, b, c, d are nonzero real numbers. The characteristic polynomial of A is given by

$$\begin{aligned} p(x) &= x^4 - (d+2)x^3 + (-a+2d)x^2 - (2b+2)x + (-2c+2d) \\ &= x^4 - p_1x^3 + p_2x^2 - p_3x + p_4. \end{aligned}$$

We need to find variables a, b, c and d say a^1, b^1, c^1 and d^1 such that the characteristic polynomial $p(x)$ corresponds to e_1. To find a^2, b^2, c^2 and d^2 such that the characteristic polynomial $p(x)$ corresponds to e_2. Similarly for vectors

e_3 and e_4. Thus we get the following system of equations.

$$\begin{array}{rcl}(d^1+2,\ -a^1+2d^1,\ 2b^1+2,\ -2c^1+2d^1) &=& e_1=(1,\ 0,\ 0,\ 0)\\ (d^2+2,\ -a^2+2d^2,\ 2b^2+2,\ -2c^2+2d^2) &=& e_2=(0,\ 1,\ 0,\ 0)\\ (d^3+2,\ -a^3+2d^3,\ 2b^3+2,\ -2c^3+2d^3) &=& e_3=(0,\ 0,\ 1,\ 0)\\ (d^4+2,\ -a^4+2d^4,\ 2b^4+2,\ -2c^4+2d^4) &=& e_4=(0,\ 0,\ 0,\ 1)\end{array}$$

Transferring constant terms to the right-hand side, we have

$$\begin{array}{rcl}(d^1,\ -a^1+2d^1,\ 2b^1,\ -2c^1+2d^1) &=& (-1,\ 0, -2,\ 0)\\ (d^2,\ -a^2+2d^2,\ 2b^2,\ -2c^2+2d^2) &=& (-2,\ 1, -2,\ 0)\\ (d^3,\ -a^3+2d^3,\ 2b^3,\ -2c^3+2d^3) &=& (-2,\ 0, -1,\ 0)\\ (d^4,\ -a^4+2d^4,\ 2b^4,\ -2c^4+2d^4) &=& (-2,\ 0, -2,\ 1)\end{array}$$

Above system of equations can be written in the matrix form as follows

$$\begin{pmatrix}0&0&0&1\\-1&0&0&2\\0&2&0&0\\0&0&-2&2\end{pmatrix}C=\begin{pmatrix}-1&-2&-2&-2\\0&1&0&0\\-2&-2&-1&-2\\0&0&0&1\end{pmatrix}$$

$$BC=D$$

where the matrix B is the coefficient matrix of a, b, c and d in p_1, p_2, p_3 and p_4. The above system has a solution

$$C=\begin{pmatrix}-2&-5&-4&-4\\-1&-1&-1/2&-1\\-1&-2&-2&-5/2\\-1&-2&-2&-2\end{pmatrix}.$$

By making use of the first column of C in A, we have a realization for e_1 as

$$A_1=\begin{pmatrix}2&2&0&-2\\0&0&1&-1\\1&0&0&-1\\1&0&0&-1\end{pmatrix}.$$

Similarly other realizations for $e_2,\ e_3,\ e_4$ respectively are as follows.

$$A_2=\begin{pmatrix}2&2&0&-5\\0&0&1&-1\\1&0&0&-2\\1&0&0&-2\end{pmatrix},\ A_3=\begin{pmatrix}2&2&0&-4\\0&0&1&-\frac{1}{2}\\1&0&0&-2\\1&0&0&-2\end{pmatrix},\ A_4=\begin{pmatrix}2&2&0&-4\\0&0&1&-1\\1&0&0&-\frac{5}{2}\\1&0&0&-2\end{pmatrix}.$$

Suppose we want to find a matrix realization for a nonzero vector (1, 1, 1, 2) lying in the first hyperoctant: Consider the curve $(t,\ t^2,\ t^3,\ 2t^4)$ for $t\in\mathbb{R}$ and the hyperplane $x_1 + x_2 + x_3 + x_4 = 1$ for $0 \leq x_i \leq 1$. The curve and a part of the hyperplane intersect only at one point. Solving equation of the curve $(t,\, t^2,\, t^3,\, 2t^4)$ for $t \in \mathbb{R}$ with the hyperplane $x_1+x_2+x_3+x_4 = 1$ for $0 \leq x_i \leq 1$, we get $t = 0.5$. The linear combination $A(x_1,\, x_2,\, x_3) = x_1A_1+x_2A_2+x_3A_3+(1-x_1-x_2-x_3)A_4$ gives a realization for all vectors on $x_1+x_2+x_3+x_4 = 1$ with $0 \leq x_i \leq 1$. Equating $(t,\, t^2,\, t^3,\, 2t^4)$ with $(x_1,\, x_2,\, x_3,\, 1-x_1-x_2-x_3)$ for $t = 0.5$ we get $x_1 = 0.5$, $x_2 = 0.125$, $x_3 = 0.125$. Consequently, the matrix realization for $(0.5, 0.25, 0.125, 0.125)$ is

$$A(0.5, 0.25, 0.125) = \begin{pmatrix} 2 & 2 & 0 & -3.25 \\ 0 & 0 & 1 & -0.9375 \\ 1 & 0 & 0 & -1.5625 \\ 1 & 0 & 0 & -1.5 \end{pmatrix}.$$

Observe that the vector $(t,\, t^2,\, t^3,\, 2t^4)$ corresponds to $(0.5,\; 0.25,\; 0.125,\, 0.125)$ for $t = 0.5$ and corresponds to $(1,\, 1,\, 1,\, 2)$ for $t = 1$. Hence the required matrix realization for $(1,\, 1,\, 1,\, 2)$ is

$$\frac{1}{t} \times A(x_1,\, x_2,\, x_3) = \frac{1}{0.5} \times A(0.5,\, 0.25,\, 0.125) = \begin{pmatrix} 4 & 4 & 0 & -6.5 \\ 0 & 0 & 2 & -1.875 \\ 2 & 0 & 0 & -3.125 \\ 2 & 0 & 0 & -3 \end{pmatrix}.$$

Now suppose we want to find a matrix realization for $(-1,\, 1, -1,\, 2)$: Consider the curve $(-t,\, t^2, -t^3,\, 2t^4)$ for $t \in \mathbb{R}$ and the same hyperplane $x_1+x_2+x_3+x_4 = 1$ for $0 \leq x_i \leq 1$ as considered above. They intersect only at one point. Solving equation of the hyperplane $x_1 + x_2 + x_3 + x_4 = 1$ for $0 \leq x_i \leq 1$ with the curve $(-t,\, t^2, -t^3,\, 2t^4)$ we get $t = -0.5$. Equating $(-t,\, t^2, -t^3,\, 2t^4)$ with $(x_1,\, x_2,\, x_3,\, 1-x_1-x_2-x_3)$ for $t = -0.5$ we get $x_1 = 0.5$, $x_2 = 0.125$, $x_3 = 0.125$. Therefore, the matrix realization for $(0.5,\, 0.25,\, 0.125,\, 0.125)$ is $A(0.5,\, 0.25,\, 0.125) = \begin{pmatrix} 2 & 2 & 0 & -3.25 \\ 0 & 0 & 1 & -0.9375 \\ 1 & 0 & 0 & -1.5625 \\ 1 & 0 & 0 & -1.5 \end{pmatrix}$.

Observe that the vector $(-t,\, t^2,\, -t^3,\, 2t^4)$ corresponds to $(0.5,\, 0.25,\, 0.125,\, 0.125)$ for $t = -0.5$ and corresponds to $(-1,\, 1, -1,\, 2)$ for $t = 1$. Hence the required matrix realization for $(-1,\, 1, -1,\, 2)$ is

$$\frac{1}{t} \times A(x_1,\, x_2,\, x_3) = -\frac{1}{0.5} \times A(0.5,\, 0.25,\, 0.125) = \begin{pmatrix} -4 & -4 & 0 & 6.5 \\ 0 & 0 & -2 & 1.875 \\ -2 & 0 & 0 & 3.125 \\ -2 & 0 & 0 & 3 \end{pmatrix}.$$

In Table 1 of Appendix, we have given matrix realizations corresponding to a zero-nonzero pattern S computed by the method as described above for a combination of eight hyperoctants. For example vectors lying in a hyperoctant surrounded by $e_1, -e_2, -e_3, -e_4$ and $-e_1, -e_2,\, e_3, -e_4$ we can make use of matrix realizations given in Appendix and Sr. no. 2 of the Table 1.

Remark 1. *In Example 1, we have computed matrix realizations for vectors* $(1, 1, 1, 2)$ *and* $(-1, 1, -1, 2)$. *The matrix realization of vector* $(-1, 1, -1, 2)$ *is the negative of the matrix realization of vector* $(1, 1, 1, 2)$. *Thus, the matrix realizations differ in a negative sign. We know that if the characteristic polynomial of a matrix* A *is* $x^n + a_1x^{n-1} + \cdots + a_n$ *then the characteristic polynomial of* $-A$ *is* $x^n - a_1x^{n-1} + \cdots + (-1)^n a_n$. *However, in zero-nonzero pattern matrices these matrix realizations are allowed. Thus, it is enough to find a matrix realization of any vector out of* $(1, 1, 1, 2)$ *and* $(-1, 1, -1, 2)$.

Remark 2. *The realizations for vectors* e_1, e_2, e_3 *and* e_4 *give matrix realizations for nonzero vectors lying in two hyperoctants namely a hyperoctant surrounded by* e_1, e_2, e_3 *and* e_4 *as well as a hyperoctant surrounded by* $-e_1$, e_2, $-e_3$ *and* e_4.

Remark 3. *In case of* $\mathbb{R}^4$, *we need to find matrix realizations for only* 8 *different hyperoctants surrounded by the following sets of unit vectors:*

$\{e_1, e_2, e_3, e_4\}$; $\{e_1, -e_2, e_3, -e_4\}$; $\{e_1, -e_2, e_3, e_4\}$; $\{e_1, e_2, e_3, -e_4\}$;
$\{e_1, e_2, -e_3, e_4\}$; $\{e_1, -e_2, -e_3, -e_4\}$; $\{e_1, -e_2, -e_3, e_4\}$; $\{e_1, e_2, -e_3, -e_4\}$

Remark 4. *In general for* $\mathbb{R}^n$, *we need to find matrix realizations for unit vectors along the coordinate axes surrounding only* 2^{n-1} *different hyperoctants.*

Construction 2.1 : A method for finding matrix realizations for a given characteristic polynomial of a spectrally arbitrary zero-nonzero pattern.

1. Find matrix realizations for vectors e_1, $\pm e_2$, $\ldots$, $\pm e_n$ with the property that matrices surrounding one hyperoctant can be allowed to differ only in one row (or column) and having the same sign pattern.

2. To identify a matrix realization for an arbitrary nonzero vector $v = (a_1, a_2, \ldots, a_n)$: Consider the curve $(ta_1, t^2a_2, \ldots, t^na_n)$ for $t \in \mathbb{R}$ which passes through the origin and the vector v. Note that this curve also passes through the vector $v' = (-a_1, a_2, \cdots, (-1)^n a_n)$.

3. If $a_1 \geq 0$, then choose a hyperoctant which contains the vector $v = (a_1, a_2, \cdots, a_n)$ with e_1 would be one of the surrounding axes. If $a_1 < 0$, then choose a hyperoctant which contains the vector $v' = (-a_1, a_2, \cdots, (-1)^n a_n)$.

4. Assume that the hyperoctant chosen in step 3, is surrounded by vectors $e_1, v_2, \cdots, v_n$ where $\{v_2, \cdots, v_n\} \subseteq \{\pm e_2, \ldots, \pm e_n\}$. Also consider the convex linear combination of these vectors $x_1e_1 + x_2v_2 + \cdots + (1 - x_1 - \cdots - x_{n-1})v_n$ where $0 \leq x_i \leq 1$ for all $1 \leq i \leq n-1$. Find the point of intersection t_0 of the curve $(ta_1, t^2a_2, \ldots, t^na_n)$ for $t \in \mathbb{R}$ with the hyperplane $x_1e_1 + x_2v_2 + \cdots + (1 - x_1 - \cdots - x_{n-1})v_n$. It should be noted that the curve and a part of the plane intersect only at one point. Equating $(ta_1, t^2a_2, \ldots, t^na_n) = x_1e_1 + x_2v_2 + \cdots + (1 - x_1 - x_2 - \cdots - x_{n-1})v_n$ for particular t_0 to obtain $x_1, x_2, \ldots, x_{n-1}$.

5. Using an affine linear combination of matrices corresponding to vectors $e_1, v_2, \cdots, v_n$, obtain the matrix realization $E(x_1, x_2, \cdots, x_{n-1})$. Finally, the required matrix realization for the vector v is $(1/t_0)E(x_1, x_2, \cdots, x_{n-1})$.

In view of the above construction, we prove our main Theorem 2. As our construction doesn't take care of the polynomial x^n, we prove the following theorem for potentially nilpotent zero-nonzero patterns.

Theorem 2. *Let S be a potentially nilpotent zero-nonzero pattern matrix of order n. Let $e_1, \pm e_2, \ldots, \pm e_n$ be unit vectors along the axes. Suppose at least $2n-1$ matrices exist in the qualitative class of S, which are realizations of these $2n-1$ vectors. If n matrices corresponding to n vectors surrounding each hyperoctant differ only in one fixed row (or column) and they have the same sign pattern, then the zero-nonzero pattern S is spectrally arbitrary. Moreover, any particular matrix realization can be constructed as an affine combination of matrices corresponding to a hyperoctant (i.e. unit vectors).*

Proof. Suppose that S is a potentially nilpotent zero-nonzero pattern matrix of order n. Then there are at least $2n-1$ matrix realizations for unit vectors $e_1, \pm e_2, \cdots, \pm e_n$ with the property that any n matrix realizations surrounding each hyperoctant vary only in one row (or column). Let $p(x) = x^n - a_1x^{n-1} + \cdots + (-1)^{n-1}x + (-1)^n a_n$ be any arbitrary monic polynomial of degree n. By Definition 2, the polynomial $p(x)$ corresponds to the vector $(a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$, let us call it as v. Our aim is to find a matrix realization for the vector $v = (a_1, a_2, \cdots, a_n)$. If $a_1 \geq 0$, then choose a hyperoctant that contains the vector v such that e_1 will be one of the surrounding axes, otherwise choose a hyperoctant which contains the vector $v' = (-a_1, a_2, \cdots, (-1)^n a_n)$.

Suppose the chosen hyperoctant is surrounded by vectors $e_1, v_2, \cdots, v_n$ where $\{v_2, v_3, \cdots, v_n\} \subseteq \{\pm e_2, \pm e_3, \cdots, \pm e_n\}$. Now consider the curve $(ta_1, t^2a_2, \cdots, t^na_n)$ for $t \in \mathbb{R}$ and also consider the convex linear combination of vectors $e_1, v_2, \cdots, v_n$ i.e. $x_1e_1 + x_2v_2 + \cdots + x_{n-1}v_{n-1} + (1 - x_1 - x_2 - \cdots - x_{n-1})v_n$ where $0 \leq x_i \leq 1$ for all $1 \leq i \leq n-1$. Note that the curve and a part of the hyperplane lying in chosen hyperoctant intersect at only one point. Let t_0 be the point of intersection of the curve $(ta_1, t^2a_2, \cdots, t^na_n)$ with the hyperplane $x_1e_1 + x_2v_2 + \cdots + x_{n-1}v_{n-1} + (1 - x_1 - x_2 - \cdots - x_{n-1})v_n$ where $0 \leq x_i \leq 1$ for all $1 \leq i \leq n-1$. Equating the point on the curve $(ta_1, t^2a_2, \cdots, t^na_n)$ with $x_1v_1 + x_2v_2 + \cdots + x_{n-1}v_{n-1} + (1 - x_1 - x_2 - \cdots - x_{n-1})v_n$ for particular t_0, obtain the coefficients $x_1, x_2, \cdots, x_{n-1}$. By hypothesis, we have matrix realizations for vectors $e_1, v_2, \cdots, v_n$ say $A_1, A_2, \cdots, A_n$ respectively such that they vary either in a fixed row or in a fixed column. Let $E(x_1, x_2, \cdots, x_{n-1}) = x_1A_1 + x_2A_2 + \cdots + x_{n-1}A_{n-1} + (1 - x_1 - \cdots - x_{n-1})A_n$ be an affine linear combination of these matrices corresponding to computed coefficients. It is easy to observe that, the characteristic polynomial of $E(x_1, x_2, \cdots, x_{n-1})$ is an affine linear combination of characteristic polynomials of matrices $A_1, A_2, \cdots, A_n$ i.e. $ch(E(x_1, x_2, \cdots, x_{n-1})) = x_1ch(A_1) + x_2ch(A_2) + \cdots + x_{n-1}ch(A_{n-1}) + (1 - x_1 - \cdots - x_{n-1})ch(A_n)$. Also $E(x_1, x_2, \cdots, x_{n-1}) \in Q(S)$ as $0 \leq x_i \leq 1$ for all $1 \leq i \leq n-1$. Therefore, the matrix $E(x_1, x_2, \cdots, x_{n-1})$ gives a matrix realization for the vector $x_1 e_1 + x_2 v_2 + \cdots + x_{n-1} v_{n-1} + (1 - x_1 - x_2 - \cdots - x_{n-1}) v_n = (t_0\, a_1, t_0^2\, a_2, \cdots, t_0^n\, a_n)$. Now as $t_0 \neq 0$, we have $(1/t_0)E(x_1, x_2, \cdots, x_{n-1}) \in Q(S)$ and $(1/t_0)E(x_1, x_2, \cdots, x_{n-1})$ gives a matrix realization for the vector $v = (a_1, a_2, \cdots, a_n)$. Thus the characteristic polynomial of $(1/t_0)E(x_1, x_2, \cdots, x_{n-1})$ is $p(x)$.

Hence S is a spectrally arbitrary zero-nonzero pattern matrix and any particular non-nilpotent matrix realization can be constructed as an affine combination. ¤

Here, finding $2n - 1$ matrix realizations corresponding to unit vectors $e_1, \pm e_2, \cdots, \pm e_n$ such that n vectors surrounding one hyperoctant are allowed to differ at most in one row (or column) is not much difficult. It amounts in solving some matrix equations. The method for computing these realizations is already depicted in Example 1.

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Table 1: Matrix Realizations

Sr.No.	Octant surrounding vectors	Matrix realizations in pattern $S = \begin{pmatrix} * & * & 0 & * \\ 0 & 0 & * & * \\ * & 0 & 0 & * \\ * & 0 & 0 & * \end{pmatrix}$
1	e_1, e_2, e_3, e_4	$e_1 \to \begin{pmatrix} 2 & 2 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$, $e_2 \to \begin{pmatrix} 2 & 2 & 0 & -5 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & -2 \end{pmatrix}$, $e_3 \to \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 0 & 1 & -1/2 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & -2 \end{pmatrix}$, $e_4 \to \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -5/2 \\ 1 & 0 & 0 & -2 \end{pmatrix}$
2	$e_1, -e_2, -e_3, -e_4$	$e_1 \to \begin{pmatrix} 2 & 2 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$, $-e_2 \to \begin{pmatrix} 2 & 2 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & -2 \end{pmatrix}$, $-e_3 \to \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 0 & 1 & -3/2 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & -2 \end{pmatrix}$, $-e_4 \to \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -3/2 \\ 1 & 0 & 0 & -2 \end{pmatrix}$

3	$e_1, -e_2,$ e_3, e_4	$e_1 \to \begin{pmatrix} 2 & 2 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad -e_2 \to \begin{pmatrix} 2 & 2 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & -2 \end{pmatrix},$ $e_3 \to \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 0 & 1 & -1/2 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & -2 \end{pmatrix}, \quad e_4 \to \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -5/2 \\ 1 & 0 & 0 & -2 \end{pmatrix}$
4	$e_1, e_2,$ $-e_3, e_4$	$e_1 \to \begin{pmatrix} 2 & 2 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad e_2 \to \begin{pmatrix} 2 & 2 & 0 & -5 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & -2 \end{pmatrix},$ $-e_3 \to \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 0 & 1 & -3/2 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & -2 \end{pmatrix}, \quad e_4 \to \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -5/2 \\ 1 & 0 & 0 & -2 \end{pmatrix}$
5	$e_1, e_2,$ $e_3, -e_4$	$e_1 \to \begin{pmatrix} 2 & 2 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad e_2 \to \begin{pmatrix} 2 & 2 & 0 & -5 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & -2 \end{pmatrix},$ $e_3 \to \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 0 & 1 & -1/2 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & -2 \end{pmatrix}, \quad -e_4 \to \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -3/2 \\ 1 & 0 & 0 & -2 \end{pmatrix}$
6	$e_1, -e_2,$ $-e_3, e_4$	$e_1 \to \begin{pmatrix} 2 & 2 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad -e_2 \to \begin{pmatrix} 2 & 2 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & -2 \end{pmatrix},$ $-e_3 \to \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 0 & 1 & -3/2 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & -2 \end{pmatrix}, \quad e_4 \to \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -5/2 \\ 1 & 0 & 0 & -2 \end{pmatrix}$
7	$e_1, -e_2,$ $e_3, -e_4$	$e_1 \to \begin{pmatrix} 2 & 2 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad -e_2 \to \begin{pmatrix} 2 & 2 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & -2 \end{pmatrix},$ $e_3 \to \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 0 & 1 & -1/2 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & -2 \end{pmatrix}, \quad -e_4 \to \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -3/2 \\ 1 & 0 & 0 & -2 \end{pmatrix}$

8	$e_1, e_2,$ $-e_3, -e_4$	$e_1 \to \begin{pmatrix} 2 & 2 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad e_2 \to \begin{pmatrix} 2 & 2 & 0 & -5 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & -2 \end{pmatrix},$ $-e_3 \to \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 0 & 1 & -3/2 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & -2 \end{pmatrix}, \quad -e_4 \to \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -3/2 \\ 1 & 0 & 0 & -2 \end{pmatrix}$

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