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EXPONENTIAL DECAY FOR THERMOELASTICITY WITH TWO POROUS STRUCTURES

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Abstract

The paper is about the spatial behaviour of an elastic continuum with two porous structures in the one-dimensional case. The equilibrium of the continuum is studied without the presence of an external body force or extrinsic equilibrated body forces.

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1 Introduction

The field equations for the equilibrium of an elastic continuum with two porous structures and the heat conduction model of Green-Naghdi type III are [3]

$$\begin{aligned} &t_x = 0, \\ &h_{1,x} + g_1 = 0, \\ &h_{2,x} + g_2 = 0, \\ &q_x = 0. \end{aligned} \tag{1}$$

The two porous structures are the macro-porosity and the micro-porosity, which are represented by the cracks or fissures in the skeleton of the elastic continuum [3].

The theory of elastic solids with voids was proposed by Nunziato and Cowin [3], who considered that the continuum has an elastic skeleton or matrix material and some interstices represented by the voids of the material. This theory was developed by writing the bulk density as the product of two fields, the matrix material density field and the volume fraction field [2]. This theory was also used in [4], [5].

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Geological materials such as rocks and soils are possible applications of this theory. Others are ceramics, pressed powders and other manufactured materials.

The constitutive equations are [3]

$$\begin{aligned} t &= \mu u_x + \gamma_1 \phi_1 + \gamma_2 \phi_2, \\ h_1 &= b_{11}\phi_{1,x} + b_{12}\phi_{2,x} + m_1 \psi_x, \\ h_2 &= b_{12}\phi_{1,x} + b_{22}\phi_{2,x} + m_2 \psi_x, \\ g_1 &= -\gamma_1 u_x - \xi_{11}\phi_1 - \xi_{12}\phi_2, \\ g_2 &= -\gamma_2 u_x - \xi_{12}\phi_1 - \xi_{22}\phi_2, \\ q &= k\psi_x + m_1\phi_{1,x} + m_2\phi_{2,x}. \end{aligned} \tag{2}$$

The field equations in the equilibrium case for the one-dimensional problem are [3]

$$\begin{aligned} 0 &= \mu u_{xx} + \gamma_1 \phi_{1,x} + \gamma_2 \phi_{2,x}, \\ 0 &= b_{11}\phi_{1,xx} + b_{12}\phi_{2,xx} + m_1\psi_{xx} - \xi_{11}\phi_1 - \xi_{12}\phi_2 - \gamma_1 u_x, \\ 0 &= b_{12}\phi_{1,xx} + b_{22}\phi_{2,xx} + m_2\psi_{xx} - \xi_{12}\phi_1 - \xi_{22}\phi_2 - \gamma_2 u_x, \\ 0 &= m_1\phi_{1,xx} + m_2\phi_{2,xx} + k\psi_{xx}. \end{aligned} \tag{3}$$

In the equations above, t is the stress, h_i are the equilibrated stresses, g_i are the equilibrated body forces, q is the heat flux, u is the displacement, ϕ_i are the volume fractions, ψ is the thermal displacement.

The thermal displacement is a new independent variable which was introduced by Green and Naghdi for the type II and III theories of heat conduction [3].

We suppose that the matrices M_1 and M_2 are positive definite, where [3]

$$M_1 = \begin{pmatrix} b_{11} & b_{12} & m_1 \\ b_{12} & b_{22} & m_2 \\ m_1 & m_2 & k \end{pmatrix}, M_2 = \begin{pmatrix} \mu & \gamma_1 & \gamma_2 \\ \gamma_1 & \xi_{11} & \xi_{12} \\ \gamma_2 & \xi_{12} & \xi_{22} \end{pmatrix}. \tag{4}$$

The boundary conditions are

$$\begin{aligned} & u(0) = u(L) = 0, \\ & \phi_{1,x}(0) = \phi_{1,x}(L) = \phi_{2,x}(0) = \phi_{2,x}(L) = 0, \\ & \psi_x(0) = \psi_x(L) = 0. \end{aligned} \tag{5}$$

2 Spatial behaviour

We study the spatial behaviour as in [2].

We define

$$M(x) = t(x)u(x) + h_1(x)\phi_1(x) + h_2(x)\phi_2(x) + q(x)\psi(x), \quad x \in [0, L]. \tag{6}$$

This function will be used in studying the spatial behaviour of the solution.

Then

$$\begin{aligned} M(x) = {} & (\mu u_x + \gamma_1 \phi_1 + \gamma_2 \phi_2) u + \\ & + (b_{11} \phi_{1,x} + b_{12} \phi_{2,x} + m_1 \psi_x) \phi_1 + \\ & + (b_{12} \phi_{1,x} + b_{22} \phi_{2,x} + m_2 \psi_x) \phi_2 + \\ & + (k \psi_x + m_1 \phi_{1,x} + m_2 \phi_{2,x}) \psi. \end{aligned} \tag{7}$$

$$\begin{aligned} M(x) = {} & \mu u_x u + \gamma_1 \phi_1 u + \gamma_2 \phi_2 u + b_{11} \phi_{1,x} \phi_1 + b_{12} \phi_{2,x} \phi_1 + \\ & + m_1 \psi_x \phi_1 + b_{12} \phi_{1,x} \phi_2 + b_{22} \phi_{2,x} \phi_2 + m_2 \psi_x \phi_2 + \\ & + k \psi_x \psi + m_1 \phi_{1,x} \psi + m_2 \phi_{2,x} \psi. \end{aligned} \tag{8}$$

In order to obtain the result about the spatial behaviour, we first need to calculate $M'(x)$ by a direct differentiation, followed by the replacement of the equilibrium equations and the constitutive equations. The function $M'(x)$ will be written as the sum of two functions w_1 and w_2 in order to separate the evaluation of these terms.

Lemma 1. *We have*

$$M'(x) = w_1 + w_2, \tag{9}$$

where

$$\begin{aligned} w_1 = {} & \mu u_x u_x + \gamma_1 \phi_1 u_x + \gamma_2 \phi_2 u_x + \gamma_1 u_x \phi_1 + \xi_{11} \phi_1 \phi_1 + \\ & + \xi_{12} \phi_2 \phi_1 + \gamma_2 u_x \phi_2 + \xi_{12} \phi_1 \phi_2 + \xi_{22} \phi_2 \phi_2, \end{aligned} \tag{10}$$

$$\begin{aligned} w_2 = {} & b_{11} \phi_{1,x} \phi_{1,x} + b_{12} \phi_{2,x} \phi_{1,x} + m_1 \psi_x \phi_{1,x} + b_{12} \phi_{1,x} \phi_{2,x} + \\ & + b_{22} \phi_{2,x} \phi_{2,x} + m_2 \psi_x \phi_{2,x} + k \psi_x \psi_x + m_1 \phi_{1,x} \psi_x + m_2 \phi_{2,x} \psi_x. \end{aligned} \tag{11}$$

Proof. We obtain

$$\begin{aligned} M'(x) = {} & t_x u + t u_x + h_{1,x} \phi_1 + h_1 \phi_{1,x} + h_{2,x} \phi_2 + h_2 \phi_{2,x} + \\ & + q_x \psi + q \psi_x = t u_x - g_1 \phi_1 + h_1 \phi_{1,x} - g_2 \phi_2 + h_2 \phi_{2,x} + q \psi_x. \end{aligned} \tag{12}$$

Then

$$\begin{aligned} M'(x) = {} & (\mu u_x + \gamma_1 \phi_1 + \gamma_2 \phi_2) u_x - (-\gamma_1 u_x - \xi_{11} \phi_1 - \xi_{12} \phi_2) \phi_1 + \\ & + (b_{11} \phi_{1,x} + b_{12} \phi_{2,x} + m_1 \psi_x) \phi_{1,x} - (-\gamma_2 u_x - \xi_{12} \phi_1 - \xi_{22} \phi_2) \phi_2 + \\ & + (b_{12} \phi_{1,x} + b_{22} \phi_{2,x} + m_2 \psi_x) \phi_{2,x} + (k \psi_x + m_1 \phi_{1,x} + m_2 \phi_{2,x}) \psi_x. \end{aligned} \tag{13}$$

$$\begin{aligned} M'(x) = {} & \mu u_x u_x + \gamma_1 \phi_1 u_x + \gamma_2 \phi_2 u_x + \gamma_1 u_x \phi_1 + \xi_{11} \phi_1 \phi_1 + \xi_{12} \phi_2 \phi_1 + \\ & + b_{11} \phi_{1,x} \phi_{1,x} + b_{12} \phi_{2,x} \phi_{1,x} + m_1 \psi_x \phi_{1,x} + \gamma_2 u_x \phi_2 + \xi_{12} \phi_1 \phi_2 + \xi_{22} \phi_2 \phi_2 + \\ & + b_{12} \phi_{1,x} \phi_{2,x} + b_{22} \phi_{2,x} \phi_{2,x} + m_2 \psi_x \phi_{2,x} + k \psi_x \psi_x + m_1 \phi_{1,x} \psi_x + m_2 \phi_{2,x} \psi_x. \end{aligned} \tag{14}$$

□

We suppose that there exist σ_m, σ_M such that

$$\sigma_m (u_x^2 + \phi_1^2 + \phi_2^2) \le w_1 \le \sigma_M (u_x^2 + \phi_1^2 + \phi_2^2). \tag{15}$$

Let

$$\begin{aligned} \mathcal{F}(\psi, \chi) = {} & \mu \psi_1 \chi_1 + \gamma_1 (\psi_2 \chi_1 + \psi_1 \chi_2) + \gamma_2 (\psi_3 \chi_1 + \psi_1 \chi_3) + \\ & + \xi_{11} \psi_2 \chi_2 + \xi_{22} \psi_3 \chi_3 + \xi_{12} (\psi_2 \chi_3 + \psi_3 \chi_2), \end{aligned} \tag{16}$$

$\forall\psi = \{\psi_1, \psi_2, \psi_3\}, \chi = \{\chi_1, \chi_2, \chi_3\}$. Then

$$\mathcal{F}(\psi, \chi) = \mathcal{F}(\chi, \psi) \tag{17}$$

and

$$\begin{aligned} \mathcal{F}(\tilde{\psi}, \tilde{\psi}) = u_x(\mu u_x + \gamma_1\phi_1 + \gamma_2\phi_2) + \phi_1(\gamma_1 u_x + \xi_{11}\phi_1 + \xi_{12}\phi_2) + \\ + \phi_2(\gamma_2 u_x + \xi_{12}\phi_1 + \xi_{22}\phi_2) = w_1, \quad \tilde{\psi} = \{u_x, \phi_1, \phi_2\}. \end{aligned} \tag{18}$$

For the result about the spatial behaviour, we need to bound some terms from the measure $M(x)$ by using the result from below.

Lemma 2. *We have*

$$t^2 + g_1^2 + g_2^2 \le \sigma_M w_1. \tag{19}$$

Proof. We use the constitutive equations

$$\begin{aligned} t^2 + g_1^2 + g_2^2 = t(\mu u_x + \gamma_1\phi_1 + \gamma_2\phi_2) - g_1(\gamma_1 u_x + \xi_{11}\phi_1 + \xi_{12}\phi_2) - \\ - g_2(\gamma_2 u_x + \xi_{12}\phi_1 + \xi_{22}\phi_2) = \mathcal{F}(T, \tilde{\psi}) \end{aligned} \tag{20}$$

for $T = \{t, -g_1, -g_2\}$, $\tilde{\psi} = \{u_x, \phi_1, \phi_2\}$.

From the Schwarz inequality and by the relations (18) and (15), we obtain

$$\begin{aligned} t^2 + g_1^2 + g_2^2 &\le \left[\mathcal{F}(\tilde{\psi}, \tilde{\psi})\right]^{\frac{1}{2}} \left[\mathcal{F}(T, T)\right]^{\frac{1}{2}} \\ &\le \left[\mathcal{F}(\tilde{\psi}, \tilde{\psi})\right]^{\frac{1}{2}} \left[\sigma_M(t^2 + g_1^2 + g_2^2)\right]^{\frac{1}{2}} \end{aligned} \tag{21}$$

□

Below we give the result of spatial behaviour, which will be followed by a discussion that presents the case of exponential decay for thermoelasticity with two porous structures.

Theorem 1. *There exists σ such that*

$$\sigma^2|M(x)| \le M'(x), x \in [0, L]. \tag{22}$$

Proof. By using the Cauchy - Schwarz inequality and the geometric-arithmetic mean inequality we obtain

$$|M(x)| \le \frac{1}{2}\left[\varepsilon_1 t^2 + \frac{1}{\varepsilon_1}u^2 + \varepsilon_2 h_1^2 + \frac{1}{\varepsilon_2}\phi_1^2 + \varepsilon_3 h_2^2 + \frac{1}{\varepsilon_3}\phi_2^2 + \varepsilon_4 q^2 + \frac{1}{\varepsilon_4}\psi^2\right] \tag{23}$$

$$\begin{aligned} h_1^2 &= (b_{11}\phi_{1.x} + b_{12}\phi_{2,x} + m_1\psi_x)(b_{11}\phi_{1,x} + b_{12}\phi_{2,x} + m_1\psi_x) \\ &= b_{11}^2\phi_{1,x}\phi_{1,x} + b_{11}b_{12}\phi_{1,x}\phi_{2,x} + b_{11}m_1\phi_{1,x}\psi_x \\ &\quad + b_{12}b_{11}\phi_{2,x}\phi_{1,x} + b_{12}^2\phi_{2,x}\phi_{2,x} + b_{12}m_1\phi_{2,x}\psi_x \\ &\quad + m_1 b_{11}\psi_x\phi_{1,x} + m_1 b_{12}\psi_x\phi_{2,x} + m_1^2\psi_x\psi_x \end{aligned} \tag{24}$$

$$h_2^2 = (b_{12}\phi_{1,x} + b_{22}\phi_{2,x} + m_2\psi_x)(b_{12}\phi_{1,x} + b_{22}\phi_{2,x} + m_2\psi_x) = \\ = b_{12}^2\phi_{1,x}\phi_{1,x} + b_{12}b_{22}\phi_{1,x}\phi_{2,x} + b_{12}m_2\phi_{1,x}\psi_x + \\ + b_{22}b_{12}\phi_{2,x}\phi_{1,x} + b_{22}^2\phi_{2,x}\phi_{2,x} + b_{22}m_2\phi_{2,x}\psi_x + \\ + m_2b_{12}\psi_x\phi_{1,x} + m_2b_{22}\psi_x\phi_{2,x} + m_2^2\psi_x\psi_x \tag{25}$$

$$q^2 = (k\psi_x + m_1\phi_{1,x} + m_2\phi_{2,x})(k\psi_x + m_1\phi_{1,x} + m_2\phi_{2,x}) = \\ = k\psi_x\psi_x + km_1\psi_x\phi_{1,x} + km_2\psi_x\phi_{2,x} + \\ + m_1k\phi_{1,x}\psi_x + m_1^2\phi_{1,x}\phi_{1,x} + m_1m_2\phi_{1,x}\phi_{2,x} + \\ + m_2k\phi_{2,x}\psi_x + m_2m_1\phi_{2,x}\phi_{1,x} + m_2^2\phi_{2,x}\phi_{2,x} \tag{26}$$

$$\sigma_m^1(\phi_{1,x}^2 + \phi_{2,x}^2 + \psi_x^2) \leq h_1^2 \leq \sigma_M^1(\phi_{1,x}^2 + \phi_{2,x}^2 + \psi_x^2) \tag{27}$$

$$\sigma_m^2(\phi_{1,x}^2 + \phi_{2,x}^2 + \psi_x^2) \leq h_2^2 \leq \sigma_M^2(\phi_{1,x}^2 + \phi_{2,x}^2 + \psi_x^2) \tag{28}$$

$$\sigma_m^3(\phi_{1,x}^2 + \sigma_{2,x}^2 + \psi_x^2) \leq q^2 \leq \sigma_M^3(\phi_{1,x}^2 + \phi_{2,x}^2 + \psi_x^2) \tag{29}$$

$$\tilde{\sigma}_m(\phi_{1,x}^2 + \phi_{2,x}^2 + \psi_x^2) \leq w_2 \leq \tilde{\sigma}_M(\phi_{1,x}^2 + \phi_{2,x}^2 + \psi_x^2) \tag{30}$$

$$\exists\lambda_1 : u_x \geq \sqrt{\lambda_1}u \quad u^2 \leq \frac{1}{\lambda_1}u_xu_x \tag{31}$$

$$\exists\lambda_2 : \psi_x \geq \sqrt{\lambda_2}\psi \quad \psi^2 \leq \frac{1}{\lambda_2}\psi_x\psi_x \tag{32}$$

$$|M(x)| \leq \frac{1}{2}\Bigg[\varepsilon_1\sigma_M w_1 + \frac{1}{\varepsilon_1\lambda_1}u_xu_x + \varepsilon_2\sigma_M^1(\phi_{1,x}^2 + \phi_{2,x}^2 + \psi_x^2) + \\ + \frac{1}{\varepsilon_2}\phi_1^2 + \varepsilon_3\sigma_M^2(\phi_{1,x}^2 + \phi_{2,x}^2 + \psi_x^2) + \frac{1}{\varepsilon_3}\phi_2^2 + \\ + \varepsilon_4\sigma_M^3(\phi_{1,x}^2 + \phi_{2,x}^2 + \psi_x^2) + \frac{1}{\varepsilon_4\lambda_2}\psi_x^2\Bigg] \tag{33}$$

$$\frac{1}{\varepsilon_1\lambda_1} = \frac{1}{\varepsilon_2} = \frac{1}{\varepsilon_3} \tag{34}$$

$$\sigma_M^4 = \max\{\sigma_M^1, \sigma_M^2\} \tag{35}$$

$$\varepsilon_2\sigma_M^4 = \varepsilon_3\sigma_M^4 = \varepsilon_4\sigma_M^3 \quad \varepsilon_4 = \varepsilon_3\frac{\sigma_M^4}{\sigma_M^3} \tag{36}$$

$$\varepsilon_4\sigma_M^3 = \frac{1}{\varepsilon_4\lambda_2} \quad \varepsilon_4^2 = \frac{1}{\sigma_M^3\lambda_2} \quad \varepsilon_4 = \frac{1}{\sqrt{\sigma_M^3\lambda_2}} \tag{37}$$

$$\varepsilon_2 = \varepsilon_3 = \varepsilon_4\frac{\sigma_M^3}{\sigma_M^4} = \frac{1}{\sqrt{\sigma_M^3\lambda_2}}\frac{\sigma_M^3}{\sigma_M^4} = \frac{1}{\sigma_M^4}\sqrt{\frac{\sigma_M^3}{\lambda_2}} \tag{38}$$

$$\varepsilon_1 = \frac{\varepsilon_2}{\lambda_1} = \frac{1}{\lambda_1 \sigma_M^4} \sqrt{\frac{\sigma_M^3}{\lambda_2}} \tag{39}$$

$$\begin{aligned} |M(x)| &\leq \frac{1}{2}\Bigg[\varepsilon_1 \sigma_M w_1 + \sigma_M^4 \sqrt{\frac{\lambda_2}{\sigma_M^3}} \frac{1}{\sigma_m} w_1 + \\ &+ \sqrt{\frac{\sigma_M^3}{\lambda_2}} \frac{1}{3\tilde{\sigma}_m} w_2 + \sqrt{\frac{\sigma_M^3}{\lambda_2}} \frac{1}{\tilde{\sigma}_m} w_2\Bigg] = \\ &= \frac{1}{2}\left[\left(\varepsilon_1 \sigma_M + \frac{\sigma_M^4}{\sigma_m}\sqrt{\frac{\lambda_2}{\sigma_M^3}}\right) w_1 + \frac{4}{3\tilde{\sigma}_M}\sqrt{\frac{\sigma_M^3}{\lambda_2}} w_2\right] \end{aligned} \tag{40}$$

$$\frac{1}{\sigma^2} = \frac{1}{2}\max\left\{\varepsilon_1 \sigma_M + \frac{\sigma_M^4}{\sigma_m}\sqrt{\frac{\lambda_2}{\sigma_M^3}}, \frac{4}{3\tilde{\sigma}_M}\sqrt{\frac{\sigma_M^3}{\lambda_2}}\right\} \tag{41}$$

$$\sigma^2 |M(x)| \leq M'(x), x \in [0, L] \tag{42}$$

□

We consider three cases for the inequality (22).

I $M(0) > 0$

$$M'(x) \geq 0 \quad M(x) > 0 \forall x \geq 0 \quad M(L) > 0 \tag{43}$$

$$M'(x) \geq \sigma^2 M(x) \tag{44}$$

$$M(0) e^{\sigma^2 x} \leq M(x) \leq M(L) e^{-\sigma^2 (L-x)}, x \in [0, L] \tag{45}$$

II $M(0) = 0$

i) $M(L) = 0$

$$M(x) = M'(x) = 0 \tag{46}$$

$$u = \phi_1 = \phi_2 = \psi = 0 \text{ in } B \tag{47}$$

ii) $M(L) > 0$

$$\hat{x} = \inf\{x \in [0, L] : M(x) > 0\} > 0 \tag{48}$$

$$M(\hat{x}) e^{\sigma^2 (x - \hat{x})} \leq M(x) \leq M(L) e^{-\sigma^2 (L-x)}, x \in [\hat{x}, L] \tag{49}$$

$$M(x) = M'(x) = 0, x \in [0, \hat{x}] \tag{50}$$

III $M(0) < 0$

i) $M(L) < 0$ $M(x) < 0, x \in [0, L]$

$$-M(L) e^{\sigma^2 (L-x)} \leq -M(x) \leq -M(0) e^{-\sigma^2 x}, x \in [0, L] \tag{51}$$

ii) $M(L) \geq 0$ similar to the cases above

3 Conclusion

We studied the model of thermoelasticity with two porous structures from [3] in the one-dimensional case with the governing equations in equilibrium following [2]. The result of spatial behaviour was followed by a discussion that presents the exponential decay. Similar results were presented in [1]. The novelty of the model is given by the introduction of the thermal displacement, a new independent variable that appears in the Green and Naghdi type II and III theories of heat conduction.

References

[1] Chirilă, A., Marin, M., *Spatial behaviour of thermoelasticity with microtemperatures and microconcentrations*, ITM Web Conf. **34** (2020), 02001.

[2] Chiriţă, S., *On some exponential decay estimates for porous elastic cylinders*, Arch. Mech. **56** (2004), no. 3, 233-246.

[3] Magana, A., Quintanilla, R., *Exponential decay in one-dimensional type II/III thermoelasticity with two porosities*, Math. Meth. Appl. Sci. **43** (2020), no. 11, 6921-6937.

[4] Marin, M., *Generalized solutions in elasticity of micropolar bodies with voids*, Rev. Acad. Canar. Cienc. **8** (1996), no. 1, 101-106.

[5] Marin, M., *A temporally evolutionary equation in elasticity of micropolar bodies with voids*, UPB Sci. Bull. A: Appl. Math. Phys. **60** (1998), no. 3-4, 3-12.