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GENERALIZATION OF SOME INEQUALITIES USING LIDSTONE INTERPOLATION VIA DIAMOND INTEGRALS

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Abstract

In present paper, several inequalities involving Csiszár divergence are established by utilizing diamond integrals and Lidstone interpolation polynomials. Consequently, new and generalized inequalities are yields. The functions involved in these inequalities are higher order convex functions. Inequalities involving Shannon entropy, Kullback-Leibler discrimination, triangle distance and Jeffrey's distance, are studied as particular instances with the help of specially chosen convex functions. The main findings are also discussed for some special time scales (both discrete and continuous). Many existing results are also obtained which established the link with existing literature.

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1 Introduction

The theory of time scales calculus and convex functions are both rapidly growing areas of research. The first encounter the both discrete and continuous structures simultaneously and the later presents the application by means of optimizations.

The class of n-convex functions is a good tool to generalize the inequalities involving convex functions. For this purpose, many interpolations can be used. The

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interaction of convexity with information theory results many divergences and entropy formulas. Hence the estimation of bounds for divergences and entropies are the outcomes of the study of mathematical inequalities involving convex functions, in both integrals and discrete cases. Whereas, the theory of time scales provides an efficient methodology to combine the both discrete and continuous structures.

Non-negative measures of dissimilarity between pairs of probability measures are very useful in data science, probability theory, statistical learning, information theory, statistical signal processing, and other related fields. It is of great interest to investigate the mathematical framework and many applications of these divergence measurements. Divergence measures are very utile and represent a critical role in different fields like sensor networks [21], economics [25], testing the order in a Markov chain [19], finance [24] and approximation of probability distributions [16]. Entropies and the divergence measures are frequently employed in statistical physics (see [15]).

S. Hilger presented the theory of time scales that provides a platform to deal with discrete and continuous cases together. In order to have a look at time scales calculus, suggested books are [11, 12]. Recently, several mathematicians have worked on this subject and constituted many results, see [3, 1, 4, 7].

In the start of time scales calculus, delta and nabla integrals were used to study the integral inequalities. In [14], authors have introduced diamond integral which is convex combination of nabla and delta integrals. Recently, Bilal *et al.* [7] extended Jensen's inequality via diamond integrals on time scales calculus (approximate symmetric integrals).

In the past few decades, many generalizations, improvements, refinements, and variants of the results for n-convex functions have been extrapolated by various investigators. Smoljak Kalamir [20] have extended some Steffensen-type inequalities to time scales by using the diamond-α integral. In [5], author have utilized Green's function and Hermite interpolating polynomial, to extend Jensen's functional for n-convex functions from Jensen's inequality involving diamond integrals.

Lidstone polynomials are useful to generalize a large number of inequalities. In [18], Gazić *et al.* have provided Jensen's inequality and its converses for $2n$-convex functions with the help of Lidstone's interpolating polynomials. In [6], author have utilized Lidstone's interpolating polynomial to extend Jensen's inequality via diamond integrals for $2n$-convex functions. In [3], authors have obtained new entropic bounds via delta integrals using Lidstone interpolating polynomial.

In Section 2, first of all some basic definitions and results of time scales calculus are given. After that Lidstone interpolating polynomial and its series representation is recalled. Section 3 contains results containing Csiszár divergence via diamond integral for the $n-$convex function. In Section 4, bounds of different divergence measures are estimated. Section 5 provides relationship of Mandelbrot entropy with other entropies. Lastly, manuscript is concluded in Section 6.

2 Preliminaries

In this section, some basics of the mathematical theory of time scales are given.

Time scale $\mathbb{T}$ is a nonempty closed subset of real numbers. Some famous examples of time scales are $\mathbb{Z}$, Cantor set and $\mathbb{N}$.

Let $r \in \mathbb{T}$, forward jump operators is

$$\sigma(r) := inf\{v \in \mathbb{T} : v > r\}, \tag{1}$$

and and backward jump operators is

$$\rho(r) := sup\{v \in \mathbb{T} : v < r\}. \tag{2}$$

Delta Integral [11, Definition 1.71]

A mapping $H : \mathbb{T} \to (-\infty, \infty)$ is called the delta antiderivative of $h : [b_1, b_2]_{\mathbb{T}} = [b_1, b_2] \bigcap \mathbb{T} \to (-\infty, \infty)$ if $H^{\triangle}(\zeta) = h(\zeta)$ holds true $\forall\ \zeta \in \mathbb{T}^{\kappa}$. The delta integral of h is

$$\int_{b_1}^{b_2} h(\zeta)\triangle\zeta = H(b_2) - H(b_1). \tag{3}$$

Nabla Integral [11, Definition 8.42]

A mapping $G : \mathbb{T} \to (-\infty, \infty)$ is called the nabla antiderivative of $g : [b_1, b_2]_{\mathbb{T}} \to (-\infty, \infty)$ if $G^{\nabla}(\zeta) = g(\zeta) \quad \forall\ \zeta \in \mathbb{T}^{\kappa}$. The nabla integral of g is

$$\int_{b_1}^{b_2} g(\zeta)\nabla\zeta = G(b_2) - G(b_1). \tag{4}$$

In [22], authors have defined diamond-alpha integral given as follows:

Consider $l : [b_1, b_2]_{\mathbb{T}} \to \mathbb{R}$ is a continuous mapping and $b_1, b_2 \in \mathbb{T}(b_1 < b_2)$. The diamond alpha integral of l is given as

$$\int_{b_1}^{b_2} l(\eta)\diamondsuit_{\alpha}\eta := \int_{b_1}^{b_2} \alpha l(\eta)\triangle\eta + \int_{b_1}^{b_2} (1-\alpha)l(\eta)\nabla\eta, \quad 0 \le \alpha \le 1, \tag{5}$$

if γl is $\triangle$ and $(1-\gamma)l$ is ∇ integrable on $[b_1, b_2]_{\mathbb{T}}$.

In case $\alpha = 0$, we have nabla-integral and for $\alpha = 1$, we have delta-integral.

In [13], real valued function γ is given as follows:

$$\gamma(x) = \lim_{y \to x} \frac{\sigma(x) - y}{\sigma(x) + 2x - 2y - \rho(x)}. \tag{6}$$

Clearly,

$$\gamma(x) = \begin{cases} \frac{1}{2}, & \text{if } x \text{ is dense;} \\ \frac{\sigma(x)-x}{\sigma(x)-\rho(x)}, & \text{if } x \text{ is not dense .} \end{cases}$$

In general $0 \le \gamma(x) \le 1$.

In [14], diamond integral is defined as follows:

Diamond Integral ([14])

Assume that $g : [b_1, b_2]_{\mathbb{T}} \to \mathbb{R}$ is a continuous function and $b_1, b_2 \in \mathbb{T}(b_1 < b_2)$. The $\diamondsuit$-integral of g is given as

$$\int_{b_1}^{b_2} g(\zeta)\diamondsuit\zeta = \int_{b_1}^{b_2} \gamma(\zeta)g(\zeta)\triangle\zeta + \int_{b_1}^{b_2} (1-\gamma(\zeta))g(\zeta)\nabla\zeta, \quad 0 \le \gamma(\zeta) \le 1, \tag{7}$$

where γg is $\triangle$ and $(1-\gamma)g$ is ∇ integrable on $[b_1, b_2]_{\mathbb{T}}$.

For monotonicity, additive, reflexive and multiplicative properties of $\diamondsuit$-integrals see [14].

Throughout the paper we assumed that:

A1: $\Theta := [b_1, b_2]_{\mathbb{T}}$, with $b_1, b_2 \in \mathbb{T}$ and $b_1 < b_2$.

A2: The set of all probability densities is denoted by $E =: \{g \quad \textit{such that} \quad g : \Theta \to \mathbb{R}, g(\eta) > 0, \int_\Theta g(\eta)\diamondsuit\eta = 1\}$.

The following Theorem is provided by Bilal *et al.* in [10].

Theorem 1. *Assume that the mapping $\phi : [0, \infty) \to (-\infty, \infty)$ is convex on $[\varrho_1, \varrho_2] \subset [0, \infty)$ and $\varrho_1 \le 1 \le \varrho_2$. If $l_1, l_2 \in E$ and*

$$\varrho_1 \le \frac{l_2(\zeta)}{l_1(\zeta)} \le \varrho_2, \quad \forall\, \zeta \in \mathbb{T},$$

then

$$\int_\Theta l_2(\zeta)\phi\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)\diamondsuit\zeta \le \frac{\varrho_2 - 1}{\varrho_2 - \varrho_1}\phi(\varrho_1) + \frac{1-\varrho_1}{\varrho_2 - \varrho_1}\phi(\varrho_2). \tag{8}$$

If suppositions of Theorem 1 remain valid, then it is possible to define the following functional $F_1(\phi)$ involving Csiszár divergence for diamond integrals:

$$F_1(\phi) = \frac{\varrho_2 - 1}{\varrho_2 - \varrho_1}\phi(\varrho_1) + \frac{1-\varrho_1}{\varrho_2 - \varrho_1}\phi(\varrho_2) - \int_\Theta l_2(\zeta)\phi\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)\diamondsuit\zeta, \tag{9}$$

where ϕ is defined on $[\varrho_1, \varrho_2]$.

Remark 1. *If suppositions of Theorem 1 remain valid, then $F_1(\phi) \ge 0$.*

In [26], Widder proved the following result:

Lemma 1. *If $\phi \in C^\infty([0, 1])$, then*

$$\phi(\mathfrak{t}) = \sum_{r=0}^{n-1} \left[\phi^{2r}(0)\Psi_r(1-\mathfrak{t}) + \phi^{2r}(0)\Psi_r(\mathfrak{t})\right] + \int_0^1 G_n(\mathfrak{t}, \mathfrak{s})\phi^{2n}(\mathfrak{t})d\mathfrak{t},$$

where Ψ_r is a polynomial of degree $2r+1$ defined by the relations

$$\Psi_0(\mathfrak{t}) = \mathfrak{t}, \Psi_n''(\mathfrak{t}) = \Psi_{n-1}(\mathfrak{t}), \Psi_n(0) = \Psi_n(1) = 0, n \ge 1$$

and

$$G_1(\mathfrak{t}, \mathfrak{s}) = G(\mathfrak{t}, \mathfrak{s}) = \begin{cases} (\mathfrak{t}-1)\mathfrak{s}\ , & \mathfrak{s} \le \mathfrak{t}, \\ (\mathfrak{s}-1)\mathfrak{t}, & \mathfrak{t} \le \mathfrak{s}, \end{cases} \tag{10}$$

is homogeneous Green's function of the differential operator $\frac{d^2}{d\mathfrak{s}^2}$ *on* $[0,1]$*, and with the successive iterates of* $G(\mathfrak{t},\mathfrak{s})$*.*

$$G_n(\mathfrak{t},\mathfrak{s}) = \int_0^1 G_1(\mathfrak{t},p)G_{n-1}(p,\mathfrak{s})dp, \quad n \geq 2. \tag{11}$$

The Lidstone polynomial can be expressed in terms of $G_n(\mathfrak{t},\mathfrak{s})$ *as*

$$\Psi_n(\mathfrak{t}) = \int_0^1 G_n(\mathfrak{t},\mathfrak{s})\mathfrak{s}d\mathfrak{s}. \tag{12}$$

In [2], Lidstone series representation of $\phi \in C^{2n}[\nu_1,\nu_2]$ is

$$\phi(x) = \sum_{r=0}^{n-1}(\nu_2-\nu_1)^{2r}\phi^{2r}(\nu_1)\Psi_r\left(\frac{\nu_2-x}{\nu_2-\nu_1}\right) + \sum_{r=0}^{n-1}(\nu_2-\nu_1)^{2r}\phi^{2r}(\nu_2)\Psi_r\left(\frac{x-\nu_1}{\nu_2-\nu_1}\right)$$
$$+ (\nu_2-\nu_1)^{2n-1}\int_{\nu_1}^{\nu_2} G_n\left(\frac{x-\nu_1}{\nu_2-\nu_1},\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\phi^{2n}(\mathfrak{t})d\mathfrak{t}. \tag{13}$$

3 Main results

In this section, first of all, an identity involving Csiszár divergence for diamond integrals is established by utilizing Lidstone polynomial:

Theorem 2. *Let the suppositions of Theorem 1 are true and* $\phi \in C^n[\nu_1,\nu_2]$*. If* G_n *is defined as in (11), then*

$$I_\phi(l_1,l_2) = \int_\Theta l_2(\vartheta)\phi\left(\frac{l_1(\vartheta)}{l_2(\vartheta)}\right)\Diamond\vartheta = \frac{\varrho_2-1}{\varrho_2-\varrho_1}\phi(\varrho_1) + \frac{1-\varrho_1}{\varrho_2-\varrho_1}\phi(\varrho_2)$$
$$-\sum_{r=0}^{n-1}(\nu_2-\nu_1)^{2r}\phi^{(2r)}(\nu_1)\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)$$
$$-\sum_{r=0}^{n-1}(\nu_2-\nu_1)^{2r}\phi^{(2r)}(\nu_2)\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)$$
$$-(\nu_2-\nu_1)^{2n-1}\int_{\nu_1}^{\nu_2}\digamma_1\left(G_n\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1},\frac{\mathfrak{s}-\nu_1}{\nu_2-\nu_1}\right)\right)\phi^{(2n)}(\mathfrak{s})\,d\mathfrak{s}, \tag{14}$$

where

$$\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right) = \frac{\varrho_2-1}{\varrho_2-\varrho_1}\psi_r\left(\frac{\varrho_1-\nu_1}{\nu_2-\nu_1}\right) + \frac{1-\varrho_1}{\varrho_2-\varrho_1}\psi_r\left(\frac{\varrho_2-\nu_1}{\nu_2-\nu_1}\right)$$
$$-\int_\Theta \psi_r\left(\frac{l_1(\vartheta)-\nu_1 l_2(\vartheta)}{\nu_2-\nu_1}\right)\Diamond\vartheta, \tag{15}$$

and

$$\digamma_1\left(G_n\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1},\frac{\mathfrak{s}-\nu_1}{\nu_2-\nu_1}\right)\right)=\frac{\varrho_2-1}{\varrho_2-\varrho_1}G_n\left(\frac{\varrho_1-\nu_1}{\nu_2-\nu_1},\frac{\mathfrak{s}-\nu_1}{\nu_2-\nu_1}\right)$$
$$+\frac{1-\varrho_1}{\varrho_2-\varrho_1}G_n\left(\frac{\varrho_2-\nu_1}{\nu_2-\nu_1},\frac{\mathfrak{s}-\nu_1}{\nu_2-\nu_1}\right)-\int_\Theta l_2(\vartheta)G_n\left(\frac{\frac{l_1(\vartheta)}{l_2(\vartheta)}-\nu_1}{\nu_2-\nu_1},\frac{\mathfrak{s}-\nu_1}{\nu_2-\nu_1}\right)\diamondsuit\vartheta. \quad (16)$$

Proof. Employ (13) in (9) and the linearity of $\digamma_1(\cdot)$ to obtain (14). □

The next Theorem is concerning with Csiszár type linear functional for $2n$-convex function.

Theorem 3. *Assume that suppositions of Theorem 2 are true and ϕ is $2n$-convex function with $\phi\in C^{2n}[\nu_1,\nu_2]$. If*

$$\digamma_1\left(G_n\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1},\frac{\mathfrak{s}-\nu_1}{\nu_2-\nu_1}\right)\right)\geq 0, \quad (17)$$

then

$$\frac{\varrho_2-1}{\varrho_2-\varrho_1}\phi(\varrho_1)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\phi(\varrho_2)-\int_\Theta l_2(\vartheta)\phi\left(\frac{l_1(\vartheta)}{l_2(\vartheta)}\right)\diamondsuit\vartheta$$
$$\geq\sum_{r=0}^{n-1}(\nu_2-\nu_1)^{2r}\left[\phi^{(2r)}(\nu_1)\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)+\phi^{(2r)}(\nu_2)\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right]. \quad (18)$$

Proof. Since the function ϕ is $2n$-convex, therefore ϕ is $2n$-times differentiable and $\phi^{(2n)}(\cdot)\geq 0$. Use Theorem 2 with (17) to obtain (18). □

Theorem 4. *Let the suppositions of Theorem 2 are true and $l\in E$ with $\phi\in C^{2n}[\Theta,\mathbb{R}]$ be an $2n$-convex function.*

(a) *If n is odd, then (18) holds.*

(b) *If (18) holds, and*

$$\sum_{r=0}^{n-1}(\nu_2-\nu_1)^{2r}\left[\phi^{(2r)}(\nu_1)\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)\right.$$
$$\left.+\phi^{(2r)}(\nu_2)\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right]\geq 0, \quad (19)$$

then

$$\frac{\varrho_2-1}{\varrho_2-\varrho_1}\phi(\varrho_1)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\phi(\varrho_2)-\int_\Theta l_2(\vartheta)\phi\left(\frac{l_1(\vartheta)}{l_2(\vartheta)}\right)\diamondsuit\vartheta\geq 0. \quad (20)$$

Proof. Since G_1 is convex and $G_n\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1},\frac{\mathfrak{s}-\nu_1}{\nu_2-\nu_1}\right)\geq 0$ for odd n, therefore (17) is valid. Furthermore, ϕ is $2n$-convex function, hence one can use Remark 1 and Theorem 3 to obtain (20). □

Remark 2. *If one select set of real numbers as time scale and $n = 1$ then (20) is same as [17, (2.1)].*

Remark 3. *Furthermore, it is possible to establish, Ostrowski and Grüss-type bounds accompanying to (14).*

4 Estimation of divergence measures

In this section, bound of different divergence measures are estimated using Theorem 2. In [8] authors have introduced differential entropy $h_{\bar{d}}(X)$ via diamond integral formalism, stated as follows:

Definition 1.

$$h_{\bar{d}}(X) := \int_{\Theta} g(\eta) \log \frac{1}{g(\eta)} \Diamond\eta \tag{21}$$

where $g \in E$ and the base of 'log' is $\bar{d}$ for some fixed $\bar{d} > 1$.

Theorem 5. *Assume that suppositions of Theorem 2 are true and ϕ is $2n$-convex function. If n is odd, then*

$$\begin{aligned} h_{\bar{c}}(Z) \geq \frac{\varrho_2 - 1}{\varrho_2 - \varrho_1} \log(\varrho_1) + \frac{1 - \varrho_1}{\varrho_2 - \varrho_1} \log(\varrho_2) - \int_{\Theta} l_1(\vartheta) \log\left(l_2(\vartheta)\right) \Diamond\vartheta \\ - \sum_{r=0}^{n-1} (2r-1)!(\nu_2 - \nu_1)^{2r} \times \left[\frac{1}{(\nu_1)^{2r}} \digamma_1\left(\psi_r\left(\frac{\nu_2 - \mathfrak{t}}{\nu_2 - \nu_1}\right)\right) \right. \\ \left. + \frac{1}{(\nu_2)^{2r}} \digamma_1\left(\psi_r\left(\frac{\mathfrak{t} - \nu_1}{\nu_2 - \nu_1}\right)\right) \right], \end{aligned} \tag{22}$$

where $h_{\bar{c}}(Z)$ is defined in (21).

Proof. Use $\phi(z) = -\log z$ in Theorem 3 to obtain (22). □

In [9], authors have defined KL divergence using diamond integral:

$$D(l_1, l_2) := \int_{\Theta} l_1(\zeta) \ln\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right) \Diamond\zeta. \tag{23}$$

Theorem 6. *Assume that suppositions of Theorem 2 are true and ϕ is $2n$-convex function. If n is odd, then*

$$\begin{aligned} D(l_1, l_2) \leq \frac{\varrho_2 - 1}{\varrho_2 - \varrho_1} \varrho_1 \ln(\varrho_1) + \frac{1 - \varrho_1}{\varrho_2 - \varrho_1} \varrho_2 \ln(\varrho_2) - \sum_{r=0}^{n-1} (2r-2)!(\nu_2 - \nu_1)^{2r} \\ \times \left[\frac{1}{(\nu_1)^{2r-1}} \digamma_1\left(\psi_r\left(\frac{\nu_2 - \mathfrak{t}}{\nu_2 - \nu_1}\right)\right) + \frac{1}{(\nu_2)^{2r-1}} \digamma_1\left(\psi_r\left(\frac{\mathfrak{t} - \nu_1}{\nu_2 - \nu_1}\right)\right) \right], \end{aligned} \tag{24}$$

where $D(l_1, l_2)$ is defined in (23).

Proof. Use $\phi(z) = z \ln z$ in Theorem 3 to obtain (24). □

In [9], Jeffrey's distance via diamond integral is stated as follows:

$$D_J(l_1,l_2) := \int_{\Theta} \big(l_1(\zeta)-l_2(\zeta)\big)\ln\left[\frac{l_1(\zeta)}{l_2(\zeta)}\right]\Diamond\zeta. \tag{25}$$

Theorem 7. *Let the suppositions of Theorem 2 are true and ϕ be $2n$-convex function. If n is odd, then*

$$\begin{aligned} D_J(l_1,l_2) \le{}& \frac{\varrho_2-1}{\varrho_2-\varrho_1}(\varrho_1-1)\ln(\varrho_1)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}(\varrho_2-1)\ln(\varrho_2)-\sum_{r=0}^{n-1}(\nu_2-\nu_1)^{2r}\\ &\times\left[\left(\frac{(2r-1)!}{(\nu_1)^{2r}}+\frac{(2r-2)!}{(\nu_1)^{2r-1}}\right)F_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)\right.\\ &\left.+\left(\frac{(2r-1)!}{(\nu_2)^{2r}}+\frac{(2r-2)!}{(\nu_2)^{2r-1}}\right)F_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right], \end{aligned} \tag{26}$$

where $D_J(l_1,l_2)$ is defined in (25).

Proof. Use $\phi(z)=(z-1)\ln z$ in Theorem 3 to obtain (26). □

Definition 2. *In [9], triangular discrimination using diamond integral is stated as follows:*

$$D_{\triangle}(l_1,l_2) := \int_{\Theta}\frac{\big(l_2(\zeta)-l_1(\zeta)\big)^2}{l_2(\zeta)+l_1(\zeta)}\Diamond\zeta. \tag{27}$$

Theorem 8. *Assume that suppositions of Theorem 2 are true and ϕ be $2n$-convex function. If n is even, then*

$$\begin{aligned} D_{\triangle}(l_1,l_2) \le{}& \frac{\varrho_2-1}{\varrho_2-\varrho_1}\frac{(\varrho_1-1)^2}{\varrho_1+1}+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\frac{(\varrho_2-1)^2}{\varrho_2+1}-\sum_{r=0}^{n-1}4(2r)!(\nu_2-\nu_1)^{2r}\\ \times&\left[\frac{1}{(\nu_1+1)^{2r+1}}F_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)+\frac{1}{(\nu_2+1)^{2r+1}}F_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right], \end{aligned} \tag{28}$$

where $D_{\triangle}(l_1,l_2)$ is defined in (27).

Proof. Use $\phi(z)=\frac{(z-1)^2}{z+1}$ in Theorem 3 to obtain (28). □

4.1 Bounds of divergence measures in classical calculus

Now, we estimate different divergence measures in classical calculus by choosing set of real numbers as time scale.

If one chooses set of real numbers as time scale in Theorem 3, then (18) provides following new bound in classical calculus for Csiszár divergence:

$$\begin{aligned} &\int_{\Theta} l_2(\vartheta)\phi\left(\frac{l_1(\vartheta)}{l_2(\vartheta)}\right)d\vartheta \le \frac{\varrho_2-1}{\varrho_2-\varrho_1}\phi(\varrho_1)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\phi(\varrho_2)\\ &-\sum_{r=0}^{n-1}(\nu_2-\nu_1)^{2r}\left[\phi^{(2r)}(\nu_1)\,F_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)+\phi^{(2r)}(\nu_2)\,F_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right], \end{aligned}$$

where

$$\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)=\frac{\varrho_2-1}{\varrho_2-\varrho_1}\psi_r\left(\frac{\varrho_1-\nu_1}{\nu_2-\nu_1}\right)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\psi_r\left(\frac{\varrho_2-\nu_1}{\nu_2-\nu_1}\right)\\-\int_\Theta\psi_r\left(\frac{l_1(\vartheta)-\nu_1 l_2(\vartheta)}{\nu_2-\nu_1}\right)d\vartheta. \quad (29)$$

If one chooses set of real numbers as time scale in Theorem 5–8 then (22), (24), (26) and (28) provide following new bounds in classical calculus for differential entropy, *KL* divergence, Jeffrey distance and Triangular discrimination, respectively:

$$\int_\Theta l_2(\vartheta)\log\frac{1}{l_2(\vartheta)}d\vartheta\geq\frac{\varrho_2-1}{\varrho_2-\varrho_1}\log(\varrho_1)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\log(\varrho_2)-\int_\Theta l_2(\vartheta)\log\big(l_2(\vartheta)\big)d\vartheta\\-\sum_{r=0}^{n-1}(2r-1)!(\nu_2-\nu_1)^{2r}\times\left[\frac{1}{(\nu_1)^{2r}}\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)\right.\\\left.+\frac{1}{(\nu_2)^{2r}}\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right],$$

$$\int_\Theta l_1(\vartheta)\ln\left(\frac{l_1(\vartheta)}{l_2(\vartheta)}\right)d\vartheta\\\leq\frac{\varrho_2-1}{\varrho_2-\varrho_1}\varrho_1\ln(\varrho_1)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\varrho_2\ln(\varrho_2)-\sum_{r=0}^{n-1}(2r-2)!(\nu_2-\nu_1)^{2r}\\\times\left[\frac{1}{(\nu_1)^{2r-1}}\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)+\frac{1}{(\nu_2)^{2r-1}}\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right],$$

$$\int_\Theta\big(l_1(\vartheta)-l_2(\vartheta)\big)\ln\left[\frac{l_1(\vartheta)}{l_2(\vartheta)}\right]d\vartheta\leq\frac{\varrho_2-1}{\varrho_2-\varrho_1}(\varrho_1-1)\ln(\varrho_1)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}(\varrho_2-1)\ln(\varrho_2)\\-\sum_{r=0}^{n-1}(\nu_2-\nu_1)^{2r}\times\left[\left(\frac{(2r-1)!}{(\nu_1)^{2r}}+\frac{(2r-2)!}{(\nu_1)^{2r-1}}\right)\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)\right.\\\left.+\left(\frac{(2r-1)!}{(\nu_2)^{2r}}+\frac{(2r-2)!}{(\nu_2)^{2r-1}}\right)\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right],$$

and

$$\int_\Theta\frac{\big(l_2(\vartheta)-l_1(\vartheta)\big)^2}{l_2(\vartheta)+l_1(\vartheta)}d\vartheta\\\leq\frac{\varrho_2-1}{\varrho_2-\varrho_1}\frac{(\varrho_1-1)^2}{\varrho_1+1}+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\frac{(\varrho_2-1)^2}{\varrho_2+1}-\sum_{r=0}^{n-1}4(2r)!(\nu_2-\nu_1)^{2r}\\\times\left[\frac{1}{(\nu_1+1)^{2r+1}}\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)+\frac{1}{(\nu_2+1)^{2r+1}}\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right],$$

where $\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)$ is defined in (29).

4.2 Bounds of divergence measures in h-discrete calculus

Now we estimate bounds of different divergence measures in h-discrete calculus. Moreover, bounds of some divergence measures in discrete calculus are estimated.

If one chooses set $h\mathbb{Z}$ as time scale, where $h > 0$, in Theorem 3, then $\vartheta = hy \in h\mathbb{Z}$ for some $y \in \mathbb{Z}$ and (18) takes the form:

$$\frac{h}{2}\Bigg[\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1} l_2(h\vartheta)\phi\bigg(\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\bigg) + \sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}} l_2(h\vartheta)\phi\bigg(\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\bigg)\Bigg]$$
$$\leq \frac{\varrho_2-1}{\varrho_2-\varrho_1}\phi(\varrho_1) + \frac{1-\varrho_1}{\varrho_2-\varrho_1}\phi(\varrho_2) - \sum_{r=0}^{n-1}(\nu_2-\nu_1)^{2r}\Bigg[\phi^{(2r)}(\nu_1)\,\digamma_1\bigg(\psi_r\bigg(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\bigg)\bigg)$$
$$+\phi^{(2r)}(\nu_2)\,\digamma_1\bigg(\psi_r\bigg(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\bigg)\bigg)\Bigg], \quad (30)$$

where

$$\digamma_1\bigg(\psi_r\bigg(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\bigg)\bigg) = \frac{\varrho_2-1}{\varrho_2-\varrho_1}\psi_r\bigg(\frac{\varrho_1-\nu_1}{\nu_2-\nu_1}\bigg) + \frac{1-\varrho_1}{\varrho_2-\varrho_1}\psi_r\bigg(\frac{\varrho_2-\nu_1}{\nu_2-\nu_1}\bigg)$$
$$-\frac{h}{2(\nu_2-\nu_1)}\Bigg[\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1}\big(l_1(h\vartheta)-\nu_1 l_2(h\vartheta)\big) + \sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}}\big(l_1(h\vartheta)-\nu_1 l_2(h\vartheta)\big)\Bigg]. \quad (31)$$

If one chooses set $h\mathbb{Z}$ as time scale, where $h > 0$ in Theorem 5–8 then (22), (24), (26) and (28) provide following new bounds in h-discrete calculus for differential entropy, *KL* divergence, Jeffrey distance and Triangular discrimination:

$$\frac{h}{2}\Bigg[\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1} l_2(h\vartheta)\log\frac{1}{l_2(h\vartheta)} + \sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}} l_2(h\vartheta)\log\frac{1}{l_2(h\vartheta)}\Bigg]$$
$$\geq \frac{\varrho_2-1}{\varrho_2-\varrho_1}\log(\varrho_1) + \frac{1-\varrho_1}{\varrho_2-\varrho_1}\log(\varrho_2)$$
$$-\frac{h}{2}\Bigg[\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1} l_2(h\vartheta)\log\big(l_1(h\vartheta)\big) + \sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}} l_2(h\vartheta)\log\big(l_1(h\vartheta)\big)\Bigg] - \sum_{r=0}^{n-1}(2r-1)!(\nu_2-\nu_1)^{2r}$$
$$\times\Bigg[\frac{1}{(\nu_1)^{2r}}\digamma_1\bigg(\psi_r\bigg(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\bigg)\bigg) + \frac{1}{(\nu_2)^{2r}}\digamma_1\bigg(\psi_r\bigg(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\bigg)\bigg)\Bigg], \quad (32)$$

$$\frac{h}{2}\Bigg[\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1} l_1(h\vartheta)\ln\left(\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\right)+\sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}} l_1(h\vartheta)\ln\left(\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\right)\Bigg]$$
$$\leq \frac{\varrho_2-1}{\varrho_2-\varrho_1}\varrho_1\ln(\varrho_1)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\varrho_2\ln(\varrho_2)-\sum_{r=0}^{n-1}(2r-2)!(\nu_2-\nu_1)^{2r}$$
$$\times\left[\frac{1}{(\nu_1)^{2r-1}}\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)+\frac{1}{(\nu_2)^{2r-1}}\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right], \quad (33)$$

$$\frac{h}{2}\Bigg[\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1}\big(l_1(h\vartheta)-l_2(h\vartheta)\big)\ln\left(\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\right)+\sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}}\big(l_1(h\vartheta)-l_2(h\vartheta)\big)\ln\left(\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\right)\Bigg]$$
$$\leq \frac{\varrho_2-1}{\varrho_2-\varrho_1}(\varrho_1-1)\ln(\varrho_1)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}(\varrho_2-1)\ln(\varrho_2)$$
$$-\sum_{r=0}^{n-1}(\nu_2-\nu_1)^{2r}\times\Bigg[\left(\frac{(2r-1)!}{(\nu_1)^{2r}}+\frac{(2r-2)!}{(\nu_1)^{2r-1}}\right)\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)$$
$$+\left(\frac{(2r-1)!}{(\nu_2)^{2r}}+\frac{(2r-2)!}{(\nu_2)^{2r-1}}\right)\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\Bigg], \quad (34)$$

and

$$\frac{h}{2}\Bigg[\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1}\frac{\big(l_2(h\vartheta)-l_1(h\vartheta)\big)^2}{l_2(h\vartheta)+l_1(h\vartheta)}+\sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}}\frac{\big(l_2(h\vartheta)-l_1(h\vartheta)\big)^2}{l_2(h\vartheta)+l_1(h\vartheta)}\Bigg]$$
$$\leq \frac{\varrho_2-1}{\varrho_2-\varrho_1}\frac{(\varrho_1-1)^2}{\varrho_1+1}+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\frac{(\varrho_2-1)^2}{\varrho_2+1}-\sum_{r=0}^{n-1}4(2r)!(\nu_2-\nu_1)^{2r}$$
$$\times\left[\frac{1}{(\nu_1+1)^{2r+1}}\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)+\frac{1}{(\nu_2+1)^{2r+1}}\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right], \quad (35)$$

where $\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)$ is defined in (31).

Remark 4. *Use $h=1$, $b_1=0$, $b_2=p$, $l_1(\vartheta)=(l_1)_\vartheta$ and $l_2(\vartheta)=(l_2)_\vartheta$, in (30) to obtain following new bound for discrete Csiszár divergence:*

$$\frac{1}{2}\Bigg[\sum_{\vartheta=0}^{p-1}(l_2)_\vartheta\phi\left(\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right)+\sum_{\vartheta=1}^{p}(l_2)_\vartheta\phi\left(\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right)\Bigg]$$
$$\leq \frac{\varrho_2-1}{\varrho_2-\varrho_1}\phi(\varrho_1)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\phi(\varrho_2)$$
$$-\sum_{r=0}^{n-1}(\nu_2-\nu_1)^{2r}\left[\phi^{(2r)}(\nu_1)\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)+\phi^{(2r)}(\nu_2)\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right],$$

where

$$\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right) = \frac{\varrho_2-1}{\varrho_2-\varrho_1}\psi_r\left(\frac{\varrho_1-\nu_1}{\nu_2-\nu_1}\right) + \frac{1-\varrho_1}{\varrho_2-\varrho_1}\psi_r\left(\frac{\varrho_2-\nu_1}{\nu_2-\nu_1}\right) - \frac{1}{2(\nu_2-\nu_1)}\left[\sum_{\vartheta=0}^{p-1}\left((l_1)_\vartheta - \nu_1(l_2)_\vartheta\right) + \sum_{\vartheta=1}^{p}\left((l_1)_\vartheta - \nu_1(l_2)_\vartheta\right)\right]. \quad (36)$$

Remark 5. *Use $h=1$, $b_1=0$, $b_2=p$, $l_1(\vartheta)=(l_1)_\vartheta$ and $l_2(\vartheta)=(l_2)_\vartheta$, in (32) to obtain following new bound for discrete Shannon entropy:*

$$\frac{1}{2}\left[\sum_{\vartheta=0}^{p-1}(l_2)_\vartheta \log\frac{1}{(l_2)_\vartheta} + \sum_{\vartheta=1}^{p}(l_2)_\vartheta \log\frac{1}{(l_2)_\vartheta}\right] \geq \frac{\varrho_2-1}{\varrho_2-\varrho_1}\log(\varrho_1) + \frac{1-\varrho_1}{\varrho_2-\varrho_1}\log(\varrho_2) - \frac{1}{2}\left[\sum_{\vartheta=0}^{p-1}(l_2)_\vartheta \log\left((l_2)_\vartheta\right) + \sum_{\vartheta=1}^{p}(l_2)_\vartheta \log\left((l_2)_\vartheta\right)\right] - \sum_{r=0}^{n-1}(2r-1)!(\nu_2-\nu_1)^{2r} \times \left[\frac{1}{(\nu_1)^{2r}}\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right) + \frac{1}{(\nu_2)^{2r}}\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right], \quad (37)$$

where $\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)$ is defined in (36).

Remark 6. *Use $h=1$, $b_1=0$, $b_2=p$, $l_1(\vartheta)=(l_1)_\vartheta$ and $l_2(\vartheta)=(l_2)_\vartheta$, in (33) to obtain following new bound for discrete Kullback-Leibler divergence:*

$$\frac{1}{2}\left[\sum_{\vartheta=0}^{p-1}(l_1)_\vartheta \ln\left(\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right) + \sum_{\vartheta=1}^{p}(l_1)_\vartheta \ln\left(\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right)\right] \leq \frac{\varrho_2-1}{\varrho_2-\varrho_1}\varrho_1\ln(\varrho_1) + \frac{1-\varrho_1}{\varrho_2-\varrho_1}\varrho_2\ln(\varrho_2) - \sum_{r=0}^{n-1}(2r-2)!(\nu_2-\nu_1)^{2r} \times \left[\frac{1}{(\nu_1)^{2r-1}}\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right) + \frac{1}{(\nu_2)^{2r-1}}\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right], \quad (38)$$

where $\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)$ is defined in (36).

Remark 7. *Use $h=1$, $b_1=0$, $b_2=p$, $l_1(\vartheta)=(l_1)_\vartheta$ and $l_2(\vartheta)=(l_2)_\vartheta$, in (34)* *to obtain following new bound for discrete Jeffrey's distance:*

$$\frac{1}{2}\left[\sum_{\vartheta=0}^{p-1}\big((l_1)_\vartheta-(l_2)_\vartheta\big)\ln\left(\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right)+\sum_{\vartheta=1}^{p}\big((l_1)_\vartheta-(l_2)_\vartheta\big)\ln\left(\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right)\right]$$
$$\leq\frac{\varrho_2-1}{\varrho_2-\varrho_1}(\varrho_1-1)\ln(\varrho_1)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}(\varrho_2-1)\ln(\varrho_2)$$
$$-\sum_{r=0}^{n-1}(\nu_2-\nu_1)^{2r}\times\left[\left(\frac{(2r-1)!}{(\nu_1)^{2r}}+\frac{(2r-2)!}{(\nu_1)^{2r-1}}\right)\mathcal{F}_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)\right.$$
$$\left.+\left(\frac{(2r-1)!}{(\nu_2)^{2r}}+\frac{(2r-2)!}{(\nu_2)^{2r-1}}\right)\mathcal{F}_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right],\quad(39)$$

where $\mathcal{F}_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)$ *is defined in (36).*

Remark 8. *Use* $h=1$, $b_1=0$, $b_2=p$, $l_1(\vartheta)=(l_1)_\vartheta$ *and* $l_2(\vartheta)=(l_2)_\vartheta$, *in (35) to obtain following new bound for discrete triangular discrimination:*

$$\frac{1}{2}\left[\sum_{\vartheta=0}^{p-1}\frac{\big((l_2)_\vartheta-(l_1)_\vartheta\big)^2}{(l_2)_\vartheta+(l_1)_\vartheta}+\sum_{\vartheta=1}^{p}\frac{\big((l_2)_\vartheta-(l_1)_\vartheta\big)^2}{(l_2)_\vartheta+(l_1)_\vartheta}\right]$$
$$\leq\frac{\varrho_2-1}{\varrho_2-\varrho_1}\frac{(\varrho_1-1)^2}{\varrho_1+1}+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\frac{(\varrho_2-1)^2}{\varrho_2+1}-\sum_{r=0}^{n-1}4(2r)!(\nu_2-\nu_1)^{2r}$$
$$\times\left[\frac{1}{(\nu_1+1)^{2r+1}}\mathcal{F}_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)+\frac{1}{(\nu_2+1)^{2r+1}}\mathcal{F}_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right],$$

where $\mathcal{F}_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)$ *is defined in (36).*

4.3 Bounds of divergence measures in q-calculus

Now, we estimate bounds of different divergence measures in q-calculus.

If one chooses set $q^{\mathbb{N}_0}$, $q>1$ as time scale, in Theorem 3, then $\vartheta=q^y\in q^{\mathbb{N}_0}$ for some $y\in\mathbb{N}_0$. Further if $b_1=q$ and $b_2=q^p$, then (18) takes the form:

$$\frac{q-1}{q+1}\left[\sum_{m=1}^{p-1}q^{m+1}l_2(q^m)\phi\left(\frac{l_1(q^m)}{l_2(q^m)}\right)+\sum_{m=2}^{p}q^{m-1}l_2(q^m)\phi\left(\frac{l_1(q^m)}{l_2(q^m)}\right)\right]$$
$$\leq\frac{\varrho_2-1}{\varrho_2-\varrho_1}\phi(\varrho_1)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\phi(\varrho_2)-\sum_{r=0}^{n-1}(\nu_2-\nu_1)^{2r}$$
$$\left[\phi^{(2r)}(\nu_1)\mathcal{F}_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)+\phi^{(2r)}(\nu_2)\mathcal{F}_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right],$$

where

$$\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)=\frac{\varrho_2-1}{\varrho_2-\varrho_1}\psi_r\left(\frac{\varrho_1-\nu_1}{\nu_2-\nu_1}\right)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\psi_r\left(\frac{\varrho_2-\nu_1}{\nu_2-\nu_1}\right)$$
$$-\frac{q-1}{(q+1)(\nu_2-\nu_1)}\left[\sum_{m=1}^{p-1}q^{m+1}\big(l_1(q^m)-\nu_1 l_2(q^m)\big)+\sum_{m=2}^{p}q^{m+1}\big(l_1(q^m)-\nu_1 l_2(q^m)\big)\right]. \tag{40}$$

If one chooses set $q^{\mathbb{N}_0}$, $q>1$ as time scale and $b_1=q$ and $b_2=q^p$ in Theorem 5–8 then (22), (24), (26) and (28) provide following new bounds in q-discrete calculus for differential entropy, Kullback-Leibler divergence, Jeffrey's distance and Triangular discrimination, respectively:

$$\frac{q-1}{q+1}\left[\sum_{m=1}^{p-1}q^{m+1}l_2(q^m)\log\frac{1}{l_2(q^m)}+\sum_{m=2}^{p}q^{m-1}l_2(q^m)\log\frac{1}{l_2(q^m)}\right]$$
$$\geq\frac{\varrho_2-1}{\varrho_2-\varrho_1}\log(\varrho_1)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\log(\varrho_2)$$
$$-\frac{q-1}{q+1}\left[\sum_{m=1}^{p-1}q^{m+1}l_2(q^m)\log\big(l_2(q^m)\big)+\sum_{m=2}^{p}q^{m-1}l_2(q^m)\log\big(l_2(q^m)\big)\right]$$
$$-\sum_{r=0}^{n-1}(2r-1)!(\nu_2-\nu_1)^{2r}\times\left[\frac{1}{(\nu_1)^{2r}}\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)\right.$$
$$\left.+\frac{1}{(\nu_2)^{2r}}\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right],$$

$$\frac{q-1}{q+1}\left[\sum_{m=1}^{p-1}q^{m+1}l_1(q^m)\ln\left(\frac{l_1(q^m)}{l_2(q^m)}\right)+\sum_{m=2}^{p}q^{m-1}l_1(q^m)\ln\left(\frac{l_1(q^m)}{l_2(q^m)}\right)\right]$$
$$\leq\frac{\varrho_2-1}{\varrho_2-\varrho_1}\varrho_1\ln(\varrho_1)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\varrho_2\ln(\varrho_2)-\sum_{r=0}^{n-1}(2r-2)!(\nu_2-\nu_1)^{2r}$$
$$\times\left[\frac{1}{(\nu_1)^{2r-1}}\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)+\frac{1}{(\nu_2)^{2r-1}}\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right],$$

$$\frac{q-1}{q+1}\left[\sum_{m=1}^{p-1}q^{m+1}\big(l_1(q^m)-l_2(q^m)\big)\ln\left(\frac{l_1(q^m)}{l_2(q^m)}\right)\right.$$
$$\left.+\sum_{m=2}^{p}q^{m-1}\big(l_1(q^m)-l_2(q^m)\big)\ln\left(\frac{l_1(q^m)}{l_2(q^m)}\right)\right]$$
$$\leq\frac{\varrho_2-1}{\varrho_2-\varrho_1}(\varrho_1-1)\ln(\varrho_1)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}(\varrho_2-1)\ln(\varrho_2)$$
$$-\sum_{r=0}^{n-1}(\nu_2-\nu_1)^{2r}\times\left[\left(\frac{(2r-1)!}{(\nu_1)^{2r}}+\frac{(2r-2)!}{(\nu_1)^{2r-1}}\right)\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)\right.$$

$$+\left(\frac{(2r-1)!}{(\nu_2)^{2r}}+\frac{(2r-2)!}{(\nu_2)^{2r-1}}\right)\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\Bigg],$$

and

$$\frac{q-1}{q+1}\left[\sum_{m=1}^{p-1}q^{m+1}\frac{\big(l_2(q^m)-l_1(q^m)\big)^2}{l_2(q^m)+l_1(q^m)}+\sum_{m=2}^{p}q^{m-1}\frac{\big(l_2(q^m)-l_1(q^m)\big)^2}{l_2(q^m)+l_1(q^m)}\right]$$
$$\leq \frac{\varrho_2-1}{\varrho_2-\varrho_1}\frac{(\varrho_1-1)^2}{\varrho_1+1}+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\frac{(\varrho_2-1)^2}{\varrho_2+1}-\sum_{r=0}^{n-1}4(2r)!(\nu_2-\nu_1)^{2r}$$
$$\times\left[\frac{1}{(\nu_1+1)^{2r+1}}\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)+\frac{1}{(\nu_2+1)^{2r+1}}\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right],$$

where $\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)$ is defined in (40).

5 Zipf-Mandelbrot law

The Zipf-Mandelbrot law is a discrete probability distribution named after the George Kingsley Zipf who proposed a simpler distribution known as Zipf's law, and the mathematician Mandelbrot, who later on generalized it. The 'Zipf-Mandelbrot law' via probability mass function can be given as

$$f(j;p,c,d)=\frac{1}{(r+c)^d\mathcal{H}_{n,c,d}},\quad r=1,\ldots,p, \tag{41}$$

where

$$\mathcal{H}_{p,c,d}=\sum_{i=1}^{p}\frac{1}{(i+c)^d}, \tag{42}$$

$p\in\mathbb{N}, c>0$ and $d\in\mathbb{R}^+$ are parameters.
The Zipf-Mandelbrot entropy defined as follows:

$$\mathcal{Z}(\mathcal{H};c,d)=\frac{1}{\mathcal{H}_{p,c,d}}\sum_{r=1}^{p}\frac{\ln(r+c)}{(r+c)^d}+\ln(\mathcal{H}_{p,c,d}). \tag{43}$$

Use $(l_2)_r=\frac{1}{(r+c)^d\mathcal{H}_{p,c,d}}$ in (37) to obtain

$$\mathcal{Z}(\mathcal{H};c,d)\geq\frac{\varrho_2-1}{\varrho_2-\varrho_1}\log(\varrho_1)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\log(\varrho_2)$$
$$-\frac{1}{2}\left[\sum_{r=0}^{p-1}\frac{1}{(r+c)^d\mathcal{H}_{p,c,d}}\log\big((l_1)_r\big)+\sum_{r=1}^{p}\frac{1}{(r+c)^d\mathcal{H}_{p,c,d}}\log\big((l_1)_r\big)\right]$$
$$-\sum_{r=0}^{p-1}(2r-1)!(\nu_2-\nu_1)^{2r}\times\left[\frac{1}{(\nu_1)^{2r}}\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)\right.$$

$$+\frac{1}{(\nu_2)^{2r}}\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\Bigg], \quad (44)$$

where

$$\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)=\frac{\varrho_2-1}{\varrho_2-\varrho_1}\psi_r\left(\frac{\varrho_1-\nu_1}{\nu_2-\nu_1}\right)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\psi_r\left(\frac{\varrho_2-\nu_1}{\nu_2-\nu_1}\right)$$
$$-\frac{1}{2(\nu_2-\nu_1)}\left[\sum_{r=0}^{p-1}((l_1)_r-\frac{\nu_1}{(r+c)^d\mathcal{H}_{p,c,d}})+\sum_{r=1}^{p}((l_1)_r-\frac{\nu_1}{(r+c)^d\mathcal{H}_{p,c,d}})\right].$$

Remark 9. *Inequality* (44) *provides relationship between Mandelbrot entropy* (43) *and discrete Shannon entropy.*

Use $(l_1)_r=\frac{1}{(r+c_1)^{d_1}\mathcal{H}_{p,c_1,d_1}}$ and $(l_2)_r=\frac{1}{(r+c_2)^{d_2}\mathcal{H}_{p,c_2,d_2}}$ in (38) to obtain

$$\mathcal{Z}(\mathcal{H};c_1,d_1)\geq\frac{d_2}{2\mathcal{H}_{p,c_1,d_1}}\left[\sum_{r=0}^{p-1}\frac{\ln(r+c_2)}{(r+c_1)^{d_1}}+\sum_{r=1}^{p}\frac{\ln(r+c_2)}{(r+c_1)^{d_1}}\right]$$
$$+\ln(\mathcal{H}_{p,c_2,d_2})-\frac{\varrho_2-1}{\varrho_2-\varrho_1}\varrho_1\ln(\varrho_1)-\frac{1-\varrho_1}{\varrho_2-\varrho_1}\varrho_2\ln(\varrho_2)+\sum_{r=0}^{n-1}(2r-2)!(\nu_2-\nu_1)^{2r}$$
$$\times\left[\frac{1}{(\nu_1)^{2r-1}}\digamma_1\left(\psi_r\left(\frac{\nu_2-\mathfrak{t}}{\nu_2-\nu_1}\right)\right)+\frac{1}{(\nu_2)^{2r-1}}\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)\right], \quad (45)$$

where

$$\digamma_1\left(\psi_r\left(\frac{\mathfrak{t}-\nu_1}{\nu_2-\nu_1}\right)\right)=\frac{\varrho_2-1}{\varrho_2-\varrho_1}\psi_r\left(\frac{\varrho_1-\nu_1}{\nu_2-\nu_1}\right)+\frac{1-\varrho_1}{\varrho_2-\varrho_1}\psi_r\left(\frac{\varrho_2-\nu_1}{\nu_2-\nu_1}\right)$$
$$-\frac{1}{2(\nu_2-\nu_1)}\Bigg[\sum_{r=0}^{p-1}(\frac{1}{(r+c_1)^{d_1}\mathcal{H}_{p,c_1,d_1}}-\frac{\nu_1}{(r+c)^d\mathcal{H}_{p,c,d}})$$
$$+\sum_{r=1}^{p}(\frac{1}{(r+c_1)^{d_1}\mathcal{H}_{p,c_1,d_1}}-\frac{\nu_1}{(r+c)^d\mathcal{H}_{p,c,d}})\Bigg].$$

Remark 10. *Inequality* (45) *provides relationship between Mandelbrot entropy* (43) *and discrete Kullback-Leibler divergence:*

Remark 11. *Moreover, use* $(l_1)_r=\frac{1}{(r+c_1)^{d_1}\mathcal{H}_{p,c_1,d_1}}$ *and* $(l_2)_r=\frac{1}{(r+c_2)^{d_2}\mathcal{H}_{p,c_2,d_2}}$ *in (39) to get relationship between Mandelbrot entropy* (43) *and discrete Jeffrey's distance.*

6 Conclusion

In this work, Lidstone polynomial is utilized to prove some inequalities containing divergence measures for diamond integrals. Bounds of different divergence measures are obtained by utilizing particular convex functions. The obtained new findings are the generalisations of the results proved in [3]. It is possible to study inequalities involving different divergence measures via Abel-Gontscharoff interpolation and Montgomery identity on time scales. Which may be included in future task.

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