Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 5(67), No. 1 - 2025, 55-70
https://doi.org/10.31926/but.mif.2025.5.67.1.4

A NOTE ON STANCU OPERATORS WITH THREE PARAMETERS

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Dedicated to the memory of Professor Dimitrie D. Stancu (1927-2014)

Abstract

In this paper, we consider the sequence of positive linear operators $L_{n,r}^{\alpha,\beta}$ depending on three non-negative parameters; an integer r and two reals α and β such that $\alpha \leq \beta$, constructed by Stancu. We consider a Kantorovich-type generalization of Stancu's operators and investigate their convergence properties in L^p-norm. Finally, we observe variation detracting property for Stancu operator and its Kantorovich modification. Moreover, we show that the Stancu operator satisfies an inequality that we call as variation p-detracting, when the attached function is of bounded p-variation in the sense of Riesz.

2000 *Mathematics Subject Classification:* 41A36, 41A25, 26A45.
Key words: Stancu operators, Kantorovich operators, variation detracting property.

1 Introduction

In [17], Stancu constructed the following Bernstein type positive linear operators

$$L_{n,r}^{\alpha,\beta}(f;x) := \sum_{k=0}^{n} w_{n,k,r}(x) f\left(\frac{k+\alpha}{n+\beta}\right), \quad x \in [0,1], \tag{1}$$

for $f \in C[0,1]$, where r is a non-negative integer parameter, $n \in \mathbb{N}$ such that $n > 2r$, α and β are real parameters satisfying $0 \leq \alpha \leq \beta$ and

$$w_{n,k,r}(x) := \begin{cases} (1-x)\, p_{n-r,k}(x); & 0 \leq k < r \\ (1-x)\, p_{n-r,k}(x) + x p_{n-r,k-r}(x); & r \leq k \leq n-r \\ x p_{n-r,k-r}(x); & n-r < k \leq n \end{cases}, \tag{2}$$

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in which $p_{n,k}$ denote the Bernstein's fundamental polynomials given by

$$p_{n,k}(x)=\begin{cases}\binom{n}{k}x^k(1-x)^{n-k}; & 0\le k\le n\\ 0; \quad k<0 \text{ or } k>n\end{cases}, \; x\in[0,1],\ n\in\mathbb{N} \tag{3}$$

It is clear that $L_{n,0}^{0,0}=L_{n,1}^{0,0}=B_n$, where B_n denote the classical Bernstein operators [4]. Moreover, $L_{n,0}^{\alpha,\beta}=L_{n,1}^{\alpha,\beta}=B_n^{\alpha,\beta}$, where $B_n^{\alpha,\beta}$ are the Bernstein-Stancu operators [16]. For some interesting works related to Bernstein-Stancu operators, we refer to [15], [8], [13] and references therein. As Stancu noticed in his paper [17], It is pertinent to mention here that the operators $L_{n,2}^{0,0}$ were constructed by Brass [7]. Gonska [10] referred to $L_{n,r}^{0,0}$ as *Brass-Stancu operators.*

Note that by the definition of fundamental functions (2), Stancu's operators (1) can be expressed as

$$L_{n,r}^{\alpha,\beta}(f;x):=\sum_{k=0}^{n-r}p_{n-r,k}(x)\left[(1-x)f\left(\frac{k+\alpha}{n+\beta}\right)+xf\left(\frac{k+r+\alpha}{n+\beta}\right)\right] \tag{4}$$

and satisfy

$$L_{n,r}^{\alpha,\beta}(f;0)=f\left(\frac{\alpha}{n+\beta}\right) \text{ and } L_{n,r}^{\alpha,\beta}(f;1)=f\left(\frac{n+\alpha}{n+\beta}\right).$$

It is well-known that since Bernstein polynomial are not suitable for approximation of discontinuous functions (see [12, Section 1.9]), Kantorovich [11] replaced the point evaluation functionals with the integral means over small intervals around the knots in the statement of the Bernstein polynomials and constructed what are called as Kantorovich polynomials $K_n:L^1[0,1]\longrightarrow C[0,1]$ given by

$$K_n(f;x)=\sum_{k=0}^{n}p_{n,k}(x)(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}f(t)\,dt, \quad x\in[0,1],\ n\in\mathbb{N}. \tag{5}$$

Lorentz [12] proved L^p-approximation of functions $f\in L^p[0,1]$ by Kantorovich polynomials, where $L^p[0,1]$, $1\le p<\infty$, denotes the space of real-valued measurable and pth power Lebesgue integrable over $[0,1]$ with norm

$$\|f\|_p=\left\{\int_0^1|f(x)|^p\,dx\right\}^{1/p}.$$

Let us denote the class of all absolutely continuous functions on $[a,b]$ by $AC[a,b]$. As is known, every absolutely continuous function is an indefinite integral of its own derivative. Hence, for $f\in AC[0,1]$, which means that $f(x)=f(0)+\int_0^x f'(t)\,dt$ with $f'\in L^1[0,1]$, Bernstein and Kantorovich operators satisfy

$$(B_{n+1}(f;x))'=K_n\left(f';x\right) \tag{6}$$

(see, e.g., [3]).

In [2], Bărbosu modified Bernstein-Stancu operators $B_n^{\alpha,\beta}$ in the sense of Kantorovich as

$$K_n^{\alpha,\beta}(f;x)=\sum_{k=0}^{n}p_{n,k}(x)(n+\beta+1)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}}f(t)\,dt,\quad x\in[0,1],\ n\in\mathbb{N}, \tag{7}$$

for $f\in L^1[0,1]$, where $0\le\alpha\le\beta$. Here, the author obtained uniform approximation of $f\in C[0,1]$ by the sequence of the operators $K_n^{\alpha,\beta}(f)$ on $[0,1]$ and gave some estimates for the rate of convergence. Çetin et al. [9] constructed and studied the following operators what are called as *Stancu-Kantorovich operators* by the authors:

$$K_{n,r}^{\alpha,\beta}(f;x)=\sum_{k=0}^{n}w_{n,k,r}(x)(n+\beta+1)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}}f(t)\,dt \tag{8}$$

for $f\in L^1[0,1]$, $x\in[0,1]$ and $n\in\mathbb{N}$. Let $e_v(t):=t^v$, $t\in[0,1]$, $v=0,1,2$. Below, we reproduce the first three moments of $K_{n,r}^{\alpha,\beta}$ from their work:

$$\begin{aligned}
K_{n,r}^{\alpha,\beta}(e_0;x) &= 1,\\
K_{n,r}^{\alpha,\beta}(e_1;x) &= \frac{nx}{n+\beta+1}+\frac{2\alpha+1}{2(n+\beta+1)},\\
K_{n,r}^{\alpha,\beta}(e_2;x) &= \frac{n^2}{(n+\beta+1)^2}\left[x^2+\left(1+\frac{r(r-1)}{n}\right)\frac{x(1-x)}{n}\right]\\
&\quad+\frac{n(2\alpha+1)x}{(n+\beta+1)^2}+\frac{3\alpha^2+2\alpha+1}{3(n+\beta+1)}.
\end{aligned}$$

The case for $K_{n,r}^{0,0}$ was studied in [6].

Unlike the relation between Bernstein and Kantorovich polynomials given by (6), it doesn't hold that

$$\left(L_{n+1,r}^{\alpha,\beta}(f;x)\right)'=K_{n,r}^{\alpha,\beta}(f';x)$$

for Stancu operators and their Kantorovich variant $K_{n,r}^{\alpha,\beta}$ for $f\in AC[0,1]$. However, for our purposes, in this study we consider an alternative Kantorovich-type generalization of the Stancu operators.

The paper is organized as follows:

In section 2; we consider a Kantorovich-type generalization $\boldsymbol{K}_{n,r}^{\alpha,\beta}$ of the Stancu operators, see (9), and observe L^p-approximation properties of these operators.

In section 3, We explore variation detracting property of the Stancu operator $L_{n,r}^{\alpha,\beta}$ and for the Stancu-Kantorovich operator $K_{n,r}^{\alpha,\beta}$ given by (8). Moreover, by using Riesz' definition of function of bounded p-variation, we obtain an estimate, what we call as *p-variation detracting property*, for the Stancu operator.

2 Kantorovich-type generalization of Stancu operators

As in the case of Kantorovich operators [12, p.30], by taking into account of the indefinite integral $F(x) = \int_0^x f(t)\,dt$ of $f \in L^1[0,1]$ in (4) and differentiating $L_{n+1,r}^{\alpha,\beta}(F;x)$ with respect to x, we get the following operators

$$\begin{aligned}
&\left(L_{n+1,r}^{\alpha,\beta}(F;x)\right)' \\
= &\sum_{k=0}^{n-r} p_{n-r,k}(x)(n+1-r)\left((1-x)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(t)\,dt + x\int_{\frac{k+r+\alpha}{n+\beta+1}}^{\frac{k+r+\alpha+1}{n+\beta+1}} f(t)\,dt\right) \\
&+\sum_{k=0}^{n+1-r} p_{n+1-r,k}(x)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+r+\alpha}{n+\beta+1}} f(t)\,dt \\
= &: \boldsymbol{K}_{n,r}^{\alpha,\beta}(f;x) \qquad (9)
\end{aligned}$$

for $x \in [0,1]$, $0 \le \alpha \le \beta$, $r \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$ such that $n > 2r$. The operators $\boldsymbol{K}_{n,r}^{\alpha,\beta}(f;x)$ can be regarded as a Kantorovich-type generalization of the Stancu operators, which involve the Stancu-Kantorovich operators $K_{n,r}^{\alpha,\beta}$ given by (8). Indeed, by (2), one has

$$\boldsymbol{K}_{n,r}^{\alpha,\beta}(f;x) = \frac{n+1-r}{n+\beta+1} K_{n,r}^{\alpha,\beta}(f;x) + \sum_{k=0}^{n+1-r} p_{n+1-r,k}(x)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+r+\alpha}{n+\beta+1}} f(t)\,dt. \qquad (10)$$

Thus, for $f \in AC[0,1]$ it holds that

$$\left(L_{n+1,r}^{\alpha,\beta}(f;x)\right)' = \boldsymbol{K}_{n,r}^{\alpha,\beta}\left(f';x\right) \qquad (11)$$

for the Stancu operators (4) and their Kantorovich-type generalization (9).

The operators $\boldsymbol{K}_{n,r}^{\alpha,\beta}$ are positive and linear and their construction is useful for the approximation of derivatives by the Stancu operators as well as for investigation of their variation detracting property. In particular, for the special case $\boldsymbol{K}_{n,r}^{0,0}$; we refer to [18] for approximation of derivatives and to [6] for variation detracting results, respectively.

It is easy to see that

$$\boldsymbol{K}_{n,0}^{\alpha,\beta} = \boldsymbol{K}_{n,1}^{\alpha,\beta} = K_{n,0}^{\alpha,\beta} = K_{n,1}^{\alpha,\beta} = K_n^{\alpha,\beta},$$

where $K_n^{\alpha,\beta}$ are the Kantorovich modification of the Bernstein-Stancu operators given by [2]. It is obvious that when $r = 0$ the second sum in (9) disappears and

$K_{n,0}^{0,0}=K_{n,1}^{0,0}=K_n$, where K_n are the well-known Kantorovich operators (5). Note that since in the cases $r=0$ and $r=1$; $L_{n,r}^{\alpha,\beta}$ reduces to the Bernstein-Stancu operator $B_n^{\alpha,\beta}$ [16], and similarly, $\boldsymbol{K}_{n,r}^{\alpha,\beta}$ reduces to $K_n^{\alpha,\beta}$ given by (7), we can take $r\in\mathbb{N}$ in the sequel without lose of generality.

2.1 L^p-Approximation properties

In this part, as in Kantorovich operators K_n (see [1]) we show that for every $f\in L^p[0,1]$, each $\boldsymbol{K}_{n,r}^{\alpha,\beta}$ is a bounded operator, mapping $L^p[0,1]$ into itself. Here, we follow the similar arguments used in [6] for $\boldsymbol{K}_{n,r}^{0,0}$. But, we obtain a finer upper bound for the norm of the operator $\boldsymbol{K}_{n,r}^{\alpha,\beta}$.

Theorem 1. *If $f\in L^p[0,1]$, $1\le p<\infty$, $0\le\alpha\le\beta$, $r\in\mathbb{N}$ is a fixed integer and $n\in\mathbb{N}$ is such that $n>2r$, then we have*

$$\left\|\boldsymbol{K}_{n,r}^{\alpha,\beta}\right\|\le 1,$$

where $\left\|\boldsymbol{K}_{n,r}^{\alpha,\beta}\right\|$ denotes the operator norm of $\boldsymbol{K}_{n,r}^{\alpha,\beta}$.

Proof. It is clear that $\boldsymbol{K}_{n,r}^{\alpha,\beta}(f;x)$, given by (9), can be written as

$$\begin{aligned}
&\boldsymbol{K}_{n,r}^{\alpha,\beta}(f;x)\\
=\ &\left(1-\frac{r}{n+1}\right)\sum_{k=0}^{n-r}p_{n-r,k}(x)(n+1)\left((1-x)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}}f(t)\,dt+x\int_{\frac{k+r+\alpha}{n+\beta+1}}^{\frac{k+r+1+\alpha}{n+\beta+1}}f(t)\,dt\right)\\
&+\frac{r}{n+1}\sum_{k=0}^{n+1-r}p_{n+1-r,k}(x)\left(\frac{n+1}{r}\right)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+r+\alpha}{n+\beta+1}}f(t)\,dt.
\end{aligned}$$

Here, noting that $\varphi(t)=|t|^p$, $1\le p<\infty$, $t\in[0,1]$, is convex, applying Jensen's inequality, since

$$\left|\int_{x_1}^{x_2}f(t)\,dt\right|^p\le(x_2-x_1)^{p-1}\int_{x_1}^{x_2}|f(t)|^p\,dt,$$

where $x_1, x_2 \in [0,1]$, we find

$$\begin{aligned}
&\left|\boldsymbol{K}_{n,r}^{\alpha,\beta}(f;x)\right|^p \\
\leq\ & \left(1-\frac{r}{n+1}\right)\sum_{k=0}^{n-r} p_{n-r,k}(x)\,(n+1)^p \\
&\times\left|(1-x)\int\limits_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(t)\,dt + x\int\limits_{\frac{k+r+\alpha}{n+\beta+1}}^{\frac{k+r+1+\alpha}{n+\beta+1}} f(t)\,dt\right|^p \\
&+\frac{r}{n+1}\sum_{k=0}^{n+1-r} p_{n+1-r,k}(x)\left(\frac{n+1}{r}\right)^p\left|\int\limits_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+r+\alpha}{n+\beta+1}} f(t)\,dt\right|^p \\
\leq\ & (n+1-r)\sum_{k=0}^{n-r} p_{n-r,k}(x)\left(\frac{n+1}{n+\beta+1}\right)^{p-1} \\
&\times\left((1-x)\int\limits_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} |f(t)|^p\,dt + x\int\limits_{\frac{k+r+\alpha}{n+\beta+1}}^{\frac{k+r+1+\alpha}{n+\beta+1}} |f(t)|^p\,dt\right) \\
&+\sum_{k=0}^{n+1-r} p_{n+1-r,k}(x)\left(\frac{n+1}{n+\beta+1}\right)^{p-1}\int\limits_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+r+\alpha}{n+\beta+1}} |f(t)|^p\,dt.
\end{aligned}$$

Since $\frac{n+1}{n+\beta+1} \leq 1$, it readily follows that $\left(\frac{n+1}{n+\beta+1}\right)^p \leq \frac{n+1}{n+\beta+1}$ for $p \geq 1$. Therefore, we can take $\left(\frac{n+1}{n+\beta+1}\right)^{p-1} \leq 1$ in the last formula and obtain that

$$\begin{aligned}
&\left|\boldsymbol{K}_{n,r}^{\alpha,\beta}(f;x)\right|^p \\
\leq\ & (n+1-r)\sum_{k=0}^{n-r} p_{n-r,k}(x)\left((1-x)\int\limits_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} |f(t)|^p\,dt + x\int\limits_{\frac{k+r+\alpha}{n+\beta+1}}^{\frac{k+r+1+\alpha}{n+\beta+1}} |f(t)|^p\,dt\right) \\
&+\sum_{k=0}^{n+1-r} p_{n+1-r,k}(x)\int\limits_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+r+\alpha}{n+\beta+1}} |f(t)|^p\,dt. \qquad (12)
\end{aligned}$$

Integrating (12) over $[0,1]$, using the well-known beta integral, we get

$$\begin{aligned}
&\int_0^1 \left|\boldsymbol{K}_{n,r}^{\alpha,\beta}(f;x)\right|^p dx \\
\leq\ & \frac{1}{n-r+2}\sum_{k=0}^{n-r}\left((n-r-k+1)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}}|f(t)|^p\,dt+(k+1)\int_{\frac{k+r+\alpha}{n+\beta+1}}^{\frac{k+r+1+\alpha}{n+\beta+1}}|f(t)|^p\,dt\right) \\
& +\frac{1}{n-r+2}\sum_{k=0}^{n+1-r}\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+r+\alpha}{n+\beta+1}}|f(t)|^p\,dt \\
=\ & : I_1+I_2. \qquad (13)
\end{aligned}$$

Since $n-r>r$, for I_1, we have

$$\begin{aligned}
I_1 &= \frac{1}{n-r+2}\left(\sum_{k=0}^{r-1}+\sum_{k=r}^{n-r}\right)(n-r-k+1)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}}|f(t)|^p\,dt \\
&\quad +\frac{1}{n-r+2}\left(\sum_{k=0}^{n-2r}+\sum_{k=n-2r+1}^{n-r}\right)(k+1)\int_{\frac{k+r+\alpha}{n+\beta+1}}^{\frac{k+r+1+\alpha}{n+\beta+1}}|f(t)|^p\,dt.
\end{aligned}$$

Replacing k with $k-r$ in the last two sums in I_1, we find

$$\begin{aligned}
I_1 &= \frac{1}{n-r+2}\left[\sum_{k=0}^{r-1}(n-r-k+1)+\sum_{k=r}^{n-r}(n-2r+2)+\sum_{k=n-r+1}^{n}(k-r+1)\right] \\
&\quad \times\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}}|f(t)|^p\,dt. \qquad (14)
\end{aligned}$$

On the other hand, for I_2, one has

$$
\begin{aligned}
I_2 &= \frac{1}{n-r+2}\sum_{k=0}^{n+1-r}\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+r+\alpha}{n+\beta+1}}|f(t)|^p\,dt \\
&= \frac{1}{n-r+2}\sum_{k=0}^{n+1-r}\left(\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}}+\int_{\frac{k+\alpha+1}{n+\beta+1}}^{\frac{k+\alpha+2}{n+\beta+1}}+\cdots+\int_{\frac{k+\alpha+r-1}{n+\beta+1}}^{\frac{k+\alpha+r}{n+\beta+1}}\right)|f(t)|^p\,dt \\
&= \frac{1}{n-r+2}\sum_{k=0}^{n+1-r}\sum_{i=0}^{r-1}\int_{\frac{k+\alpha+i}{n+\beta+1}}^{\frac{k+\alpha+1+i}{n+\beta+1}}|f(t)|^p\,dt \\
&= \frac{1}{n-r+2}\sum_{i=0}^{r-1}\sum_{k=0}^{n+1-r}\int_{\frac{k+\alpha+i}{n+\beta+1}}^{\frac{k+\alpha+1+i}{n+\beta+1}}|f(t)|^p\,dt. \qquad (15)
\end{aligned}
$$

Now, in the last line of (15); replacing k with $k-i$ and changing the order of the summations, it readily follows that

$$
\begin{aligned}
I_2 &= \frac{1}{n-r+2}\sum_{i=0}^{r-1}\sum_{k=i}^{n-r+1+i}\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}}|f(t)|^p\,dt \\
&= \frac{1}{n-r+2}\left(\sum_{k=0}^{r-1}\sum_{i=0}^{k}+\sum_{k=r}^{n-r}\sum_{i=0}^{r-1}+\sum_{k=n-r+1}^{n}\sum_{i=k-n+r-1}^{r-1}\right)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}}|f(t)|^p\,dt \\
&= \frac{1}{n-r+2}\left[\sum_{k=0}^{r-1}(k+1)+\sum_{k=r}^{n-r}r+\sum_{k=n-r+1}^{n}(n-k+1)\right] \\
&\quad\times\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}}|f(t)|^p\,dt. \qquad (16)
\end{aligned}
$$

Substituting (14) and (16) into (13), we arrive at

$$\begin{aligned}\int_0^1 \left|\boldsymbol{K}_{n,r}^{\alpha,\beta}(f;x)\right|^p dx &\leq \left(\sum_{k=0}^{r-1}+\sum_{k=r}^{n-r}+\sum_{k=n-r+1}^{n}\right)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} |f(t)|^p\,dt \\ &= \sum_{k=0}^{n}\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} |f(t)|^p\,dt \\ &= \int_{\frac{\alpha}{n+\beta+1}}^{\frac{n+\alpha+1}{n+\beta+1}} |f(t)|^p\,dt \leq \int_0^1 |f(t)|^p\,dt.\end{aligned}$$

Consequently, passing to L^p-norm, we obtain $\left\|\boldsymbol{K}_{n,r}^{\alpha,\beta}(f)\right\|_p \leq \|f\|_p$ for every $f \in L^p[0,1]$, which completes the proof. □

Remark 1. *Taking into account of (8), the first three moments of the operators* $\boldsymbol{K}_{n,r}^{\alpha,\beta}$ *can be obtained from (10) by direct substitution. Using density of* $C[0,1]$ *in* $L^p[0,1]$*, and the well-known Lusin theorem, we obtain* $\lim_{n\to\infty}\left\|\boldsymbol{K}_{n,r}^{\alpha,\beta}(f)-f\right\|_p = 0$ *for* $f \in L^p[0,1]$ *and for a fixed* $r \in \mathbb{N}$.

3 Variation detracting property

Let, as usual, $V_{[0,1]}[f]$ denote the total variation of the function of f . Also, let $TV[0,1]$ denote the class of all functions of bounded variation on $[0,1]$ with seminorm $\|f\|_{TV[0,1]} := V_{[0,1]}[f]$. As is well-known, Lorentz [12] proved that each Bernstein operator satisfies the inequality

$$V_{[0,1]}[B_n(f)] \leq V_{[0,1]}[f] \text{ for } f \in TV[0,1],$$

which is called as variation detracting property. Öksüzer et al. [13] obtained that Bernstein-Stancu operator $B_n^{\alpha,\beta} = L_{n,0}^{\alpha,\beta} = L_{n,1}^{\alpha,\beta}$ is also variation detracting.

In this section, we show that each Stancu operator $L_{n,r}^{\alpha,\beta}$ and its Kantorovich modification $K_{n,r}^{\alpha,\beta}$ given by (8) are variation detracting as well.

Theorem 2. *If* $f \in TV[0,1]$, $r \in \mathbb{N}$ *is fixed and* $0 \leq \alpha \leq \beta$, *then*

$$V_{[0,1]}\left[L_{n,r}^{\alpha,\beta}(f)\right] \leq V_{[0,1]}[f]$$

for every $n \in \mathbb{N}$ *such that* $n > 2r$.

Proof. Suppose that $r \in \mathbb{N}$ is fixed and $f \in TV[0,1]$. Then, since $L_{n,r}^{\alpha,\beta}(f;x)$ is continuous on $[0,1]$ and $\left(L_{n,r}^{\alpha,\beta}(f;x)\right)'$ is bounded on $(0,1)$, we have $L_{n,r}^{\alpha,\beta}(f;x)$ is absolutely continuous on $[0,1]$. Differentiating (4), we get

$$\begin{aligned}
&\left(L_{n,r}^{\alpha,\beta}(f;x)\right)' \\
= \;& (n-r)\sum_{k=0}^{n-1-r} p_{n-1-r,k}(x)\left\{(1-x)\left[f\left(\frac{k+1+\alpha}{n+\beta}\right)-f\left(\frac{k+\alpha}{n+\beta}\right)\right]\right. \\
&\left.+x\left[f\left(\frac{k+r+1+\alpha}{n+\beta}\right)-f\left(\frac{k+r+\alpha}{n+\beta}\right)\right]\right\} \\
&+\sum_{k=0}^{n-r} p_{n-r,k}(x)\left[f\left(\frac{k+r+\alpha}{n+\beta}\right)-f\left(\frac{k+\alpha}{n+\beta}\right)\right]. \qquad (17)
\end{aligned}$$

Since the total variation of an absolutely continuous function is the integral of the absolute value of its derivative, one has

$$V_{[0,1]}\left[L_{n,r}^{\alpha,\beta}(f)\right]=\int_0^1\left|\left(L_{n,r}^{\alpha,\beta}(f;x)\right)'\right|dx.$$

Making use of (17) in the last formula, using beta integral, we find

$$\begin{aligned}
&V_{[0,1]}\left[L_{n,r}^{\alpha,\beta}(f)\right]\le\frac{1}{n-r+1}\sum_{k=0}^{n-1-r}\left[(n-r-k)\left|f\left(\frac{k+1+\alpha}{n+\beta}\right)-f\left(\frac{k+\alpha}{n+\beta}\right)\right|\right. \\
&\left.+(k+1)\left|f\left(\frac{k+r+1+\alpha}{n+\beta}\right)-f\left(\frac{k+r+\alpha}{n+\beta}\right)\right|\right] \\
&+\frac{1}{n-r+1}\sum_{k=0}^{n-r}\sum_{i=0}^{r-1}\left|f\left(\frac{k+\alpha+i+1}{n+\beta}\right)-f\left(\frac{k+\alpha+i}{n+\beta}\right)\right|. \qquad (18)
\end{aligned}$$

Proceeding as in the evaluation of the summations in (14) and (16) in the proof of Theorem 1, if we decompose the first and second sums in (18) into three sums, we arrive at

$$\begin{aligned}
&V_{[0,1]}\left[L_{n,r}^{\alpha,\beta}(f)\right] \\
\le\;& \sum_{k=0}^{n-1}\left|f\left(\frac{k+1+\alpha}{n+\beta}\right)-f\left(\frac{k+\alpha}{n+\beta}\right)\right| \\
\le\;& \left|f\left(\frac{\alpha}{n+\beta}\right)-f(0)\right|+\sum_{k=0}^{n-1}\left|f\left(\frac{k+1+\alpha}{n+\beta}\right)-f\left(\frac{k+\alpha}{n+\beta}\right)\right| \\
&+\left|f(1)-f\left(\frac{n+\alpha}{n+\beta}\right)\right| \\
\le\;& V_{[0,1]}[f] \qquad (19)
\end{aligned}$$

for $0\le\alpha\le\beta$. According to the special choices of α and β, we encounter with the following cases for the particular partition of $[0,1]$ in (19):

(i) For the case $\alpha=\beta=0$, the terms $\left|f\left(\frac{\alpha}{n+\beta}\right)-f(0)\right|$ and $\left|f(1)-f\left(\frac{n+\alpha}{n+\beta}\right)\right|$ disappear (see [6]).

(ii) In the case $0=\alpha<\beta$, the term $\left|f\left(\frac{\alpha}{n+\beta}\right)-f(0)\right|$ disappears.

(iii) In the case $\alpha>0$ and $\alpha=\beta$, the term $\left|f(1)-f\left(\frac{n+\alpha}{n+\beta}\right)\right|$ disappears.

This completes the proof. □

Next result shows that each Stancu-Kantorovich operator $K_{n,r}^{\alpha,\beta}$ given by (8) is also variation detracting. Note that the same result for $K_{n,r}^{0,0}$ was given in [5].

Theorem 3. *If $f\in TV[0,1]$, $r\in\mathbb{N}$ is fixed and $0\le\alpha\le\beta$, then*

$$V_{[0,1]}\left[K_{n,r}^{\alpha,\beta}(f)\right]\le V_{[0,1]}[f] \tag{20}$$

for every $n\in\mathbb{N}$ such that $n>2r$.

Proof. Suppose that $r\in\mathbb{N}$ be fixed and $f\in TV[0,1]$. Then, we have $K_{n,r}^{\alpha,\beta}(f)\in AC[0,1]$. As in [3, Proposition 3.3], we set

$$F_{n,k}^{\alpha,\beta}:=(n+\beta+1)\int\limits_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+1+\alpha}{n+\beta+1}} f(t)\,dt=\int\limits_0^1 f\left(\frac{k+\alpha+u}{n+\beta+1}\right)du,\quad 0\le k\le n.$$

By the definition of Stancu's fundamental polynomials (2), $K_{n,r}^{\alpha,\beta}(f;x)$ can be expressed as

$$K_{n,r}^{\alpha,\beta}(f;x)=\sum_{k=0}^{n-r}p_{n-r,k}(x)\left[(1-x)F_{n,k}^{\alpha,\beta}+xF_{n,k+r}^{\alpha,\beta}\right].$$

Therefore, proceeding as in (17), we have

$$\begin{aligned}&\left(K_{n,r}^{\alpha,\beta}(f;x)\right)'\\&=(n-r)\sum_{k=0}^{n-r-1}p_{n-r-1,k}(x)\left\{(1-x)\left[F_{n,k+1}^{\alpha,\beta}-F_{n,k}^{\alpha,\beta}\right]+x\left[F_{n,k+r+1}^{\alpha,\beta}-F_{n,k+r}^{\alpha,\beta}\right]\right\}\\&+\sum_{k=0}^{n-r}p_{n-r,k}(x)\left[F_{n,k+r}^{\alpha,\beta}-F_{n,k}^{\alpha,\beta}\right].\end{aligned} \tag{21}$$

Reasoning as in the proof of Theorem 2, from (21), we find

$$\begin{aligned} & V_{[0,1]}\left[K_{n,r}^{\alpha,\beta}(f)\right] \\ = & \int_0^1 \left|\left(K_{n,r}^{\alpha,\beta}(f;x)\right)'\right| dx \\ \leq & \frac{1}{n-r+1}\sum_{k=0}^{n-r-1}\left\{(n-r-k)\left|F_{n,k+1}^{\alpha,\beta}-F_{n,k}^{\alpha,\beta}\right|+(k+1)\left|F_{n,k+r+1}^{\alpha,\beta}-F_{n,k+r}^{\alpha,\beta}\right|\right\} \\ & +\frac{1}{n-r+1}\sum_{k=0}^{n-r}\sum_{i=0}^{r-1}\left|F_{n,k+1+i}^{\alpha,\beta}-F_{n,k+i}^{\alpha,\beta}\right| \end{aligned} \tag{22}$$

Now, again, proceeding as in the formulas for (14) and (16) in the proof of Theorem 1, from (22), one has

$$V_{[0,1]}\left[K_{n,r}^{\alpha,\beta}(f)\right] \leq \sum_{k=0}^{n-1}\left|F_{n,k+1}^{\alpha,\beta}-F_{n,k}^{\alpha,\beta}\right|.$$

Now, it remains only to show that $\sum\limits_{k=0}^{n-1}\left|F_{n,k+1}^{\alpha,\beta}-F_{n,k}^{\alpha,\beta}\right| \leq V_{[0,1]}[f]$, which is obtained by using similar argument to the proof of in [3, Proposition 3.3]. Indeed, setting $u_{-1}^{\alpha,\beta} := 0$, $u_k^{\alpha,\beta} := \frac{k+\alpha+u}{n+\beta+1}$, $0 \leq k \leq n$, and $u_{n+1}^{\alpha,\beta} := 1$, and noting that $\left\{u_k^{\alpha,\beta}\right\}_{k=-1}^{n+1}$ is a particular partition of $[0,1]$, it readily follows that

$$\begin{aligned} \sum_{k=0}^{n-1}\left|F_{n,k+1}^{\alpha,\beta}-F_{n,k}^{\alpha,\beta}\right| & \leq \int_0^1 \sum_{k=0}^{n-1}\left|f\left(u_{k+1}^{\alpha,\beta}\right)-f\left(u_k^{\alpha,\beta}\right)\right| du \\ & \leq \int_0^1 \sum_{k=-1}^{n}\left|f\left(u_{k+1}^{\alpha,\beta}\right)-f\left(u_k^{\alpha,\beta}\right)\right| du \\ & \leq V_{[0,1]}[f], \end{aligned}$$

which verifies (20), and completes the proof. □

4 p-Variation detracting

In [14], Riesz introduced the concept of bounded p-variation ($1 \leq p < \infty$) for a function $f:[a,b]\to\mathbb{R}$ as

$$V_{[a,b]}^p[f] := \sup\left\{\sum_{k=0}^{n-1}\frac{\left|f(x_{k+1})-f(x_k)\right|^p}{(x_{k+1}-x_k)^{p-1}} : \{x_k\}_{k=0}^n \in \mathcal{P}[a,b]\right\} < \infty,$$

where $\mathcal{P}[a,b]$ denotes the set of all possible partitions $\{x_k\}_{k=0}^n$ of $[a,b]$ such that $a = x_0 < x_1 < \cdots < x_n = b$. Let us denote the class of functions of bounded p-variation on $[a,b]$ by $TV_p[a,b]$. In the case $1 < p < \infty$, Riesz proved that

$$f \in TV_p[a,b] \Longleftrightarrow f \in AC[a,b] \text{ and } f' \in L^p[a,b] \tag{23}$$

and that

$$V^p_{[a,b]}[f] = \|f'\|^p_p. \tag{24}$$

In this part, we show that the Stancu operator is p-variation detracting, namely we have

Theorem 4. *If $f \in TV_p[0,1]$, $1 < p < \infty$, $r \in \mathbb{N}$ is fixed and $0 \le \alpha \le \beta$, then*

$$V^p_{[0,1]}\left[L^{\alpha,\beta}_{n,r}(f)\right] \le V^p_{[0,1]}[f]$$

for every $n \in \mathbb{N}$ such that $n > 2r$.

Proof. Suppose that $f \in TV_p[0,1]$. Then, we have $f \in AC[a,b]$ and $f' \in L^p[a,b]$ by (23). Hence, we immediately have $L^{\alpha,\beta}_{n,r}(f;x) \in AC[0,1]$ and

$$\left(L^{\alpha,\beta}_{n,r}(f;x)\right)' = \boldsymbol{K}^{\alpha,\beta}_{n-1,r}(f';x) \in L^p[0,1]$$

by (11) and Theorem 1. Thus, from (23), we have $L^{\alpha,\beta}_{n,r}(f) \in TV_p[0,1]$. In view of (17) and (24), we get

$$V^p_{[0,1]}\left[L^{\alpha,\beta}_{n,r}(f)\right] = \int_0^1 \left|\left(L^{\alpha,\beta}_{n,r}(f;x)\right)'\right|^p dx. \tag{25}$$

Now, as in Theorem 1, from Jensen's inequality, we get

$$\begin{aligned}
&\left|\left(L^{\alpha,\beta}_{n,r}(f;x)\right)'\right|^p \\
\le\ & \left(1-\frac{r}{n}\right)\sum_{k=0}^{n-1-r} p_{n-1-r,k}(x)\,n^p \\
&\times\left|(1-x)\left[f\left(\frac{k+1+\alpha}{n+\beta}\right)-f\left(\frac{k+\alpha}{n+\beta}\right)\right]\right. \\
&\left.+x\left[f\left(\frac{k+r+1+\alpha}{n+\beta}\right)-f\left(\frac{k+r+\alpha}{n+\beta}\right)\right]\right|^p \\
&+\frac{r}{n}\sum_{k=0}^{n-r} p_{n-r,k}(x)\,n^p\left|\sum_{i=0}^{r-1}\frac{1}{r}\left[f\left(\frac{k+i+1+\alpha}{n+\beta}\right)-f\left(\frac{k+i+\alpha}{n+\beta}\right)\right]\right|^p \\
\le\ & n^{p-1}\left\{(n-r)\sum_{k=0}^{n-1-r} p_{n-1-r,k}(x)\left[(1-x)\left|f\left(\frac{k+1+\alpha}{n+\beta}\right)-f\left(\frac{k+\alpha}{n+\beta}\right)\right|^p\right.\right. \\
&\left.+x\left|f\left(\frac{k+r+1+\alpha}{n+\beta}\right)-f\left(\frac{k+r+\alpha}{n+\beta}\right)\right|^p\right] \\
&\left.+\sum_{i=0}^{r-1}\sum_{k=0}^{n-r} p_{n-r,k}(x)\left|f\left(\frac{k+i+1+\alpha}{n+\beta}\right)-f\left(\frac{k+i+\alpha}{n+\beta}\right)\right|^p\right\},
\end{aligned} \tag{26}$$

where, by means of the fact that $\sum\limits_{i=0}^{r-1}\frac{1}{r}=1$, we have used Jensen's inequality for the last sum. Now, taking into account the fact that $n^{p-1}\leq(n+\beta)^{p-1}$ for $p>1$ and $\beta\geq0$, and integrating (26) over $[0,1]$, using beta integral, from (25) we reach to

$$\begin{aligned}
&V_{[0,1]}^{p}\left[L_{n,r}^{\alpha,\beta}(f)\right]\\
\leq\ &\frac{(n+\beta)^{p-1}}{n-r+1}\left\{\sum_{k=0}^{n-1-r}\left[(n-r-k)\left|f\left(\frac{k+1+\alpha}{n+\beta}\right)-f\left(\frac{k+\alpha}{n+\beta}\right)\right|^{p}\right.\right.\\
&\left.+(k+1)\left|f\left(\frac{k+r+1+\alpha}{n+\beta}\right)-f\left(\frac{k+r+\alpha}{n+\beta}\right)\right|^{p}\right]\\
&\left.+\sum_{i=0}^{r-1}\sum_{k=0}^{n-r}\left|f\left(\frac{k+i+1+\alpha}{n+\beta}\right)-f\left(\frac{k+i+\alpha}{n+\beta}\right)\right|^{p}\right\}.
\end{aligned}$$

Again, reasoning exactly as in the evaluation for the summations in (14) and (16) in Theorem 1, we conclude

$$\begin{aligned}
&V_{[0,1]}^{p}\left[L_{n,r}^{\alpha,\beta}(f)\right]\leq\\
\leq\ &\sum_{k=0}^{n-1}(n+\beta)^{p-1}\left|f\left(\frac{k+1+\alpha}{n+\beta}\right)-f\left(\frac{k+\alpha}{n+\beta}\right)\right|^{p}\\
\leq\ &(n+\beta)^{p-1}\left\{\left|f\left(\frac{\alpha}{n+\beta}\right)-f(0)\right|^{p}\alpha^{1-p}\right.\\
&\left.+\sum_{k=0}^{n-1}\left|f\left(\frac{k+1+\alpha}{n+\beta}\right)-f\left(\frac{k+\alpha}{n+\beta}\right)\right|^{p}+\left|f(1)-f\left(\frac{n+\alpha}{n+\beta}\right)\right|^{p}(\beta-\alpha)^{1-p}\right\}\\
\leq\ &V_{[0,1]}^{p}[f],
\end{aligned}$$

which completes the proof.

□

We note here that in the submission version of the paper, the formula (26) was evaluated by multiplying-dividing the term $(n+\beta)^{\ p}$. The anonymous referee provided a supplementary remark in his/her report that we adopted and used in the current version, which made the formula simpler.

Acknowledgement *The author would like to thank the anonymous reviewer for his/her helpful comments that improved the quality of the manuscript.*

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