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PROJECTILE MOTION UNDER QUADRATIC AIR RESISTENCE VIA MULTISTEP-DIFFERENTIAL TRANSFORMATION METHOD

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Abstract

The projectile motion system was examined in this study when air resistance was present (we only looked at the situation of quadratic air resistance). First, we used Newtonian mechanics to derive the equation governing the system. Second, the study focused on the projectile motion's characteristics, namely its maximum height and maximum horizontal range. Additionally, we used simulation plots to apply the Differential Transform Method (DTM) and the Multistep Differential Transform Method (Ms-DTM) to solve the equations found. This allowed us to discuss the projectile's trajectory. We computed the flying time and the maximum range by comparing our findings with experimental data from published research. In the end, the absolute errors were computed to find the optimal numerical approach to solve this problem.

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Key words: projectile motion, quadratic air resistance, differential transform method (DTM), multistep differential transform method (Ms-DTM).

1 Introduction

Projectile motion has numerous real-world uses, including conflicts and sports (particularly baseball, tennis, javelin throw, and many others) [20]. The equations governing projectile motion systems can be set up using either Lagrangian or Newtonian mechanics. The effect of air resistance is typically ignored in undergraduate texts, although it must be included in order to produce more accurate

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results. We look for numerical methods to solve the resulting equations because they cannot be solved analytically in the case of air resistance.

Galileo was the first scientist to study projectile motion, and he held that gravity was the only factor that could effectively affect a projectile [7]. Unlike Galileo, Harriot [12, 19] made an effort to create models that take into consideration how air resistance affects projectiles. Drawing from the firsthand knowledge of rescuers and artillerymen during his day, Harriot postulated that the asymmetry resulting from the impact of air was a common feature of all feasible trajectories.

Furthermore, by using Newtonian mechanics, differential equations that describe the projectiles' motion—with or without air resistance—can be obtained. These equations can be solved precisely in the situation of no air resistance to determine certain projectile characteristics including time of flight, maximum height, and maximum horizontal range. However, adding air resistance leads to more complex equations, particularly for quadratic air resistance, and calculating them analytically is no longer as simple as it once was. After translating the equations to polar coordinates as in the study [21], some researchers choose to solve these equations numerically, for example, using the Lambert W function. A more thorough analysis of projectile motion with squared air resistance was later provided [22], with a particular emphasis on low-angle trajectory approximation. Apart from the aforementioned references, there exist several works that have been conducted utilizing diverse approaches; for instance, one may consult [1, 3–5, 8, 10, 11, 14, 18] and the references included therein.

In this work, we focus our research on quadratic air resistance and its effect on projectile motion. We have organized the remainder of the paper as follows: Part 2 presents a mathematical description of projectile motion under quadratic air resistance. In Part 3, the system describing the motion of the bullet is solved by the differential transform method (DTM) and the multi-step differential transform method (Ms-DTM). Some numerical results with discussions are presented in Sect. 4. Finally, the paper was concluded with the conclusion in Section 5.

2 Mathematical modeling of the projectile motion under the influence of air resistance

Consider a projectile of mass (m) projected in space (in two dimensions); one along the horizontal axis ($x-axis$) and the other along the vertical axis ($y-axis$). Assuming the launching angle with horizontal range is θ. In addition, we choose the projection position to be ($x_o = 0, y_o$)and initial velocity be $\vec{v}_o$ at a time$t = 0$. Thus, we can write:

$$\vec{v}_o = v_{xo}\hat{i} + v_{yo}\hat{j}. \tag{1}$$

where $v_{xo} = v_o \cos\theta$, and $v_{yo} = v_o \sin\theta$

This motion ends when$y = y_o$ with$t \neq 0$, this means that projectile returns back to the same level of projection. Note here that the angle of projection is in the first quartile.

It was proven in many texts that the air resistance $\overrightarrow{F}(\overrightarrow{v})$ is not a simple function but in general it takes the form [2]

$$\overrightarrow{F}(\overrightarrow{v}) = -B_1\overrightarrow{v} - B_2\overrightarrow{v}\left|\overrightarrow{v}\right|. \tag{2}$$

in which B_1, and $B_2$are constants whose values depend on the size and the shape of the projectile. One can see from (2) that air resistance either be a linear or quadratic or more complicated a combination of them. Dealing with linear term is a trivial problem, so we will focus our attention on the case where air resistance is quadratic one.

For spherical objects projected in air the values forB_1, and $B_2$are related to the diameter (D in meter) of the spherical objects projected and they are given as [7]

$$B_1 = 1.55 \times 10^{-4} D;$$
$$B_2 = 0.22 D^2. \tag{3}$$

Below, we will discuss some properties of the projectile motion in the presence of quadratic air resistance.

2.1 Quadratic air resistance

Assuming air resistance is quadratic, and then applying Newton's second law leads to:

$$a_x = -Cv_x^2. \tag{4}$$

$$a_y = -g - Cv_y^2. \tag{5}$$

Where $C = \frac{B_2}{m}$.

As a result, the horizontal velocity and position at any time respectively read:

$$v_x(t) = \frac{v_{x0}}{Cv_{x_0}t + 1}. \tag{6}$$

$$x(t) = \frac{1}{C} Ln\left|Cv_{x_0}t + 1\right|. \tag{7}$$

On the other hand, the vertical velocity and position at any time respectively read:

$$v_y(t) = \sqrt{\frac{g}{C}}\tan\left(-\sqrt{Cg}t + \tan^{-1}\sqrt{\frac{C}{g}}v_{yo}\right). \tag{8}$$

$$y(t) = \frac{1}{C}\left[Ln\left|cos\left(-\sqrt{Cg}t + tan^{-1}\sqrt{\frac{C}{g}}\ v_{y_0}\right)\right| - Ln\left|cos\left(tan^{-1}\sqrt{\frac{C}{g}}\ v_{y_0}\right) + y_o\right|\right]. \tag{9}$$

Furthermore, time needed to reach maximum height (i.e, when$v_y = 0$) is:

$$t = \frac{1}{\sqrt{Cg}} tan^{-1}\sqrt{\frac{C}{g}}\ v_{y_0}. \tag{10}$$

Also, the maximum height ($y_{max.}$) reads:

$$y_{max} = \frac{-Ln\left|cos\left(tan^{-1}\sqrt{\frac{C}{g}}\, v_{y_0}\right)\right|}{C} + y_o. \tag{11}$$

3 Numerical methods

3.1 Differential transform method (DTM)

One of the mathematical techniques for resolving ordinary differential equations (ODEs) problems is the DTM. It is important to remember that Zhou in 1986 [23] and Pukhov in 1982 [17] were the first to present this technique for solving linear and nonlinear initial value issues in electrical circuit analysis. This approach is based on Taylor series expansion and is both analytical and numerical at the same time. A definition of DTM has been added in this subsection [2, 9]. A few significant theories were also mentioned.

3.1.1 Definition:

the differential of a function and $Y(k)$ is defined as

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(t)}{dt^k}\right]_{t=to}. \tag{12}$$

Where $y(t)$ is the original function and Y(k) is the transformed function.
The inverse transformation is defined by

$$y(t) = \sum_{k}^{\infty} \frac{(t-to)^k}{k!}\frac{d^k y(t)}{dt^k}|_{t=t_0}. \tag{13}$$

In Table 1 below we presented some basic functions that can be deduced using the above definition [13], [16].

Table 1: Transformed functions and basic operation.

Original function	Transformed function
(1) $y(t) = y_1(t) \pm y_2(t)$	$Y(k) = Y_1(k) \pm Y_2(k)$
(2) $y(t) = \alpha y_i(t)$	$Y(k) = \alpha Y_i(k)$ *where* α *is constant*
(3) $y(t) = \frac{dy_i(t)}{dt}$	$Y(k) = (k+1)\,Y_i(k+1)$
(4) $y(t) = \frac{d^2y_i(t)}{dt^2}$	$Y(k) = (k+1)(k+2)Y_i(k+2)$.
(5) $y(t) = \frac{d^n y_i(t)}{dt^n}$	$Y(k) = \frac{(k+n)!}{k!}Y_i(k+1)$
(6) $y(t) = y_1(t)\, y_2(t)$	$Y(k) = \sum_{k_1=0}^{k} Y_1(k_1)Y_2(k-k_1)$
(7) $y(t) = t^n$	$Y(k) = \delta(k-n)$ where $\delta(k-n) = \begin{matrix} 1 & k=n \\ 0 & k \neq n \end{matrix}$

Now, we will apply the DTM to solve our model. To do this, our first step is writing our equations in a system of first order differential equations as follow:

From equation 4 we find the system for $x-dimension$ quadratic air resistance as:

$$\frac{dx}{dt} = v_x. \tag{14}$$

$$\frac{dv_x}{dt} = -Cv_x^2. \tag{15}$$

Let x=$x_1$and $\dot{x} = x_2$ we get:

$$\frac{dx_1}{dt} = x_2. \tag{16}$$

$$\frac{dx_2}{dt} = -Cx_2^2 \tag{17}$$

From equation 5 we get the system for $y-dimension$ quadratic air resistance:

$$\frac{dy}{dt} = v_y. \tag{18}$$

$$\frac{dv_y}{dt} = -g - Cv_y^2. \tag{19}$$

Let y=$y_1$and $\dot{y} = y_2$

$$\frac{dy_1}{dt} = y_2. \tag{20}$$

$$\frac{dy_2}{dt} = -g - Cy_2^2. \tag{21}$$

Applying the DTM method to x-direction system we get:

$$X_1\left(k+1\right) = \frac{1}{k+1} X_2(k). \tag{22}$$

$$X_2\left(k+1\right) = \frac{-C}{(k+1)} \sum_{r=0}^{k} X_2(r) X_2(k-r). \tag{23}$$

Then the solution can be written as:

$$x_1\left(t\right) = \sum_{k=0}^{n} X_1\left(k\right) * t^k. \tag{24}$$

$$x_2\left(t\right) = \sum_{k=0}^{n} X_2\left(k\right) * t^k. \tag{25}$$

While for y-direction system we get:

$$Y_1\left(k+1\right) = \frac{1}{k+1} Y_2(k). \tag{26}$$

$$Y_2\left(k+1\right)=\frac{-1}{\left(k+1\right)}\left[g\delta\left(k\right)+C\sum_{r=0}^{k}Y_2(r)Y_2(k-r)\right]. \quad (27)$$

Then the solutions can be written as:

$$y_1\left(t\right)=\sum_{k=0}^{n}Y_1\left(k\right)*t^k. \quad (28)$$

$$y_2\left(t\right)=\sum_{k=0}^{n}Y_2\left(k\right)*t^k. \quad (29)$$

3.2 Multistep differential transformation method (Ms-DTM)

By using DTM, we were able to identify precise and user-friendly solutions. However, we also observed that DTM provides a good estimate of the correct answer in short time intervals while exhibiting true behavior from the problem. It provides series solutions over extended time intervals and increases accuracy in compared to the Ms-DTM [6, 15].

3.2.1 Definition

Let [0, T] be the interval for nonlinear initial value problem

$$f\left(t,y,y',\ldots,y^{(r)}\right)=0, y^{(r)}\left(0\right)=c_k \text{ for } k=0,1,\ldots,r-1. \quad (30)$$

can be expressed by finite series

$$y\left(t\right)=\sum_{n=0}^{N}a_nt^n. \quad (31)$$

We assume the interval [0,T] is divided into M equal length subintervals: $\left[t_0,t_1\right],\left[t_1,t_2\right],\ldots,\left[t_{M-1},t_M\right]$ with step size h=$\frac{T}{M}$, by using the nodes $t_m=m.h$.

At the first interval [0, t_1], we consider initial conditions $y_1^{(k)}\left(0\right)=c_k$, at this term the DTM is applied to 11, and the solution is given as

$$y_1\left(t\right)=\sum_{n=0}^{N}a_{1n}t^n, t\in\left[0,t_1\right]. \quad (32)$$

Then, when m $\geq$ 2 and at each subinterval $[t_{m-1},t_m]$, we consider the initial conditions $y_m^{(k)}(t_{m-1})=y_{m-1}^{(k)}(t_{m-1})$and applied the DTM 21 at $[t_{m-1},t_m]$, where t_0 in 12 is altered with t_{m-1}.

The operation is repeated to generate a sequence of approximations $y_m\left(t\right)$, m = 1, 2, $\cdots$, M, for the solution of y(t)

$$y_m\left(t\right)=\sum_{n=0}^{N}a_{mn}(t-t_{m-1})^n, t\in\left[t_{m-1},t_m\right]. \quad (33)$$

where N = K.M. In general, the Ms- DTM solution given as:

$$y(t) = \begin{matrix} y_1(t), t \in [0, t_1] \\ y_2(t), t \in [t_1, t_2] \\ \vdots \\ y_m(t), t \in [t_{M-1}, t_M] \end{matrix} \quad . \tag{34}$$

When we examine the step size h = T, we will see that the Ms-DTM technique and the standard DTM approach are identical. In actuality, there is no distinction between the two. Nonetheless, the goal of employing Ms-DTM is to provide a longer-lasting, more precise solution. Now, when we apply the Ms-DTM method to the system in the x-direction, we get:

$$X_{1m}(k+1) = \frac{1}{k+1} X_{2m}(k). \tag{35}$$

$$X_{2m}(k+1) = \frac{-C}{(k+1)} \sum_{r=0}^{k} X_{2m}(r) X_{2m}(k-r). \tag{36}$$

When $m = 1$ we use $x_{11}(0) = c_1$, $x_{12}(0) = c_2$. If $m > 1$ we use $x_{1m}(t_{m-1}) = x_{1m-1}(t_{m-1})$, $x_{2m}(t_{m-1}) = x_{2m-1}(t_{m-1})$.

Then we get the following series solution

$$x_i(t) = \begin{matrix} \sum_{k=0}^{n} X_{i1}(k) t^k, t \in [0, t_1] \\ \sum_{k=0}^{n} X_{i2}(k) (t-t_1)^k, t \in [t_1, t_2] \\ . \\ . \\ . \\ \sum_{k=0}^{n} X_{iM}(k) (t-t_{M-1})^k, t \in [t_{M-1}, t_M] \end{matrix} \qquad i = 1, 2. \tag{37}$$

While, applying the Ms-DTM method to the y-direction system we obtain:

$$Y_{1m}(k+1) = \frac{1}{k+1} Y_{2m}(k). \tag{38}$$

$$Y_{2m}(k+1) = \frac{-1}{(k+1)} \left[g\delta(k) + C \sum_{r=0}^{k} Y_{2m}(r) Y_{2m}(k-r) \right]. \tag{39}$$

When $m = 1$ we use $y_{11}(0) = a_1$, $y_{12}(0) = a_2$. If $m > 1$ we use $y_{1m}(t_{m-1}) = y_{1m-1}(t_{m-1})$, $y_{2m}(t_{m-1}) = y_{2m-1}(t_{m-1})$.

and we get the following series solution

$$y_i(t) = \begin{matrix} \sum_{k=0}^{n} Y_{i1}(k) t^k, t \in [0, t_1] \\ \sum_{k=0}^{n} Y_{i2}(k) (t-t_1)^k, t \in [t_1, t_2] \\ . \\ . \\ . \\ \sum_{k=0}^{n} Y_{iM}(k) (t-t_{M-1})^k, t \in [t_{M-1}, t_M] \end{matrix} \qquad i = 1, 2. \tag{40}$$

4 Simulation results and discussion

In this section we will study some specific cases, where reliable experimental data for them can be found in Table 2 below.

Table 2: The experimental data of the American Ministry of Defense, tables of fire of one mortar [10] at $\theta = 45^o$, g =9.81 m/s^2, B=0.0005, x_0=y_0= 0.

Case	Initial value of velocity (v_0) [m\s]	Time of flight (T) [s]	Maximum Range (R) [m]
1	101.8	14.4	971.96
2	112.16	15.7	1159.40
3	121.91	17	1348.67
4	131.36	18.2	1538.86

Using the two methods mentioned above, we will substitute the beginning values and compare them with the exact answer 7, 9 . We note that the horizontal and vertical velocities are equivalent because the angle = 45^o.

Below we will discuss four different cases with different initial conditions.

The first case: projection with initial speed v_0=101.8.

For this case, the positions x(t) and y(t) are calculated at certain value of time. In Table 3 we reported the absolute error for these positions. As one can see we consider the systematic increase in time as: 2, 4, 6, 8, 10, 12 and 14 seconds

Table 3: Absolute error calculated for the outcomes of the two techniques applied in relation to the precise x- and y-dimensional solutions for scenario 1.

	Absolute Error in (x)		Absolute Error in (y)	
Time (s)	DTM	Ms-DTM	DTM	Ms-DTM
2	2.6919×10^{-6}	1.233×10^{-10}	1.4061×10^{-4}	2.8534×10^{-9}
4	3.2534×10^{-4}	1.504×10^{-10}	0.0163	1.8749×10^{-9}
6	0.0053	1.2307×10^{-10}	0.2548	5.6068×10^{-10}
8	0.0375	6.5484×10^{-11}	1.7561	3.3583×10^{-9}
10	0.1702	8.1×10^{-12}	7.7451	5.7133×10^{-9}
12	0.5820	8.95×10^{-11}	25.7799	6.4518×10^{-9}
14	1.6379	1.738×10^{-10}	70.6896	2.9466×10^{-9}

The second case: projection with initial speed v_0=112.16

Again, the positions x(t) and y(t) are calculated at certain value of time. In Table 4 we reported the absolute error for these positions. As one can see we consider the systematic increase in time as: 2, 4, 6, 8, 10, 12 and 14 seconds.

Table 4: Absolute error calculated for the outcomes of the two techniques applied in relation to the precise x- and y-dimensional solutions for scenario 2.

	x-dimension		y-dimension	
Time (s)	DTM	Ms-DTM	DTM	Ms-DTM
2	5.2733×10^{-6}	2.1078×10^{-10}	1.8323×10^{-4}	3.3951×10^{-9}
4	6.3396×10^{-4}	2.417×10^{-10}	0.0212	1.8954×10^{-9}
6	0.0102	1.7621×10^{-10}	0.3308	1.4659×10^{-9}
8	0.0724	6.03×10^{-11}	2.2773	5.4134×10^{-9}
10	0.3274	7.97×10^{-11}	10.0395	9.1735×10^{-9}
12	1.1161	2.292×10^{-10}	33.4206	1.1805×10^{-8}
14	3.1312	3.811×10^{-10}	91.7065	1.1339×10^{-8}

The third case: projection with initial speed v_0=121.91

For this third case, the positions x(t) and y(t) are calculated at certain value of time. In Table 5 we reported the absolute error for these positions. While, for the fourth case we show our results in Table 6 As one can see we consider the systematic increase in time as: 2, 4, 6, 8, 10, 12 and 14 seconds

Table 5: Absolute error calculated for the outcomes of the two techniques applied in relation to the precise x- and y-dimensional solutions for scenario 3.

	x-dimension		y-dimension	
Time (s)	DTM (5)	Ms-DTM (6,5)	DTM (5)	Ms-DTM (6,5)
2	9.3984×10^{-6}	3.3350×10^{-10}	2.3426×10^{-4}	4.0072×10^{-9}
4	0.0011	3.5817×10^{-10}	0.0271	1.8939×10^{-9}
6	0.0180	2.2771×10^{-10}	0.4211	2.5093×10^{-9}
8	0.1273	2.12×10^{-11}	2.8954	7.6997×10^{-9}
10	0.5741	2.186×10^{-10}	12.7528	1.2899×10^{-8}
12	1.9509	4.684×10^{-10}	42.4331	1.7325×10^{-8}
14	5.4582	7.17364×10^{-10}	116.4324	1.9464×10^{-8}
16	13.2506	9.5769×10^{-10}	277.5415	1.5447×10^{-8}

The fourth case: projection with initial speed v_0=131.36

For this final case, the positions x(t) and y(t) are calculated at certain value of time. In Table 6 we reported the absolute error for these positions. While, for the fourth case we show our results in Table6. As one can see we consider the systematic increase in time as: 2, 4, 6, 8, 10, 12 and 14 seconds

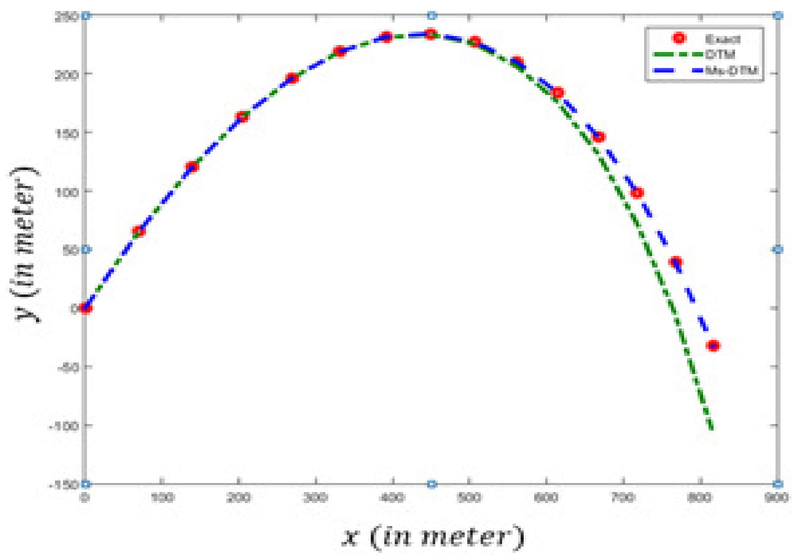

Figure 1: The relationship between x and y for case 1

Table 6: Absolute error computed for the results of the methods with respect to the exact solution for x-dimension and y-dimension for case 4.

	x-dimension		y-dimension	
Time (s)	DTM (5)	Ms-DTM (6,5)	DTM (5)	Ms-DTM (6,5)
2	1.5764×10^{-5}	5.0082×10^{-10}	2.9634×10^{-4}	4.7067×10^{-9}
4	0.0019	5.0403×10^{-10}	0.0341	1.8530×10^{-9}
6	0.02998	2.6995×10^{-10}	0.5302	3.7564×10^{-9}
8	0.2109	7.28×10^{-11}	3.6403	1.0341×10^{-8}
10	0.9481	4.574×10^{-10}	16.0156	1.7090×10^{-8}
12	3.2130	8.5×10^{-10}	53.2476	2.3337×10^{-8}
14	8.9666	1.2338×10^{-9}	146.0418	2.7907×10^{-8}
16	21.7179	1.6005×10^{-9}	348.0935	2.7848×10^{-8}
18	47.2268	1.9452×10^{-9}	745.5160	1.4672×10^{-8}

In Figs.1,2,3and4, we simulated the relationship between the x position and the y position of the projectile for the exact, DTM and Ms-DTM results. It is clear from these figures that Ms-DTM is in an excellent agreement with the exact results for all ranges, while DTM is an excellent agreement for short intervals only.

In Figs.5,6,7and8, we plotted the absolute errors in x and y of DTM and Ms-DTM in comparison with exact results. The figures show that Ms-DTM has a very small (neglected) absolute errors in comparison with the exact results while as time goes on absolute errors between DTM and exact results increasing.

In Tables 7 and 8 we reported a comparison between the results obtained from

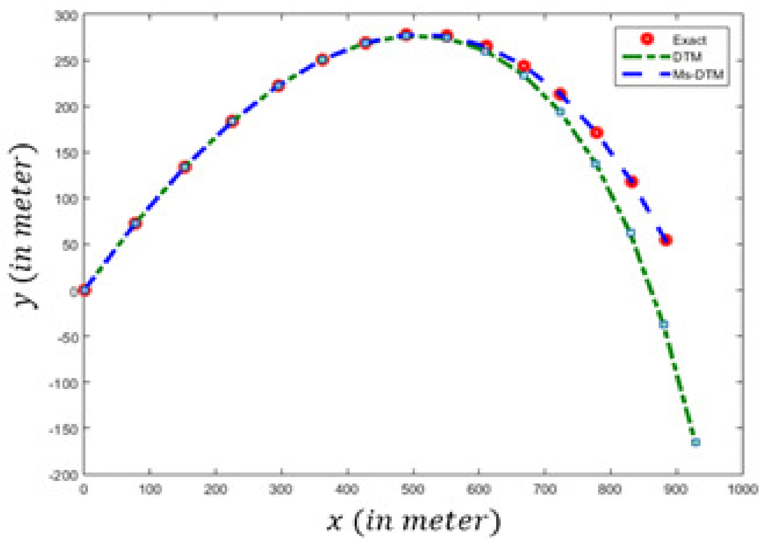

Figure 2: The relationship between x and y for case 2

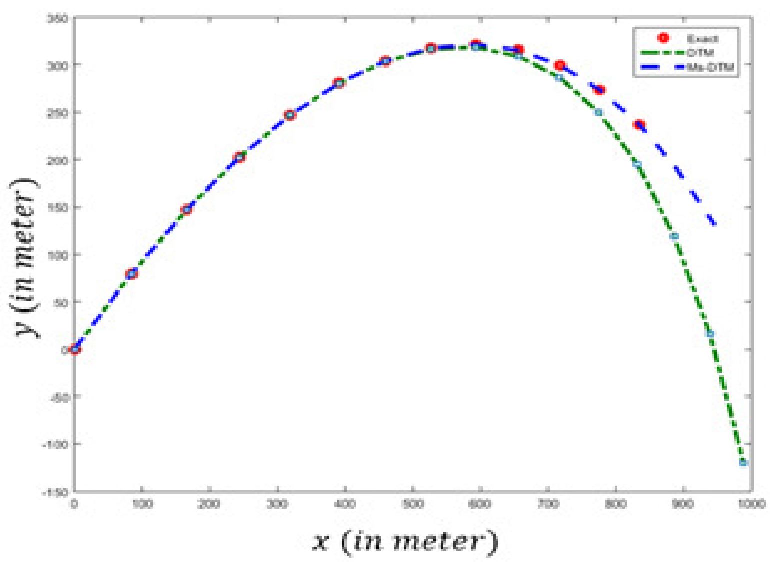

Figure 3: The relationship between x and y for case 3.

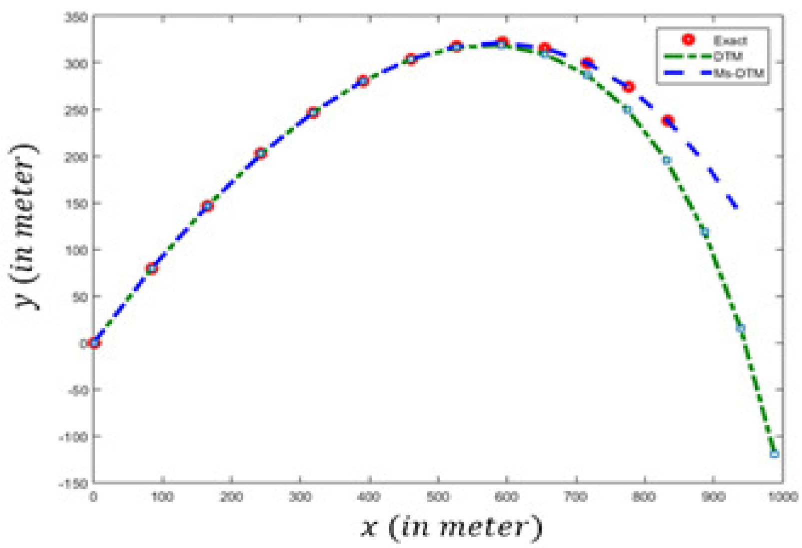

Figure 4: The relationship between x and y for case 4.

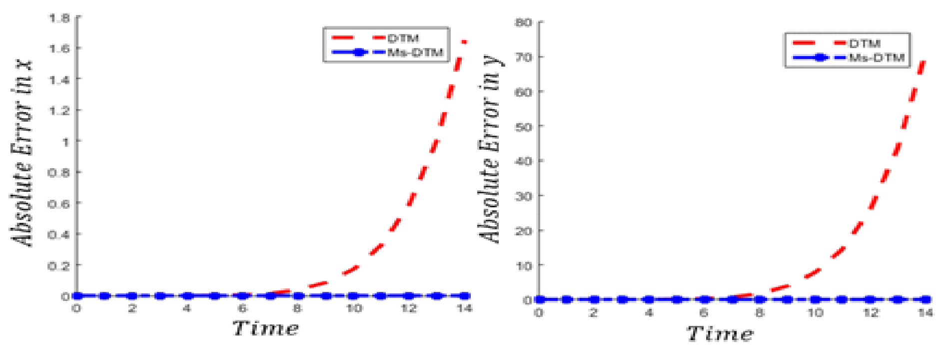

Figure 5: Absolute error computed for the results of the DTM and Ms-DTM methods compared with the exact solution for case 1.

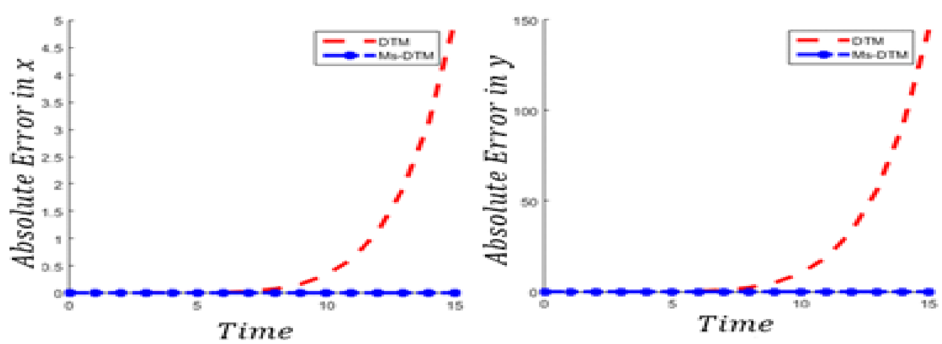

Figure 6: Absolute error computed for the results of the DTM and Ms-DTM methods compared with the exact solution for case 2.

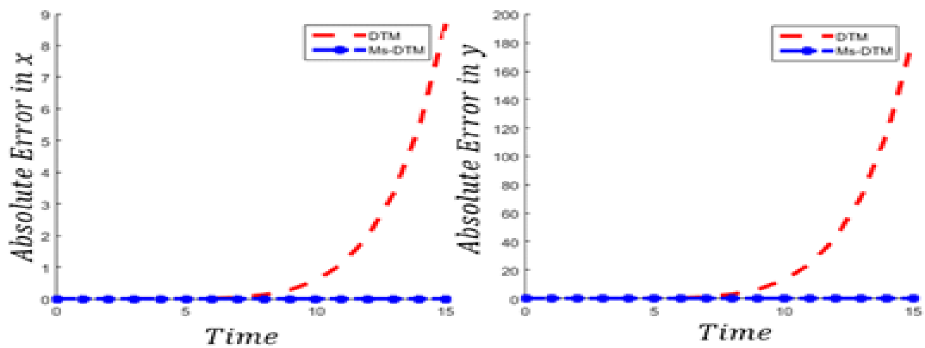

Figure 7: Absolute error computed for the results of the DTM and Ms-DTM methods compared with the exact solution for case 3.

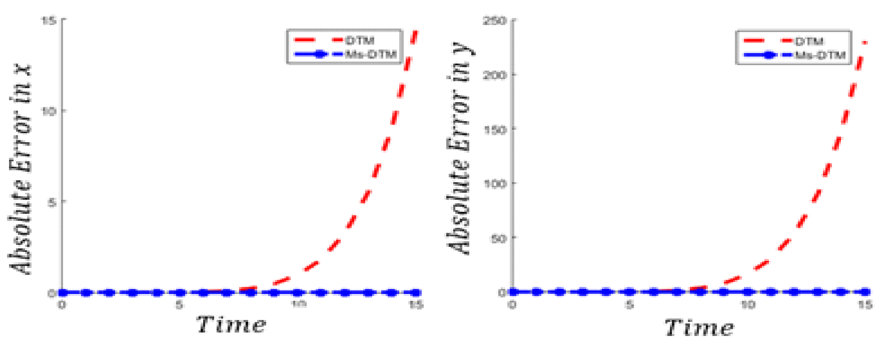

Figure 8: Absolute error computed for the results of the DTM and Ms-DTM methods compared with the exact solution for case 4.

Table 7: Comparison between the results obtained from the experimental time-of-flight data with the results of the both methods and calculates the absolute error.

Initial velocity	Time of flight (s)			Error	
	Experimental result	DTM	Ms-DTM	DTM	Ms-DTM
101.8	14.4	12.94	13.54	1.46	.86
112.16	15.7	13.62	14.7	2.08	1
121.91	17.0	14.11	15.75	2.89	1.25
131.36	18.2	14.40	16.72	3.8	1.48

the experimental time-of-flight data and maximum range data respectively with the results of the both methods and calculates the absolute error.

We note from the Table 7 that the errors rate increase as the velocity increases, and by comparing the both methods, Ms-DTM is considered more accurate than DTM, taking into account that the errors rate were calculated compared to the experimental data.

Table 8: Comparison between the results obtained from the experimental maximum range data with the results of the both methods and calculates the percentage error.

Initial velocity	Maximum Range (m)			Error %	
	Experimental result	DTM	Ms-DTM	DTM	Ms-DTM
101.8	971.96	763.805	794.695	21.412	18.24
112.16	1159.40	861.001	918.912	25.74	20.74
121.91	1348.67	944.63	1036.33	29.91	23.11
131.36	1538.86	1013.02	1149.31	34.17	25.31

We note that in the table that the error percentage is rather high, and DTM is considered higher than Ms-DTM.

5 Conclusions

In the present study, we have analyzed analytically the motion of the projectiles under Quadratic Air Resistance , and numerically, using DTM and Ms-DTM methods. The motion of the projectiles is subject to many interesting properties such as time of flight, maximum height, maximum horizontal range, etc. Due to the fact that the velocity of air resistance is quadratic, it was very difficult to find analytical solutions for these properties. We have compared our numerical results with the real data from the U.S. Department of Defense. We observed that DTM provides very accurate results in very small-time intervals, while Ms- DTM provides very high-accurate results in all time intervals as one can see from the figures and tables presented in this work. In conclusion, we recommend to study the projectile motion under another numerical method to get the best method. In real situation, the air resistance isn't exactly linear or quadratic, but mixed between the two, which is the cause of large percentage errors.

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