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ETA-RICCI SOLITONS ON LORENTZIAN PARA-KENMOTSU MANIFOLDS

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Abstract

This work introduces the investigation of ETA(η)- Ricci solitons on a Lorentzian para-Kenmotsu manifold. In this study, we investigate η- Ricci solitons on Lorentzian para-Kenmotsu manifolds satisfying the condition $C.D = 0$. Additionally, we have constructed and thoroughly shown the findings about the harmonic and Weyl harmonic curvature tensor. Furthermore, our study focuses on the outcomes obtained by investigating Lorentzian para-kenmotsu manifolds that possess a η- Ricci soliton meeting the curvature requirement $N \cdot \psi = 0$. Finally, we provide an illustrative case of a manifold that demonstrates the presence of proper η- Ricci solitons.

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1 Introduction

The idea of Ricci flow, as well as the introduction of the notion of Ricci soliton and its existence, were proposed by Hamilton [13]. The use of this particular idea is necessary to address Thurston's geometric conjecture, which posits that a three-dimensional manifold may be decomposed geometrically if it is closed. Hamilton's classification provides a comprehensive categorization of compact four-dimensional manifolds exhibiting positive curvature. The equation governing the Ricci flow may be expressed as follows

$$\frac{\partial}{\partial t} g_{ij}(t) = -2C_{ij}. \tag{1}$$

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A Ricci soliton is a mathematical concept that serves as an extension of an Einstein metric. It has been defined precisely on a Riemannian manifold denoted as (M, g) [3]. A Ricci soliton is defined as a triple (g, Ω, Λ), where g represents a Riemannian metric, Ω represents a vector field (referred to as a potential vector field), and Λ represents a real scalar such that

$$\pounds_\Omega g + 2D + 2\Lambda g = 0, \tag{2}$$

where D represent the Ricci tensor of the manifold M, and let $\pounds_\Omega$ symbolize the Lie derivative operator applied to the vector field Ω. According to the [13], the Ricci soliton exhibits three distinct behaviours depending on the value of the Λ shrinking, steady, and expanding for negative, zero, and positive values of Λ, respectively. A Ricci soliton in which the vector field Ω is identically zero may be simplified to the Einstein equation. The metrics that satisfy the equation (2) are of significant interest and use in the field of physics. They are sometimes referred to as quasi-Einstein metrics [5], [6]. The equation of Ricci soliton has also been the subject of investigation by theoretical physicists in the context of string theory.

Friedmann is responsible for the earliest contributions made in this approach, and he is the one who addresses certain elements of it [11]. Ricci solitons were first presented in the field of Riemannian geometry [14] as solutions of the Ricci flow that exhibit self-similarity. These solitons have shown to be significant in the study of singularities associated with the Ricci flow. Ricci solitons have been extensively investigated in several scholarly works by multiple writers, including references [9], [10], [16], [19], [20], [21] and numerous others.

The concept of η-Ricci solitons was developed by Cho and Kimura [7] as an extension of Ricci solitons, specifically for Hopf hypersurfaces in complex space forms. An η-Ricci soliton is defined as a tuple $(g, \Omega, \Lambda, \varrho)$, where Ω represents a vector field on the manifold M, and Λ and ϱ are real scalars. The Riemannian metric g associated with the soliton satisfies a certain equation

$$\pounds_\Omega g + 2D + 2\Lambda g + 2\varrho\eta \otimes \eta = 0, \tag{3}$$

where D is the Ricci tensor corresponding to the metric tensor g.

In most cases

(i). In the scenario where $\varrho = 0$, the η-Ricci soliton will be simplified to the Ricci soliton.

(ii). If the condition $\varrho \neq 0$ is satisfied, the η-Ricci soliton is referred to as a proper η-Ricci soliton.

Several authors recently carried out a study on a η-Ricci soliton, as published in multiple publications such as [2], [4], [17], [18], [22], [23]. Their investigations have revealed numerous intriguing geometric characteristics.

In their work, Gray [12] presents the concept of cyclic parallel Ricci tensor and Codazzi type of Ricci tensor. A Riemannian manifold or semi-Riemannian manifold is considered to possess a cyclic parallel Ricci tensor denoted as D of type $(0, 2)$, which is non-zero and fulfills a certain condition

$$(\nabla_{M_1} D)(M_2, M_3) + (\nabla_{M_2} D)(M_3, M_1) + (\nabla_{M_3} D)(M_1, M_2) = 0 \tag{4}$$

Let us consider the scenario where the curvature tensor is harmonic, meaning that its divergence is zero, i.e., $divC = 0$. This condition implies

$$(\nabla_{M_3} D)(M_1, M_2) = (\nabla_{M_1} D)(M_2, M_3) \tag{5}$$

where, the symbol *div* represents the mathematical operation known as divergence. This implies that the Levi-Civita connection ∇ associated with the metric is a Yang-Mills connection, but the metric itself remains unchanged on the manifold. The (5) suggests that the Ricci tensor D exhibits the property of being of codazzi type.
In addition to this, the Weyl tensor is harmonic, thus we obtain

$$\begin{aligned}(\nabla_{M_1} D)(M_2, M_3) - (\nabla_{M_2} D)(M_1, M_3) = \frac{1}{2(n-1)}\Big[&(M_1 a^1) g(M_2, M_3) - \\ &(M_2 a^1) g(M_1, M_3)\Big],\end{aligned} \tag{6}$$

where a^1 represents the scalar curvature.
The projective curvature tensor N [24] in a manifold (M, g) is defined by the following

$$N(M_1, M_2)M_3 = C(M_1, M_2)M_3 - \frac{1}{n-1}\Big[g(M_2, M_3)GX - g(M_1, M_3)GM_2\Big] \tag{7}$$

Let G be the Ricci tensor operator, which is defined by the equation $D(M_1, M_2) = g(GM_1, M_2)$, where M_1, M_2, and M_3 are elements of the $\Gamma(M)$, which represents the vector fields of the manifold M.

2 Preliminaries

Consider a Lorentzian metric manifold denoted by M^n, equipped with a $(1, 1)$ tensor field ψ. In this analysis, we will examine a vector field denoted as δ, a Lorentzian metric denoted as g, and a 1-form denoted as η on the manifold M. We make the assumption that the structure tensor (ψ, δ, η, g) adheres to the following conditions [1]:

$$\psi^2 M_1 = M_1 + \eta(M_1)\delta, \quad \eta(\delta) = -1, \tag{8}$$

which implies

$$(a)\, \psi\delta = 0, \quad (b)\, \eta(\psi M_1) = 0, \quad (c)\, rank(\psi) = n - 1. \tag{9}$$

It follows that the manifold M has a Lorentzian metric g such that

$$g(\psi M_1, \psi M_2) = g(M_1, M_2) + \eta(M_1)\eta(M_2), \tag{10}$$

and the manifold M is deemed to possess a Lorentzian nearly para contact structure denoted by (ψ, δ, η, g). In the present scenario, we possess

$$(a)\, g(M_1, \delta) = \eta(M_1), \quad (b)\, \nabla_{M_1}\delta = \psi M_1, \tag{11}$$

$$(\nabla_{M_1}\psi)M_2 = g(M_1, M_2)\delta + \eta(M_2)M_1 + 2\eta(M_1)\eta(M_2)\delta, \tag{12}$$

The symbol ∇ represents the covariant differentiation operator as it pertains to the Lorentzian metric g.
If one were to place

$$\Phi(M_1, M_2) = g(M_1, \psi M_2) = g(\psi M_1, M_2) = \Phi(M_2, M_1), \tag{13}$$

for arbitrary vector fields M_1 and M_2, the tensor field $\Phi(M_1, M_2)$ is a symmetric $(0, 2)$ tensor field.
In a recent study, Haseeb and Prasad introduced an intriguing manifold known as the Lorentzian para-Kenmotsu manifold [15].

Definition 1. *A Lorentzian almost para contact manifold M is referred to as a Lorentzian para Kenmotsu manifold if, for any vector fields M_1 and M_2 defined on M, the following condition holds:*

$$(\nabla_{M_1}\psi)M_2 = -g(\psi M_1, M_2)\delta - \eta(M_2)\psi M_1, \tag{14}$$

$$\nabla_{M_1}\delta = -M_1 - \eta(M_1)\delta, \tag{15}$$

$$(\nabla_{M_1}\eta)(M_2) = -g(M_1, M_2) - \eta(M_1)\eta(M_2). \tag{16}$$

The symbol ∇ denotes the covariant differentiation.

Remark 1. *In the context of a Lorentzian para-Kenmotsu manifold M, the following relations are valid:*

$$g(C(M_1, M_2)M_3, \delta) = \eta(C(M_1, M_2)M_3) = g(M_2, M_3)\eta(M_1) - g(M_1, M_3)\eta(M_2), \tag{17}$$

$$C(\delta, M_1)M_2 = g(M_1, M_2)\delta - \eta(M_2)M_1, \tag{18}$$

$$C(M_1, M_1)\delta = \eta(M_2)M_1 - \eta(M_1)M_2, \tag{19}$$

$$C(\delta, M_1)\delta = M_1 + \eta(M_1)\delta, \tag{20}$$

$$D(M_1, \delta) = (n-1)\eta(M_1), \tag{21}$$

$$G\delta = (n-1)\delta, \tag{22}$$

$$D(\psi M_1, \psi M_2) = D(M_1, M_2) + (n-1)\eta(M_1)\eta(M_2), \tag{23}$$

where C be the Riemannian curvature tensor and D denote the Ricci tensor.
In the context of a three-dimensional Riemannian manifold M, it is observed that

$$C(M_1, M_2)M_3 = g(M_2, M_3)GM_1 - g(M_1, M_2)GM_2 + D(M_2, M_3)M_1 - D(M_1, M_3)M_2 - \frac{r}{2}\Big[g(M_2, M_3)M_1 - g(M_1, M_3)M_2\Big] \tag{24}$$

where $G(g(GM_1, M_2)) = D(M_1, M_2)$ and a^1 represent, respectively, the Ricci operator and the scalar curvature. Also

$$GM_1 = \frac{1}{2}\Big[(a^1 - 2)M_1 + (a^1 - 6)\eta(M_1)\eta(M_2)\Big]$$

3 Ricci and η - Ricci solitons in the context of Lorentzian para - Kenmotsu manifolds

Considering a para-contact metric manifold symbolized by $(M, \psi, \delta, \eta, g)$. Let us consider the given equation

$$\pounds_\delta g + 2D + 2\Lambda g + 2\varrho\eta \otimes \eta = 0, \tag{25}$$

where Lie derivative operator, denoted as $\pounds_\delta$, pertains to a vector field δ. The Ricci curvature tensor field is represented by D, while Λ and ϱ are real constants. By expressing $\pounds_\delta g$ in terms of the Levi-Civita connection ∇, we may get the following expression:

$$2D(M_1, M_2) = -g(\nabla_{M_1}\delta, M_2) - g(M_1, \nabla_{M_2}\delta) - 2\Lambda g(M_1, M_2) - 2\varrho\eta(M_1)\eta(M_2), \tag{26}$$

for every given M_1, M_2, $\in \Gamma(M)$.
An expression $(g, \delta, \Lambda, \varrho)$ that accomplishes the relation (26) is referred to as a η-Ricci soliton over the manifold M. If the condition $\varrho = 0$ is satisfied, the triple (g, δ, Λ) may be identified as a Ricci soliton [13]. Furthermore, the soliton can be classified as either shrinking, steady, or expanding based on the values of Λ, specifically when $\Lambda < 0$, $\Lambda = 0$, or $\Lambda > 0$, respectively [8].
Now it is understood that in a Lorentzian para-Kenmotsu manifold $\nabla_{M_1}\delta = -M_1 - \eta(M_1)\delta$, consequently (26) turns out to be

$$2D(M_1, M_2) = -g(M_1 - \eta(M_1)\delta, M_2) - g(M_1, -M_2 - \eta(M_2)\delta) - 2\Lambda g(M_1, M_2) - 2\varrho\eta(M_1)\eta(M_2),$$

Upon solving this equation, we get the solution

$$D(M_1, M_2) = (1 - \Lambda)g(M_1, M_2) + (1 - \varrho)\eta(M_1)\eta(M_2), \tag{27}$$

for any two elements, M_1 and M_2 are members of the set $\Gamma(M)$.
By substituting $M_2 = \delta$ into the aforementioned equation, we get

$$D(M_1, \delta) = (\varrho - \Lambda)\eta(M_1), \tag{28}$$

In the given context, the Ricci operator G is characterized by its definition

$$GM_1 = (1 - \Lambda)M_1 + (1 - \varrho)\eta(M_1). \tag{29}$$

In light of the fact that (21) is contrasted with (29), it is evident that we possess

$$\varrho - \Lambda = n - 1. \tag{30}$$

The set of data $(M, \delta, \Lambda, \varrho)$ that accommodates (25) is referred to as a η-Ricci soliton on the manifold M [7].
Consequently, the following proposition may be asserted.

Proposition 1. *Consider a manifold M that is n-dimensional and has the structure of a Lorentzian para-Kenmotsu manifold. If a manifold allows for the existence of a η-Ricci soliton $(g, \delta, \Lambda, \varrho)$, then the manifold M is a η-Einstein manifold in the form (27), and the scalar values λ and ϱ are connected by the equation $\varrho - \Lambda = n - 1$.*

Specifically, by substituting $\varrho = 0$ into equations (27) and (30), we get the expressions $D(M_1, M_2) = (1 - \Lambda)g(M_1, M_2) + \eta(M_1)\eta(M_2)$ and $\Lambda = 1 - n$, respectively. Consequently, we have

Corollary 1. *Consider a manifold M that is n-dimensional and has the structure of a Lorentzian para-Kenmotsu manifold. In the case when the manifold exhibits a Ricci soliton (g, δ, Λ), it may be concluded that M is a η-Einstein manifold. Furthermore, the expanding or shrinking of the manifold is determined by whether the vector field δ is spacelike or timelike, respectively.*

4 η-Ricci solitons on Lorentzian para-Kenmotsu manifolds exhibiting the condition $C \cdot D = 0$

In this section, we will examine a Lorentzian para-Kenmotsu manifold of dimension n that has a η-Ricci soliton satisfying the condition $C \cdot D = 0$. This condition indicates that

$$(C(M_1, M_2) \cdot D)(M_3, M_4) = 0 \tag{31}$$

According to (31), we may derive

$$D(C(M_1, M_2)M_3, M_4) + D(M_3, C(M_1, M_2)M_4) = 0. \tag{32}$$

By replacing $M_1 = \delta$ into equation (32), we may get the following expression

$$D(C(\delta, M_2)M_3, M_4) + D(M_3, C(\delta, M_2)M_4) = 0. \tag{33}$$

By substituting the expression of D from equation (27) and using the symmetries of C, we can determine

$$(\varrho - 1)[\eta(M_2)g(M_1, M_3) + \eta(M_3)g(M_1, M_2) + 2\eta(M_1)\eta(M_2)\eta(M_3)] = 0, \tag{34}$$

Through the process of substitution, $M_3 = \delta$ into equation (34), we get the following expression

$$(\varrho - 1)[g(M_1, M_2) + \eta(M_1)\eta(M_2)] = 0, \tag{35}$$

Based on the above information, it can be deduced that the value of ϱ is equal to 1. The value of Λ may be determined as Λ=$(2 - n)$ based on the relationship given in (30).
Consequently, it is possible to assert the following proposition:

Proposition 2. *Consider a Lorentzian para-Kenmotsu manifold of dimension n, denoted by $(g, \xi, \Lambda, \varrho)$. Suppose this manifold admits a valid η-Ricci soliton satisfying $C \cdot D = 0$. In this case, it may be concluded that $\varrho = 1$ and $\Lambda = (2 - n)$.*

Based on the aforementioned proposition, we obtain:

Corollary 2. *In the context of a Lorentzian para-Kenmotsu manifold M that satisfies the condition $C \cdot D = 0$, it may be shown that there does not exist a Ricci soliton with the potential vector field δ.*

5 Harmonic and Weyl harmonic curvature tensors of a Lorentzian para-Kenmotsu manifold equipped with a η-Ricci soliton

In this article, we examine the harmonic and Weyl harmonic curvature tensors of a Lorentzian para-Kenmotsu manifold with a η-Ricci soliton.
By performing a covariant differentiation on (28) with regard to the variable M_3, we may get the following expression

$$(\nabla_{M_3} D)(M_1, M_2) = (1 - \varrho)[(\nabla_{M_3}\eta)M_1\eta(M_2) + (\nabla_{M_3}\eta)M_2\eta(M_1)]$$

Utilizing (16) in the aforementioned expression yields

$$(\nabla_{M_3} D)(M_1, M_2) = -(1 - \varrho)[g(M_3, M_1)\eta(M_2) + g(M_3, M_2)\eta(M_1) + 2\eta(M_1)\eta(M_2)\eta(M_3)] \tag{36}$$

Based on equation (36), it may be inferred that

$$(\nabla_{M_3} D)(M_1, M_2) - (\nabla_{M_1} D)(M_2, M_3) = (1 - \varrho)[g(M_1, M_2)\eta(M_3) - g(M_3, M_2)\eta(M_1)] \tag{37}$$

According to the assigned hypothesis, the manifold exhibits harmonic curvature, in other words,

$$(\nabla_{M_3} D)(M_1, M_2) = (\nabla_{M_1} D)(M_2, M_3)$$

Therefore, as derived from (37), we obtain

$$(1 - \varrho)\big[g(M_1, M_2)\eta(M_3) - g(M_3, M_2)\eta(M_1)\big] = 0$$

Consequently, it can be deduced that the value of ϱ is equal to 1. As a result, (30) demonstrates that $\Lambda = 2 - n$.
Subsequently, using (27), we may get the following result

$$D(M_1, M_2) = (n - 1)g(M_1, M_2).$$

On the contrary, let us assume that the manifold under consideration is an Einstein manifold. It is evident that the Codazzi-type Ricci tensor is existing. Hence, it may be concluded that the manifold exhibits harmonic curvature.
Therefore, the following proposition may be asserted.

Proposition 3. *A manifold that is Lorentzian para-Kenmotsu and admits a η-Ricci soliton is deemed to have harmonic curvature if and only if the manifold is an Einstein manifold. Furthermore, it can be determined that $\varrho = 1$ and $\Lambda = 2-n$.*

Assume that the manifold M is a Lorentzian para-Kenmotsu manifold with a harmonic Weyl tensor. Equation (6) provides the following result

$$(\nabla_{M_1} D)(M_2, M_3) - (\nabla_{M_2} D)(M_1, M_3) = \frac{1}{2(n-1)}\Big[(M_1 a^1)g(M_2, M_3) - (M_2 a^1)g(M_1, M_3)\Big] \quad (38)$$

Utilizing (36) throughout expression (38), we get

$$(1-\varrho)[g(M_2, M_3)\eta(M_1) - g(M_1, M_3)\eta(M_2)] = \frac{1}{2(n-1)}\Big[(M_1 a^1)g(M_2, M_3) - (M_2 a^1)g(M_1, M_3)\Big]$$

Substituting $M_2 = \delta$ into the aforementioned equation

$$(1-\varrho)[g(M_1, M_3) + \eta(M_3)\eta(M_1)] = \frac{1}{2(n-1)}\Big[(M_1 a^1)\eta(M_3) - (\delta a^1)g(M_1, M_3)\Big]$$

By substituting $M_3 = \phi M_3$ into the aforementioned equation, we get

$$(1-\varrho)[g(M_1, \psi M_3)] = \frac{1}{2(n-1)}\Big[-(\delta a^1)g(M_1, \psi M_3)\Big] \quad (39)$$

Considering the scenario $\delta a^1 = 0$, it follows from the aforementioned equation that

$$(1-\varrho)g(M_1, \psi M_3) = 0$$

Consequently, it can be deduced that the value of ϱ is equal to 1. Subsequently, by referring to equation (30), we may deduce that Λ is equal to $2-n$. Consequently, equation (27) can be inferred from this result

$$D(M_1, M_2) = (n-1)g(M_1, M_2).$$

On the contrary, in the case when the manifold is an Einstein manifold, it exhibits the property of Ricci symmetry, denoted by $\nabla D = 0$, and has a constant scalar curvature. Therefore, it may be concluded that the Weyl tensor exhibits harmonic behavior.
Therefore, the following proposition may be stated.

Proposition 4. *Consider a Lorentzian para-Kenmotsu manifold denoted by M, which also admits a η-Ricci soliton. The manifold M exhibits a harmonic Weyl tensor if and only if it is an Einstein manifold, given that the scalar curvature a^1 remains unaltered under the influence of the characteristic vector field δ. Furthermore, let it be known that the value of ϱ is equal to 1, and the value of Λ is equal to $2-n$.*

6 η - Ricci solitons satisfying the condition on Lorentzian para-Kenmotsu manifold $N \cdot \psi = 0$

It is presupposed that the LP-Kenmotsu manifold, which admits a η-Ricci soliton, accomplishes the curvature requirement

$$N \cdot \psi = 0$$

This demonstrates that

$$N(M_1, M_2)\psi M_3 - \psi(N(M_1, M_2)M_3) = 0$$

By substituting $M_3 = \delta$, the resulting expression is obtained

$$\psi(N(M_1, M_2)\delta) = 0 \tag{40}$$

By substitution of $M_3 = \delta$ into equation (7), we get the following expression

$$N(M_1, M_2)\delta = C(M_1, M_2)\delta - \frac{1}{(n-1)}\Big[\eta(M_2)GM_1 - \eta(M_1)GM_2\Big]$$

Now, using equations (19) and (29) in the aforementioned equation, we derive

$$N(M_1, M_2)\delta = \eta(M_2)M_1 - \eta(M_1)M_2 - \frac{1}{(n-1)}\Big[\eta(M_2)\{(1-\Lambda)M_1 + (1-\varrho)\eta(M_1)\} - \eta(M_1)\{(1-\Lambda)M_2 + (1-\varrho)\eta(M_2)\}\Big],$$

thus implying

$$N(M_1, M_2)\delta = \left[1 - \frac{(1-\Lambda)}{(n-1)}\right]\big[\eta(M_2)M_1 - \eta(M_1)M_2\big] \tag{41}$$

By utilizing equation (41) into equation (40), we obtain

$$\left[1 - \frac{(1-\Lambda)}{(n-1)}\right]\big[\eta(M_2)\psi M_1 - \eta(M_1)\psi M_2\big] = 0.$$

substituting $M_1 = \psi M_1$ into the aforementioned equation leads to

$$\left[1 - \frac{(1-\Lambda)}{(n-1)}\right]\eta(M_2)\psi^2 M_1 = 0.$$

By substituting $M_2 = \delta$, the resulting equation is obtained

$$-\left[1 - \frac{(1-\Lambda)}{(n-1)}\right]\big[M_1 + \eta(M_1)\delta\big] = 0.$$

By using the substitution $M_2 = \delta$, we get the resultant equation

$$\left[1 - \frac{(1-\Lambda)}{(n-1)}\right]\big[\psi M_1 + \eta(\psi M_1)\delta\big] = 0.$$

$$\left[1-\frac{(1-\Lambda)}{(n-1)}\right]\psi M_1 = 0. \tag{42}$$

By computing the inner product of equation (42) with regard to the variable M_4, we get

$$\left[1-\frac{(1-\Lambda)}{(n-1)}\right]g(\psi M_1, M_4) = 0.$$

Consequently, the equation $\left[1-\frac{(1-\Lambda)}{(n-1)}\right] = 0$ leads to the conclusion that Λ is equal to $2-n$. Hence, by referring to equation (30) , it can be seen that the value of ϱ is equal to 1.
Therefore, equation (27) indicates

$$D(M_1, M_2) = (n-1)g(M_1, M_2).$$

Based on our analysis, it can be inferred that

Proposition 5. *In the case when a Lorentzian para-Kenmotsu manifold has a η-Ricci soliton and fulfills the curvature requirement $N \cdot \psi = 0$, it can be deduced that the parameters ϱ and Λ take the values of 1 and $2-n$ respectively. Additionally, it can be concluded that the manifold in consideration is an Einstein manifold.*

Corollary 3. *In the case when a Lorentzian para-Kenmotsu manifold has a η-Ricci soliton and fulfils the curvature requirement $G \cdot N = 0$, it can be deduced that the parameters ϱ and Λ take on the values of 1 and $2-n$ respectively. Furthermore, it can be concluded that the manifold in consideration is an Einstein manifold.*

7 An instance of a three-dimensional LP-Kenmotsu manifold that allows for a η-Ricci soliton

In this analysis, we will now direct our attention towards the three- dimensional manifold.

$$M = \{(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3 : \gamma_3 \neq 0\} \tag{43}$$

Let γ_1, γ_2, and γ_3 be the standard coordinates in $\mathbb{R}^3$.
The vector fields mentioned

$$\sigma^1 = \frac{\partial}{\partial\gamma_1}, \quad \sigma^2 = \frac{\partial}{\partial\gamma_2}, \ \ and \ \ \sigma^3 = \gamma_1\frac{\partial}{\partial\gamma_1} + \gamma_2\frac{\partial}{\partial\gamma_2} + \frac{\partial}{\partial\gamma_3} = \delta, \tag{44}$$

are linearly independent at every point in the set M.
Let g be the Lorentzian metric that is defined by
$g(\sigma^1, \sigma^3) = g(\sigma^2, \sigma^3) = g(\sigma^1, \sigma^2) = 0,$
$g(\sigma^1, \sigma^1) = g(\sigma^2, \sigma^2) = 1, \quad g(\sigma^3, \sigma^3) = -1$
Let η be the 1- form that is defined by

$$\eta(M_3) = g(M_3, \sigma^3) = g(M_3, \delta), \tag{45}$$

for a given vector field M_3 on manifold M.
Consider the $(1,1)$- tensor field to be denoted by ψ

$$\psi(\sigma^1) = \sigma^2, \;\; \psi(\sigma^2) = \sigma^1, \;\; \psi(\sigma^3) = 0. \tag{46}$$

Subsequently, by using the linearity properties of both ψ and g, we may deduce

$$\eta(\sigma^3) = -1, \quad \psi^2 M_3 = M_3 + \eta(M_3)\sigma^3,$$

$$g(\psi M_3, \psi M_4) = g(M_3, M_4) + \eta(M_3)\eta(M_4),$$

for every vector field M_3 and M_4 defined on the manifold M.
It is evident that

$$\eta(\sigma^1) = 0, \quad \eta(\sigma^2) = 0, \quad \eta(\sigma^3) = -1. \tag{47}$$

Consequently, for $\sigma^3 = \delta$, the structure (ψ, δ, η, g) establishes a Lorentzian almost paracontact metric structure on M [15].
Let us assume that the Levi-Civita connection is denoted by ∇ with regard to the Lorentzian metric g. Then there is also

$$[\sigma^1, \sigma^2] = 0, \quad [\sigma^1, \sigma^3] = \sigma^1, \quad [\sigma^2, \sigma^3] = \sigma^2, \tag{48}$$

By using Koszul's formula, we may express the Levi-Civita connection ∇ in terms of the metric tensor g that is,

$$\begin{aligned} 2g(\nabla_{M_1} M_2, M_3) = M_1 g(M_2, M_3) + M_2 g(M_3, M_1) - M_3 g(M_1, M_2) - \\ g(M_1, [M_2, M_3]) - g(M_2, [M_1, M_3]) + g(M_3, [M_1, M_2]), \end{aligned}$$

It is possible to easily compute

$$\nabla_{\sigma^1}\sigma^3 = \sigma^1, \quad \nabla_{\sigma^1}\sigma^2 = 0, \quad \nabla_{\sigma^1}\sigma^1 = -\sigma^3, \tag{49}$$

$$\nabla_{\sigma^2}\sigma^3 = \sigma^2, \quad \nabla_{\sigma^2}\sigma^2 = \sigma^3, \quad \nabla_{\sigma^2}\sigma^1 = 0, \tag{50}$$

$$\nabla_{\sigma^3}\sigma^3 = 0, \quad \nabla_{\sigma^3}\sigma^2 = 0, \quad \nabla_{\sigma^3}\sigma^1 = 0. \tag{51}$$

In light of what has been discussed so far, it is clear that the manifold under investigation satisfies ∇, which may be written as

$$\nabla_{M_3}\delta = -M_3 - \eta(M_3)\delta, \quad \textit{and} \quad (\nabla_{M_3}\psi)M_4 = -g(\psi M_3, M_4) - \eta(M_4)\psi M_3 \tag{52}$$

In addition to this, the Riemannian curvature tensor, denoted by D may be expressed as

$$C(M_1, M_2)M_3 = \nabla_{M_1}\nabla_{M_2} M_3 - \nabla_{M_2}\nabla_{M_1} M_3 - \nabla_{[M_1, M_2]} M_3. \tag{53}$$

Following that

$$C(\sigma^1, \sigma^2)\sigma^3 = 0, \;\; C(\sigma^2, \sigma^3)\sigma^3 = -\sigma^2, \;\; C(\sigma^1, \sigma^3)\sigma^3 = -\sigma^1, \tag{54}$$

$$C(\sigma^1, \sigma^2)\sigma^2 = \sigma^1, \;\; C(\sigma^2, \sigma^3)\sigma^2 = -\sigma^3, \;\; C(\sigma^1, \sigma^3)\sigma^2 = 0, \tag{55}$$

$$C(\sigma^1, \sigma^2)\sigma^1 = \sigma^2, \; C(\sigma^2, \sigma^3)\sigma^1 = 0, \; C(\sigma^1, \sigma^3)\sigma^1 = \sigma^3, \tag{56}$$

The Ricci tensor, denoted by D, may then be obtained by

$$D(\sigma^1, \sigma^1) = -2 \;\; D(\sigma^2, \sigma^2) = 2, \; D(\sigma^3, \sigma^3) = -2. \tag{57}$$

$$D(\sigma^1, \sigma^2) = 0 \;\; D(\sigma^1, \sigma^3) = 0, \; D(\sigma^2, \sigma^3) = 0. \tag{58}$$

By referring to equation (27), it can be shown that $D(\sigma^1, \sigma^1)$ and $D(\sigma^2, \sigma^2)$ are both equal to $1 - \Lambda$. The equation $D(\sigma^3, \sigma^3) = \Lambda - \varrho$ implies that $\Lambda = -1$ and $\varrho = 1$. The collection of data $(g, \delta, \Lambda, \varrho)$, where $\Lambda = -1$ and $\varrho = 1$, may be used to establish the existence of a η-Ricci soliton on the Lorentzian para Kenmotsu manifold M.

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