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# SOLVING SPLIT EQUALITY FIXED POINT PROBLEM OF GENERALIZED DEMIMETRIC MAPPING AND CERTAIN OPTIMIZATION PROBLEM VIA DYNAMIC STEP-SIZE TECHNIQUE

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**Abstract**

In this article, we study the split equality problem of certain optimization problem in real Hilbert spaces. We propose a new viscosity iterative algorithm for approximating solution for finite families of split equality variational inequality and split equality fixed point problems of generalized demimetric mapping in real Hilbert spaces. Using our iterative method, we establish a strong convergence result for finding a common element for finite families of variational inequality and fixed point problems of generalized demimetric mapping. We present some consequences and application to convex minimization ptoblem to validate our main result. Our result complements and generalizes some related results in literature.

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## 1 Introduction

Let $X_1$, $X_2$ and $X_3$ be real Hilbert spaces. The Multiple-set Split Equality Common Fixed Point Problem (MSECFP) is to $(\overline{x}, \overline{y})$ such that

$$\overline{x} \in \bigcap_{i=1}^{m} Fix(U_i), \ \overline{y} \in \bigcap_{j=1}^{r} Fix(V_j) \text{ with } F\overline{x} = G\overline{y}, \tag{1}$$

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where $m, r \geq 1$ are integers, $\{U_i\}_{i=1}^m : X_1 \to X_1$ and $\{V_j\}_{j=1}^r : X_2 \to X_2$ are nonlinear mappings, $F : X_1 \to X_3$ and $G : X_2 \to X_3$ are two bounded linear operators. If $U_i (1 \leq i \leq m)$ and $V_j (1 \leq j \leq r)$ are projection mappings, then the MSECFP reduces to the Multiple-set Split Equality Problem (MSEP) which is to find

$$\overline{x} \in \bigcap_{i=1}^{m} C_i, \overline{y} \in \bigcap_{j=1}^{r} K_j \text{ with } F\overline{x} = G\overline{y}, \tag{2}$$

where $\{C_i\}_{i=1}^m$ and $\{K_j\}_{j=1}^r$ are nonempty, closed and convex susbets of $X_1$ and $X_2$, respectively. If $m = r = 1$, the MSECFP and MSEP becomes the Split Equality Common Fixed Point Problem (SECFP) and Split Equality Problem (SEP), respectively. The SECFP and SEP allows symmetric and partial relation with respect to both variables. Both problems have some important applications in game theory, variational inequality problems and intensity modulated therapy (see [3, 8, 38]). Several authors have proposed different iterative methods for solving SECFP (see [1, 2, 17, 15, 28] and other references there in). In 2015, Chidume *et al.* [15] introduced the following Krasnoselskii-type method for solving split equality fixed point problem of demicontractive mappings: For arbitrary $x_1 \in X_1$ and $y_1 \in X_2$, define the iterative method by

$$\begin{cases} x_{n+1} = (1-\alpha)(x_n - \gamma F^*(Fx_n - Gy_n)) + \alpha U(x_n - \gamma F^*(Fx_n - Gy_n)) \\ y_{n+1} = (1-\alpha)(y_n + \gamma G^*(Fx_n - Gy_n)) + \alpha V(y_n + \gamma G^*(Fx_n - Gy_n)), \end{cases}$$

where $U : X_1 \to X_1$ and $V : X_2 \to X_2$ are demicontractive mappings with constants $k_1$ and $k_2$, respectively, $\alpha \in (0, 1-k)$ and $\gamma \in (0, \frac{2}{\lambda_{F^*F} + \lambda_{G^*G}})$, where $\lambda_{F^*F}$ and $\lambda_{G^*G}$ denote the spectral radii of $F^*F$ and $G^*G$ respectively and $k = \max\{k_1, k_2\}$. They obtained both weak and strong convergence results.

In 2018, Zhao and Zong [43] proposed the following parallel and cyclic algorithms for solving the multiple-set split equality common fixed point problem of firmly quasi-nonexpansive mappings: Let $x_0 \in X_1,\ y_0 \in X_2$ be arbitrary. For $n \geq 0$, let

$$\begin{cases} u_n = x_n - (\alpha^1 U_1(x_n) + \cdots + \alpha_n^p U_p(x_n)) + F^*(Fx_n - Gy_n), \\ x_{n+1} = x_n - \tau_n u_n, \\ v_n = y_n - (\beta_n^1 T_1(y_n) + \cdots + \beta_n^r T_r(y_n)) - G^*(Fx_n - Gy_n), \\ y_{n+1} = y_n - \tau_n v_n, \end{cases}$$

where $U_i$ and $T_j$ are firmly quasi-nonexpansive mappings, $\{\alpha_n^i\}_{i=1}^p, \{\beta_n\}_{j=1}^r \subset [0,1]$ such that $\sum\limits_{i=1}^{p} \alpha_n^i = 1$ and $\sum\limits_{j=1}^{r} \beta_n^j = 1$ for every $n \geq 0$ and the step size $\tau_n$ is chosen as

$$\tau_n \in \left(\epsilon, \min\left\{1, \frac{\|Fx_n - Gy_n\|^2}{\|F^*(Fx_n - Gy_n)\|^2 + \|G^*(Fx_n - Gy_n)\|^2}\right\} - \epsilon\right),\ n \in \Pi.$$

Under some mild conditions, they established a strong convergence result. The Variational Inequality Problem (in short, VIP) introduced by Lions and Stampacchia [23, 33] finds its applications in mechanics and potential theory respectively. The VIP has been used as an analytical tool for studying differential equations in infinite dimensional spaces with applications. The VIP is very useful as it combines major concepts in applied mathematics such as systems of nonlinear equations, obstacle problems, network equilibrium problems, necessary optimality conditions for optimization problems and fixed point problems (see [16, 18, 23]).

The VIP is to find a point $x^* \in C$ such that

$$\langle hx^*, x - x^* \rangle \geq 0, \ \forall \ x \in C, \tag{3}$$

where $h : C \to X$ is a nonlinear mapping. We denote by $VI(C, h)$ the solution set of (3). It is known that if h is $\varphi$-inverse strongly mapping and $0 < \eta \leq 2\varphi$, then $I - \eta h$ is nonexpansive. We also have that, for $\eta > 0, \ u = P_C(I - \eta h)u$ if and only if $u \in VI(C, h)$.

In 2012, Censor *et al.* [11] introduced the Split Variational Inequality Problem (SVIP) which is to find $x^* \in C$ such that

$$\langle h_1 x^*, x - x^* \rangle \geq 0, \ \forall \ x \in C, \tag{4}$$

and

$$y^* = Fx^* \in K \text{ solves } \langle h_2 y^*, y - y^* \rangle \geq 0, \ \forall \ y \in K, \tag{5}$$

where $h_1 : C \to X_1$ and $h_2 : K \to X_2$ are nonlinear mappings and $F : X_1 \to X_2$ are bounded linear operator. The SVIP can be reduce to split minimization problem between two spaces such that the image of a solution point of one minimization problem under a given bounded linear operator is a solution of another minimization problem. Using the idea of the split equality problem and VIP (3), we define the split equality variational inequality problem (in short, SEVIP), which is to find

$$x^* \in C \text{ such that } \langle h_1 x^*, x - x^* \rangle \geq 0, \ \forall \ x \in C, \tag{6}$$

and

$$y^* \in K \text{ such that } \langle h_2 y^*, y - y^* \rangle \geq 0, \ \forall \ y \in K \text{ and } Fx^* = Gy^*, \tag{7}$$

where $F : X_1 \to X_3$ and $G : X_2 \to X_3$ are bounded linear operators. The SEVIP has attracted many authors working in this direction due to its broad applications in many areas of applied mathematics (most notably, inverse problems which arise from phase retrieval and in medical image reconstruction [6]).

Several authors have considered approximation solution of SEVIP. For instance, in 2021, Chaichuay [14] proposed the following iterative method for approximating solution of SEVIP: For $u, x_1 \in C$ and $v, y_1 \in K$

$$\begin{cases} u_n = x_n - \gamma_n F^*(Fx_n - Gy_n), \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) P_C(I - \lambda_1 h_1) u_n \\ v_n = y_n + \gamma_n G^*(Fx_n - Gy_n) \\ y_{n+1} = \alpha_n v + (1 - \alpha_n) P_K(I - \lambda_2 h_2) v_n, \ \forall \ n \geq 1, \end{cases}$$

where $h_1 : C \to X_1$ and $h_2 : K \to X_2$ are inverse strongly monotone mappings, $\gamma_n \subset (\epsilon, \frac{2}{\lambda_F+\lambda_G} - \epsilon)$ for all $n \in \mathbb{N}$, $\lambda_F$ and $\lambda_G$ are spectral radii of $F^*F$ and $G^*G$ respectively. They proved a strong convergence result of their proposed algorithm.

Let $X_j, j = 1, 2, 3$ are real Hilbert spaces and $C, K$ be nonempty, closed and convex subsets of $X_1$ and $X_2$ respectively. Let $f^{(j)} : X_1 \to X_1$ and $g^{(j)} : X_2 \to X_2$ be $k_j$ and $\phi_j$-inverse strongly monotone mapping. Let $\{U^{(j)}\}_{j=1}^N : X_1 \to X_1$ and $\{V^{(j)}\}_{j=1}^N : X_2 \to X_2$ be $\varphi^{(j)}$ and $\lambda^{(j)}$ generalized demimetric mapping. We consider the following problem:

$$\begin{cases} \overline{x} \in \bigcap\limits_{j=1}^{N} (Fix(U^{(j)})) \bigcap VI(C_j, f^{(j)}), \\ \text{and} \\ \overline{y} \in \bigcap\limits_{j=1}^{N} (Fix(V^{(j)})) \bigcap VI(K_j, g^{(j)}), \text{where } F\overline{x} = G\overline{y}. \end{cases} \tag{8}$$

It is obvious that the problem discussed in this article generalizes the problems in (1)-(5). We denote by $\Omega$, the solution set of (6).

Motivated by the results of [15, 17, 16] and other related results in literature, we propose a new and efficient method for finding a common element of the set of solution of finite familes of split equality variational inequality problem and split equality fixed point problem of generalized demimetric mapping in the framework of real Hilbert spaces. We state and prove a strong convergence result for solving the aforementioned problems without prior knowledge of the operator norm. Consequences and application were illustrated to validate the importance of our main result. Our main result generalizes and improves the results of Censor *et al.* [11], Eslamian [17] and many other related results in the literature.

We highlight some of the contributions as follows:

(i) Our iterative method is govern by a self adaptive step-size which does not require prior information of the operator norms of $\|F\|$ and $\|G\|$, whereas the results of Chidume *et al.* [15] and Chaichuay [14] requires the knowledge of the operator norm.

(ii) We established a strong convergence result which is desirable to weak convergence result obtained in [15, 43] for it translates the physically tangible property that the energy $\|x_n - x\|$ of the error between the iterate $x_n$ and the solution $x$ eventually becomes arbitrary small. During the course of establishing a strong convergence result, we were able to dispense with the compactness conditions on the iterative method.

(iii) The class of generalized demimetric considered in this article generalizes the class of firmly quasi-nonexpansive mappings employed in [43] and demicontractive mappings considered in [15].

## 2 Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively.

Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $X$. Let $T : C \to C$ be a single-valued mapping, then a point $x \in C$ is called a fixed point of $T$ if $Tx = x$. We denote by $F(T)$, the set of all fixed points of $T$.

A nonlinear mapping $T : X \to X$ is called

(i) nonexpansive, if

$$||Tx - Ty|| \leq ||x - y||, \ \forall \ x, y \in X; \tag{9}$$

(ii) strongly nonexpansve, if $T$ satisfies (i) and

$$\lim_{n\to\infty} ||(x_n - y_n) - (Tx_n - y_n)|| = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are bounded sequences in $X$ and

$$\lim_{n\to\infty} (||x_n - y_n|| - ||Tx_n - Ty_n||) = 0;$$

(iii) averaged nonexpansive, if it can be written as

$$T = (1 - \alpha)I + \alpha S,$$

where $\alpha \in (0, 1)$, $I$ is the identity operator on $X$, and $S : X \to X$ is a nonexpansive mapping;

(iv) firmly nonexpansive , if

$$||Tx - Ty||^2 \leq \langle x - y, Tx - Ty\rangle, \ \forall \ x, y \in X;$$

(v) $k$-strictly pseudocontractive, if for $0 \leq k < 1$,

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2, \ \forall \ x, y \in X;$$

(vi) monotone, if

$$\langle Tx - Ty, x - y\rangle \geq 0, \ \forall \ x, y \in X;$$

(vii) $\alpha$-inverse strongly monotone ($\alpha$-ism) if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha||Tx - Ty||^2, \ \forall \ x, y \in X.$$

For a real Hilbert space $H$, we can easily see that (v) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq ||x-y||^2 - \frac{1-\lambda}{2}||(I-T)x-(I-T)y||^2.$$

**Definition 1.** *[19, 25] The mapping $T : X \to X$ is said to be demicontractive, if there exists $\phi \in [0,1)$ such that*

$$\|Tu - p\|^2 \leq \|u-p\|^2 + \phi\|u - Tp\|^2, \ \forall \ u \in X, \ \forall \ p \in F(T). \tag{10}$$

It is obvious that (10) can be re-written as

$$\langle u - p, u - Tu\rangle \geq \frac{1-\phi}{2}\|u - Tu\|^2.$$

It is well-known that the class of demicontractive mappings generalizes many types of nonlinear mappings which includes nonexpansive and quasi-nonexpansive mappings. The class of demicontractive mappings have been studied by different authors (see [39, 42]) and it is known to find its applications in applied mathematics and optimization.

Recently, Takahashi [34] introduced a new class of nonlinear mappings which generalizes the class of demicontractive mappings as follows:

**Definition 2.** *Let $\phi \in (-\infty, 1)$. A mapping $T : X \to X$ with $F(T) \neq \emptyset$ is called $\phi$-demimetric, if for any $u \in X$ and $p \in F(T)$,*

$$\langle u - p, u - Tu\rangle \geq \frac{1-\phi}{2}\|u - Tu\|^2. \tag{11}$$

Very recently, Kawasaki and Takahashi [21] generalizes the concept of demimetric mappings as follows:

**Definition 3.** *Let $\theta$ be a real number with $\theta \neq 0$. A mapping $T : X \to X$ with $F(T) \neq \emptyset$ is called $\theta-$ generalized demimetric, if*

$$\theta\langle u - p, u - Tu\rangle \geq \|u - Tu\|^2, \ \forall \ u \in X \ \textit{and} \ p \in F(T).$$

It can be seen that the class of generalized demimetric mappings includes the well-known nonlinear mappings such as strict pseudocontraction, quasi - nonexpansive and demicontractive (see [21, 35]).

**Example 1.** *[16] Let $X$ be the real line. Define $T$ on $\mathbb{R}$ by $T(u) = \frac{3}{2}u$. Clearly, $0$ is the only fixed point of $T$. We have $T$ is $(\frac{-1}{2})$-generalized demimetric mapping. Indeed, for each $u \neq 0$, we have*

$$\Big(-\frac{1}{2}\Big)(u)\big(-\frac{1}{2}u\big) = \theta\langle u - p, u - Tu\rangle = \|u - Tu\|^2 = \frac{1}{4}u^2.$$

*Substituting $p = 0$ and $u = 1$, we can see that $T$ is not demicontractive mapping.*

Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. For every point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$ such that

$$||x - P_C x|| \leq ||x - y||, \quad \forall\, y \in C.$$

$P_C$ is called the metric projection of $H$ onto $C$ and it is well known that $P_C$ is a nonexpansive mapping of $H$ onto $C$ that satisfies the inequality:

$$||P_C x - P_C y|| \leq \langle x - y, P_C x - P_C y\rangle.$$

Moreover, $P_C x$ is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \leq 0,$$

and

$$||x - y||^2 \geq ||x - P_C x||^2 + ||y - P_C x||^2, \quad \forall\, x \in H,\; y \in C.$$

We now state some of the results needed to establish our strong convergence result.

**Lemma 1.** *[13] Let $H$ be a real Hilbert space, then for all $x, y \in H$ and $\alpha \in (0,1)$, the following inequalities hold:*

$$||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x - y||^2.$$
$$2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x - y||^2 = ||x + y||^2 - ||x||^2 - ||y||^2.$$

**Lemma 2.** *[36] Let $H_1$ and $H_2$ be real Hilbert spaces. Let $B : H_1 \to H_2$ be a bounded linear operator with $B \neq 0$, and $S : H_2 \to H_2$ be a nonexpansive mapping. Then $B^*(I - S)B$ is $\frac{1}{2||B||^2}$-ism.*

**Definition 4.** *Let $T : X \to X$ be a mapping, then $I - T$ is said to be demiclosed at the $0$ if for any sequence $\{x_n\}$ in $X$, the conditions $x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, imply $x = Tx$.*

**Lemma 3.** *[40] Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$ and $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at $0$ (i.e., if $\{x_n\}$ converges weakly to $x \in C$ and $\{x_n - Tx_n\}$ converges strongly to $0$, then $x = Tx$.*

**Lemma 4.** *[21] Let $X$ be a real Hilbert space and let $\theta$ be a real number with $\theta \neq 0$. Let $T : X \to X$ be a $\theta$-generalized demimetric mapping. Then the fixed point set $F(T)$ of $T$ is closed and convex.*

**Lemma 5.** *[7] Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if the following inequality holds.*

$$\langle x - z, z - y\rangle \geq 0, \;\forall\, y \in C.$$

**Lemma 6.** *[31] Let $\{\alpha_n\}$ be sequence of nonnegative real numbers, $\{a_n\}$ be sequence of real numbers in $(0,1)$ such that $\sum\limits_{n=1}^{\infty} a_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n, \ \forall \ n \geq 1.$$

*If $\limsup\limits_{k\in\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition*

$$\liminf_{k\to\infty}(a_{n_{k+1}} - a_{n_k}) \geq 0,$$

*then $\lim\limits_{k\to\infty} a_k = 0$.*

## 3 Main results

In this section, we present our algorithm and its convergence analysis.

(L1) Let $X_1, X_2$ and $X_3$ be real Hilbert spaces, $F : X_1 \to X_3$ and $G : X_2 \to X_3$ be bounded linear operators. Suppose $\{C_j\}_{j=1}^N$ and $\{K_j\}_{j=1}^N$ be finite families of nonempty, closed and convex subsets of $X_1$ and $X_2$, respectively.

(L2) For $j \in \{1, 2, \cdots, N\}$, let $f^{(j)} : X_1 \to X_1$ be a finite family of $\pi_j$-inverse strongly monotone mapping and $g^{(j)} : X_2 \to X_2$ be a finite family of $\phi_j$-inverse strongly monotone mapping. Suppose $\{U^{(j)}\}_{j=1}^N : X_1 \to X_1$ is a $\varphi^{(j)}$-generalized demimetric mapping such that $I - U^{(j)}$ is demiclosed at 0 and $\{V^{(j)}\}_{j=1}^N : X_2 \to X_2$ is a $\lambda^{(j)}-$ generalized demimetric mapping such that $I - V^{(j)}$ is demiclosed at 0.

(L3) Let $h_i, i = 1, 2$ be contraction mappings with constants $\psi_i \in [0, \frac{1}{2}), i = 1, 2$ and $\psi = \max\{\psi_i, i = 1, 2\}$. Assume that the step-size $\Phi_k$ is chosen in such a way that

$$\Phi_k \in \left(\epsilon, \frac{2\|Fp^k - Gq^k\|^2}{\|G^*(Fp^k - Gq^k)\|^2 + \|F^*(Fp^k - Gq^k)\|^2} - \epsilon\right), \ k \in \Pi,$$

otherwise $\Phi_k = \Phi$ ($\Phi$ being any nonnegative value), where the index set $\Pi = \{k : Fp^k - Gq^k \neq 0\}$.

Let the sequences $\{\delta^k\}, \{\gamma_k^j\}, \{\rho_k^j\}, \{\mu_k^j\}$ and $\{\omega_k^j\}$ satisfy the following conditions:

(Q1) $\{\delta^k\} \in (0,1)$, $\lim\limits_{k\to\infty} \delta^k = 0$ and $\sum\limits_{k=1}^{\infty} \delta^k = \infty$,

(Q2) $\{\gamma_k^{(j)}\} \subset [a^{(j)}, b^{(j)}] \subset (0, 2\pi_j)$,

(Q3) $\{\mu_k^{(j)}\} \subset [d^{(j)}, e^{(j)}] \subset (0, 2\phi_j)$, (Q4) $\{\rho_k^{(j)}\} \subset [m^{(j)}, n^{(j)}] \subset (0, \frac{2\ell^{(j)}}{\varphi^{(j)}})$,

(Q5) $\{\omega_k^{(j)}\} \subset [r^{(j)}, s^{(j)}] \subset (0, \frac{2\tau^{(j)}}{\lambda^{(j)}})$.

Let $\{p^k\}$ and $\{q^k\}$ be sequences generated by $p^1 \in X_1,\ q^1 \in X_2$ and

$$\begin{cases} w^k = p^k - \Phi_k F^*(Fp^k - Gq^k) \\ u^k = H_k^{(N)} H_k^{(N-1)} \cdots H_k^{(1)} w^k \\ p^{k+1} = \delta^k h_1(p^k) + (1-\delta^k)u^k \\ r^k = q^k + \Phi_k G^*(Fp^k - Gq^k) \\ y^k = S_k^{(N)} S_k^{(N-1)} \cdots S_k^{(1)} r^k \\ q^{k+1} = \delta^k h_2(q^k) + (1-\delta^k)y^k, \end{cases} \tag{12}$$

where $H_k^{(j)} = P_{C_j}(I - \gamma_k^{(j)} f^{(j)})U_k^{(j)},\ U_k^{(j)} = I + \ell^j \rho_k^{(j)}(U^{(j)} - I)$ and $\ell^{(j)} = \frac{\varphi^{(j)}}{|\varphi^{(j)}|}$, and $S_k^{(j)} = P_{K_j}(I - \mu_k^{(j)} g^{(j)})V_k^{(j)},\ V_k^{(j)} = I + \tau^{(j)}\omega_k^{(j)}(V^{(j)} - I)$ and $\tau^{(j)} = \frac{\lambda^{(j)}}{|\lambda^{(j)}|}$. Then the sequences $\{(p^k, q^k)\}$ generated iteratively by (12) strongly converges to $(\overline{x}, \overline{y}) \in \Gamma$, where

$$\Gamma := \left\{ \begin{array}{l|l} \overline{x} \in \bigcap_{j=1}^{N} (F(U^{(j)}) \bigcap VI(C_j, f^{(j)})) & \\ & F\overline{x} = G\overline{y} \\ \overline{y} \in \bigcap_{j=1}^{N} (F(V^{(j)}) \bigcap VI(K_j, g^{(j)})) & \end{array} \right\} \text{ is nonempty.}$$

*Proof.* Let $(\overline{x}, \overline{y}) \in \Gamma$, then since $f^{(1)} : X_1 \to X_1$ is $\pi_1$-inverse strongly monotone mapping, we have for any $x, y \in X_1$

$$\begin{aligned} \|(I - \gamma_k^{(1)} f^{(1)})x - (I - \gamma_k^{(1)} f^{(1)})y\|^2 &= \|(x-y) - \gamma_k^{(1)}(f^{(1)}x - f^{(1)}y)\|^2 \\ &\le \|x-y\|^2 - \gamma_k^{(1)}(2\pi_{(1)} - \gamma_k^{(1)})\|f^{(1)}x\|^2 \\ &\le \|x-y\|^2. \end{aligned} \tag{13}$$

Also, using the fact that $U^{(1)} : X_1 \to X_1$ is a $\varphi^{(1)}-$ generalized demimetric mapping, we get that

$$\begin{aligned} \|U_k^{(1)} w^k - \overline{x}\|^2 &= \|w^k - \ell^{(1)}\rho_k^{(1)}(U^{(1)}w^k - w^k) - \overline{x}\|^2 \\ &= \|w^k - \overline{x}\|^2 + 2\langle w^k - \overline{x}, \ell^{(1)}\rho_k^{(1)}(U^{(1)}w^k - w^k)\rangle \\ &+ \|\ell^{(1)}\rho_k^{(1)}(U^{(1)}w^k - w^k)\|^2 \\ &\le \|w^k - \overline{x}\|^2 - 2(\ell^{(1)}\rho_k^{(1)})(\frac{1}{\varphi^{(1)}})\|U^{(1)}w^k - w^k\|^2 \\ &+ (\rho_k^{(1)})^2\|U^{(1)}w^k - w^k\|^2 \\ &= \|w^k - \overline{x}\|^2 - \rho_k^{(1)}\left(2\frac{\ell^{(1)}}{\varphi^{(1)}} - \rho_k^{(1)}\right)\|U^{(1)}w^k - w^k\|^2. \end{aligned} \tag{14}$$

Following the same process as in (14), we have

$$\|V_k^{(1)}r^k - \overline{y}\|^2 = \|r^{(k)} - \overline{y}\|^2 - \omega_k^{(1)}\left(2\frac{\tau^{(1)}}{\lambda^{(1)}} - \omega_k^{(1)}\right)\|V^{(1)}r^k - r^k\|^2. \tag{15}$$

On adding (14) and (15), and applying (Q3) and (Q4), we obtain

$$\|U_k^{(1)}w^k - \overline{x}\|^2 + \|V_k^{(1)}r^k - \overline{y}\|^2 = \|w^k - \overline{x}\|^2 + \|r^k - \overline{y}\|^2 \tag{16}$$

$$-\rho_k^{(1)}\left(2\frac{\ell^{(1)}}{\varphi^{(1)}} - \rho_k^{(1)}\right)\|U^{(1)}w^k - w^k\|^2 - \omega_k^{(1)}\left(2\frac{\tau^{(1)}}{\lambda^{(1)}} - \omega_k^{(1)}\right)\|V^{(1)}r^k - r^k\|^2$$

$$\leq \|w^k - \overline{x}\|^2 + \|r^k - \overline{y}\|^2. \tag{17}$$

Put $z_k^{(1)} = U_k^{(1)}w^k,\ a_k^{(1)} = P_{C_1}(I - \gamma_k^{(1)}f^{(1)})z_k^{(1)}$ and $m_k^{(1)} = V_k^{(1)}r^k,\ b_k^{(1)} = P_{K_1}(I - \mu_k^{(1)}g^{(1)})m_k^{(1)}$. Then using (13), we get

$$\begin{aligned}\|a_k^{(1)} - \overline{x}\|^2 &= \|P_{C_1}(I - \gamma_k^{(1)}f^{(1)})z_k^{(1)} - P_{C_1}(I - \gamma_k^{(1)}f^{(1)})\overline{x}\|^2\\ &\leq \|(I - \gamma_k^{(1)}f^{(1)})z_k^{(1)} - (I - \gamma_k^{(1)}f^{(1)})\overline{x}\|^2\\ &\leq \|z_k^{(1)} - \overline{x}\|^2 - \gamma_k^{(1)}(2\pi_1 - \gamma_k^{(1)})\|f^{(1)}z_k^{(1)}\|^2,\end{aligned} \tag{18}$$

and

$$\|b_k^{(1)} - \overline{y}\|^2 \leq \|m_k^{(1)} - \overline{y}\|^2 - \mu_k^{(1)}(2\phi_1 - \mu_k^{(1)})\|g^{(1)}m_k^{(1)}\|^2. \tag{19}$$

On adding (17), (18) and (19), we get

$$\begin{aligned}\|a_k^{(1)} - \overline{x}\|^2 + \|b_k^{(1)} - \overline{y}\|^2 &\leq \|z_k^{(1)} - \overline{x}\|^2 + \|m_k^{(1)} - \overline{y}\|^2\\ &\quad - \gamma_k^{(1)}(2\pi_1 - \gamma_k^{(1)})\|f^{(1)}z_k^{(1)}\|^2 - \|^2\\ &\quad - \mu_k^{(1)}(2\phi_1 - \mu_k^{(1)})\|g^{(1)}m_k^{(1)}\|^2 && (20)\\ &\leq \|z_k^{(1)} - \overline{x}\|^2 + \|m_k^{(1)} - \overline{y}\|^2 && (21)\\ &\leq \|w^k - \overline{x}\|^2 + \|r^k - \overline{y}\|^2. && (22)\end{aligned}$$

Since $U^{(2)} : X_1 \to X_1$ is a $\varphi^{(2)}-$ generalized demimetric mapping, we have

$$\begin{aligned}\|U_k^{(2)}a_k^{(1)} - \overline{x}\|^2 &= \|a_k^{(1)} - \ell^{(2)}\rho_k^{(2)}(U^{(2)}a_k^{(1)} - a_k^{(1)}) - \overline{x}\|^2\\ &= \|a_k^{(1)} - \overline{x}\|^2 + 2\langle a_k^{(1)} - \overline{x}, \ell^{(2)}\rho_k^{(2)}(U^{(2)}a_k^{(1)} - a_k^{(1)})\rangle\\ &+ \|\ell^{(2)}\rho_k^{(2)}(U^{(2)}a_k^{(1)} - a_k^{(1)})\|^2\\ &\leq \|a_k^{(1)} - \overline{x}\|^2 - 2(\ell^{(2)}\rho_k^{(2)})(\frac{1}{\varphi^{(2)}})\|U^{(2)}a_k^{(1)} - a_k^{(1)}\|^2\\ &+ (\rho_k^{(2)})^2\|U^{(2)}a_k^{(1)} - a_k^{(1)}\|^2\\ &= \|a_k^{(1)} - \overline{x}\|^2 - \rho_k^{(2)}(2\frac{\ell^{(2)}}{\varphi^{(2)}} - \rho_k^{(2)})\|U^{(2)}a_k^{(1)} - a_k^{(1)}\|^2.\end{aligned} \tag{23}$$

Following the same approach as in (23), we get

$$\|V_k^{(2)} b_k^{(1)} - \overline{y}\|^2 = \|b_k^{(1)} - \overline{y}\|^2 - \omega_k^{(2)}\big(2\frac{\tau^{(2)}}{\lambda^{(2)}} - \omega_k^{(2)}\big)\|V^{(2)} b_k^{(1)} - b_k^{(1)}\|^2. \tag{24}$$

By adding (23), (24) and applying (Q4) and (Q5), we get

$$\begin{aligned}\|U_k^{(2)} a_k^{(1)} - \overline{x}\|^2 + \|V_k^{(2)} b_k^{(1)} - \overline{y}\|^2 &\le \|a_k^{(1)} - \overline{x}\|^2 + \|b_k^{(1)} - \overline{y}\|^2 \\ &\quad - \rho_k^{(2)}\big(2\frac{\ell^{(2)}}{\varphi^{(2)}} - \rho_k^{(2)}\big)\|U^{(2)} a_k^{(1)} - a_k^{(1)}\|^2 \\ &\quad - \omega_k^{(2)}\big(2\frac{\tau^{(2)}}{\lambda^{(2)}} - \omega_k^{(2)}\big)\|V^{(2)} b_k^{(1)} - b_k^{(1)}\|^2 \qquad (25) \\ &\le \|a_k^{(1)} - \overline{x}\|^2 + \|b_k^{(1)} - \overline{y}\|^2. \qquad (26)\end{aligned}$$

Put $z_k^{(2)} = U_k^{(2)} a_k^{(1)}$, $a_k^{(2)} = P_{C_2}(I - \gamma_k^{(2)} f^{(2)}) z_k^{(2)}$ and $m_k^{(2)} = V_k^{(2)} b_k^{(1)}$, $b_k = P_{K_2}(I - \mu_k^{(2)} g^{(2)}) m_k^{(2)}$. Then using (14), we get

$$\|a_k^{(2)} - \overline{x}\|^2 \le \|z_k^{(2)} - \overline{x}\|^2 - \gamma_k^{(2)}(2\pi_2 - \gamma_k^{(2)})\|f^{(2)} z_k^{(2)}\|^2, \tag{27}$$

and

$$\|b_k^{(2)} - \overline{y}\|^2 \le \|m_k^{(2)} - \overline{y}\|^2 - \mu_k^{(2)}(2\phi_2 - \mu_k^{(2)})\|g^{(2)} m_k^{(2)}\|^2. \tag{28}$$

On adding (27) and (28), we get

$$\begin{aligned}\|a_k^{(2)} - \overline{x}\|^2 + \|b_k^{(2)} - \overline{y}\|^2 &\le \|z_k^{(2)} - \overline{x}\|^2 + \|m_k^{(2)} - \overline{y}\|^2 \\ &\quad - \gamma_k^{(2)}(2\pi_2 - \gamma_k^{(2)})\|f^{(2)} z_k^{(2)}\|^2 \\ &\quad - \mu_k^{(2)}(2\phi_2 - \mu_k^{(2)})\|g^{(2)} m_k^{(2)}\|^2. \qquad (29)\end{aligned}$$

For $j = 3, \cdots, N$, we put $z_k^{(j)} = U_k^{(j)} a_k^{(j-1)}$, $a_k^{(j)} = P_{C_j}(I - \gamma_k^{(j)} f^{(j)}) z_k^{(j)}$ and $m_k^{(j)} = V_k b_k^{(j-1)}$, $b_k^{(j)} = P_{K_j}(I - \mu_k^{(j)} g^{(j)}) m_k^{(j)}$. Using a similar argument for $j = \{3, 4, \cdots, N\}$, we have

$$\begin{aligned}\|U_k^{(j)} a_k^{(j-1)} - \overline{x}\|^2 + \|V_k^{(j)} b_k^{(j-1)} - \overline{y}\|^2 &\le \|a_k^{(j-1)} - \overline{x}\|^2 + \|b_k^{(j-1)} - \overline{y}\|^2 \\ &\quad - \rho_k^{(j)}\big(2\frac{\ell^{(j)}}{\varphi^{(j)}} - \rho_k^{(j)}\big)\|U^{(j)} a_k^{(j-1)} - a_k^{(j-1)}\|^2 \\ &\quad - \omega_k^{(j)}\big(2\frac{\tau^{(j)}}{\lambda^{(j)}} - \omega_k^{(j)}\big)\|V^{(j)} b_k^{(j-1)} - b_k^{(j-1)}\|^2 \qquad (30) \\ &\le \|a_k^{(j-1)} - \overline{x}\|^2 + \|b_k^{(j-1)} - \overline{y}\|^2. \qquad (31)\end{aligned}$$

Also, following the same process as in (29), we have

$$\begin{aligned}\|a_k^{(j)} - \overline{x}\|^2 + \|b_k^{(j)} - \overline{y}\|^2 &\le \|z_k^{(j)} - \overline{x}\|^2 + \|m_k^{(j)} - \overline{y}\|^2 \\ &\quad - \gamma_k^{(j)}(2\pi_j - \gamma_k^{(j)})\|f^{(j)} z_k^{(j)}\|^2 \\ &\quad - \mu_k^{(j)}(2\phi_j - \mu_k^{(j)})\|g^{(j)} m_k^{(j)}\|^2 \qquad (32) \\ &\le \|z_k^{(j)} - \overline{x}\|^2 + \|m_k^{(j)} - \overline{y}\|^2. \qquad (33)\end{aligned}$$

Put $a_k^{(0)} = w_k$ and $b_k = r^{(k)}$. Then we obtain from (12) that

$$\begin{aligned}
\|u^k - \overline{x}\|^2 + \|y^k - \overline{y}\|^2 &= \|a_k^{(N)} - \overline{x}\|^2 + \|b_k^{(N)} - \overline{y}\|^2 \\
&\leq \|z_k^{(N)} - \overline{x}\|^2 + \|m_k^{(N)} - \overline{y}\|^2 \\
&\quad - \gamma_k^{(m)}(2\pi_N - \gamma_k^{(m)})\|f^{(N)} z_k^{(m)}\|^2 \\
&\quad\quad - \mu_k^{(N)}(2\pi_N - \mu_k^{(N)})\|g^{(N)} m_k^{(N)}\|^2 \\
&\leq \cdots \\
&\leq \|a_k^{(N-1)} - \overline{x}\|^2 + \|b_k^{(N-1)} - \overline{y}\|^2 \\
&\quad - \gamma_k^{(N)}(2\pi_N - \gamma_k^{(N)})\|f^{(N)} z_k^{(N)}\|^2 \\
&\quad - \mu_k(2\phi_N - \mu_k^{(N)})\|g^{(N)} m_k^{(N)}\|^2 \\
&\quad - \rho_k^{(N)}\big(2\frac{\ell^{(N)}}{\varphi^{(N)}} - \rho_k^{(N)}\big)\|U^{(N)} a_k^{(N-1)} - a_k^{(N-1)}\|^2 \\
&\quad - \omega_k^{(N)}\big(2\frac{\tau^{(N)}}{\lambda^{(N)}} - \omega_k^{(N)}\big)\|V^{(N)} b_k^{(N-1)} - b_k^{(N-1)}\|^2 \\
&\leq \cdots \\
&\leq \|w^k - \overline{x}\|^2 + \|r^k - \overline{y}\|^2 - \sum_{j=1}^{N} \gamma_k^{(j)}\big(2\pi_j - \gamma_k^{(j)}\big)\|f^{(j)} z_k^{(j)}\|^2 \\
&\quad - \sum_{j=1}^{N} \mu_k^{(j)}\big(2\phi_j - \mu_k^{(j)}\big)\|g^{(j)} m_k^{(j)}\|^2 \\
&\quad - \sum_{j=1}^{N} \rho_k^{(j)}\big(2\frac{\ell^{(j)}}{\varphi^{(j)}} - \rho_k^{(j)}\big)\|U^{(j)} a_k^{(j-1)} - a_k^{(j-1)}\|^2 \\
&\quad - \sum_{j=1}^{N} \omega_k^{(j)}\big(2\frac{\tau^{(j)}}{\lambda^{(j)}} - \omega_k^{j}\big)\|V^{(j)} b_k^{(j-1)} - b_k^{(j-1)}\|^2. \quad (34)
\end{aligned}$$

Thus, using $(Q2) - (Q5)$, we arrive at

$$\|u^k - \overline{x}\|^2 + \|y^k - \overline{y}\|^2 \leq \|w^k - \overline{x}\|^2 + \|r^k - \overline{y}\|^2. \quad (35)$$

From (12), we get

$$\begin{aligned}
\|w^k - \overline{x}\|^2 &= \|p^k - \Phi_k F^*(Fp^k - Gq^k) - \overline{x}\|^2 \\
&= \|p^k - \overline{x}\|^2 + \Phi_k^2\|F^*(Fp^k - Gq^k)\|^2 - 2\Phi_K\langle p^k - \overline{x}, F^*(Fp^k - Gq^k)\rangle \\
&= \|p^k - \overline{x}\|^2 + \Phi_k^2\|F^*(Fp^k - Gq^k)\|^2 - 2\Phi_K\langle Fp^k - F\overline{x}, Fp^k - Gq^k\rangle \\
&= \|p^k - \overline{x}\|^2 + \Phi_k^2\|F^*(Fp^k - Gq^k)\|^2 - \Phi_k\|Fp^k - F\overline{x}\|^2 \quad (36) \\
&\quad\quad - \Phi_k\|Fp^k - Gq^k\|^2 + \Phi_k\|Gq^k - F\overline{x}\|^2.
\end{aligned}$$

In a similar way, we obtain that

$$\begin{aligned}\|r^k - \overline{y}\|^2 &= \|q^k + \Phi_k G^*(Fp^k - Gq^k) - \overline{y}\|^2 \\ &= \|q^k - \overline{y}\|^2 + \Phi_k^2 \|G^*(Fp^k - Gq^k)\|^2 - \Phi_k \|Gq^k - G\overline{y}\|^2 \\ &- \Phi_k \|Fp^k - Gq^k\|^2 + \Phi_k \|Fp^k - G\overline{y}\|^2. \end{aligned} \tag{37}$$

On adding (36) and (37), and using the fact that $\mathrm{F}\overline{x} = G\overline{y}$, we have

$$\begin{aligned}\|w^k - \overline{x}\|^2 + \|r^k - \overline{y}\|^2 &= \|p^k - \overline{x}\|^2 + \|q^k - \overline{y}\|^2 - \Phi_k \big[2\|Fp^k - Gq^k\|^2 \\ &- \Phi_k(\|G^*(Fp^k - Gq^k)\|^2 + \|F^*(Fp^k - Gq^k)\|^2)\big] \qquad (38) \\ &\leq \|p^k - \overline{x}\|^2 + \|q^k - \overline{y}\|^2. \qquad (39)\end{aligned}$$

This implies from (12), (35) and (39) that

$$\begin{aligned}&\|p^{k+1} - \overline{x}\|^2 + \|q^{k+1} - \overline{y}\|^2 \\ &\leq \delta^k \|h_1(p^k) - \overline{x}\|^2 + (1 - \delta^k)\|u^k - \overline{x}\|^2 + \delta^k \|h_2(q^k) - \overline{y}\|^2 + (1 - \delta^k)\|y^k - \overline{y}\|^2 \\ &\leq \delta^k \|h_1(p^k) - h_1(\overline{x}) + h_1(\overline{x}) - \overline{x}\|^2 + (1 - \delta^k)\|p^k - \overline{x}\|^2 \\ &+ \delta^k \|h_2(q^k) - h_2(\overline{y}) + h_2(\overline{y}) - \overline{y}\|^2 + (1 - \delta^k)\|q^k - \overline{y}\|^2 \\ &\leq 2\delta^k \big(\|h_1(p^k) - h_1(\overline{x})\|^2 + \|h_1(\overline{x}) - \overline{x}\|^2\big) \\ &\quad + (1 - \delta^k)\|p^k - \overline{x}\|^2 + 2\delta^k \big(\|h_2(q^k) - h_2(\overline{y})\|^2 + \|h_2(\overline{y}) - \overline{y}\|^2\big) \\ &\qquad + (1 - \delta^k)\|q^k - \overline{y}\|^2 \\ &\leq 2\delta^k \psi_1^{(2)} \|p^k - \overline{x}\|^2 + 2\delta^k \|h_1(\overline{x}) - \overline{x}\|^2 + (1 - \delta^k)\|p^k - \overline{x}\|^2 + 2\delta^k \psi_2^2 \|q^k - \overline{y}\|^2 \\ &\quad + 2\delta^k \|h_2(\overline{y}) - \overline{y}\|^2 + (1 - \delta^k)\|q^k - \overline{y}\|^2 \\ &\leq (1 - \delta^k(1 - 2\psi^2))\big[\|p^k - \overline{x}\|^2 + \|q^k - \overline{y}\|^2\big] \\ &\quad + \frac{2}{(1 - 2\psi^2)}\big(\|h_1(\overline{x}) - \overline{x}\|^2 + \|h_2(\overline{y}) - \overline{y}\|^2\big) \\ &\leq \max\Big\{\Delta^k, \frac{2}{(1 - 2\psi^2)}\big(\|h_1(\overline{x}) - \overline{x}\|^2 + \|h_2(\overline{y}) - \overline{y}\|^2\big)\Big\},\end{aligned}$$

where $\Delta^k = \|p^k - \overline{x}\|^2 + \|q^k - \overline{y}\|^2$, so

$$\begin{aligned}\Delta^{k+1} &\leq (1 - \delta^k(1 - 2\psi^2))\Delta^k + \delta^k(1 - 2\psi^2)\frac{2}{(1 - 2\psi^2)}\big(\|h_1(\overline{x}) - \overline{x}\|^2 + \|h_2(\overline{y}) - \overline{y}\|^2\big) \\ &\leq \max\{\Delta^k, \frac{2}{(1 - 2\psi^2)}(\|h_1(\overline{x}) - \overline{x}\|^2 + \|h_2(\overline{y}) - \overline{y}\|^2)\}.\end{aligned}$$

Thus, $\{\Delta^k\}$ is bounded, which also implies that $\{p^k\}$ and $\{q^k\}$ are bounded. Consequently, $\{w^k\}, \{r^k\}, \{u^k\}$ and $\{y^k\}$ are bounded.

Now from (12), (34) and (39), we get

$$
\begin{aligned}
&\|p^{k+1}-\overline{x}\|^2+\|q^{k+1}-\overline{y}\|^2\\
&\quad\le \delta^k\big[\|h_1(p^k)-\overline{x}\|^2+\|h_2(q^k)-\overline{y}\|^2\big]+(1-\delta^k)\big[\|u^k-\overline{x}\|^2+\|y^k-\overline{y}\|^2\big]\\
&\quad\le \delta^k\big[\|h_1(p^k)-\overline{x}\|^2+\|h_2(q^k)-\overline{y}\|^2\big]+(1-\delta^k)\big[\|w^k-\overline{x}\|^2+\|r^k-\overline{y}\|^2\\
&\qquad -\sum_{j=1}^{N}\gamma_k^{(j)}\big(2\pi_j-\gamma_k^{(j)}\big)\|f^{(j)}z_k^{(j)}\|^2-\sum_{j=1}^{N}\mu_k^{(j)}\big(2\phi_j-\mu_k^{(j)}\big)\|g^{(j)}m_k^{(j)}\|^2\\
&\qquad -\sum_{j=1}^{N}\rho_k^{(j)}\big(2\frac{\ell^{(j)}}{\varphi^{(j)}}-\rho_k^{(j)}\big)\|U^{(j)}a_k^{(j-1)}-a_k^{(j-1)}\|^2\\
&\qquad -\sum_{j=1}^{N}\omega_k^{(j)}\big(2\frac{\tau^{(j)}}{\lambda^{(j)}}-\omega_k^{j}\big)\|V^{(j)}b_k^{(j-1)}-b_k^{(j-1)}\|^2\big]\\
&\quad\le \delta^k\big[\|h_1(p^k)-\overline{x}\|^2+\|h_2(q^k)-\overline{y}\|^2\big]+(1-\delta^k)\big[\|p^k-\overline{x}\|^2+\|q^k-\overline{y}\|^2\\
&\qquad -\sum_{j=1}^{N}\gamma_k^{(j)}\big(2\pi_j-\gamma_k^{(j)}\big)\|f^{(j)}z_k^{(j)}\|^2-\sum_{j=1}^{N}\mu_k^{(j)}\big(2\phi_j-\mu_k^{(j)}\big)\|g^{(j)}m_k^{(j)}\|^2\\
&\qquad -\sum_{j=1}^{N}\rho_k^{(j)}\big(2\frac{\ell^{(j)}}{\varphi^{(j)}}-\rho_k^{(j)}\big)\|U^{(j)}a_k^{(j-1)}-a_k^{(j-1)}\|^2\\
&\qquad -\sum_{j=1}^{N}\omega_k^{(j)}\big(2\frac{\tau^{(j)}}{\lambda^{(j)}}-\omega_k^{j}\big)\|V^{(j)}b_k^{(j-1)}-b_k^{(j-1)}\|^2\big].
\end{aligned}
\tag{40}
$$

Suppose that there exists a subsequence $\{p^{k_l}\}$ of $\{p^k\}$, then in view of Lemma 6, we get

$$
\limsup_{l\to\infty}\{\triangle^{k_l}-\triangle^{k_{l+1}}\}\le 0. \tag{41}
$$

By considering (40) and (41), we have for each $j\in\{1,2,\cdots,N\}$ that

$$
\begin{aligned}
&\limsup_{l\to\infty}\Big((1-\delta^{k_l})\sum_{j=1}^{N}\Big[\gamma_{k_l}^{(j)}(2\pi_j-\gamma_{k_l}^{(j)})\|f^{(j)}z_{k_l}^{(j)}\|^2+\mu_{k_l}^{(j)}(2\phi_j-\mu_{k_l}^{(j)})\|g^{(j)}m_{k_l}^{(j)}\|^2\Big]\\
&\qquad +\rho_{k_l}^{(j)}\big(2\frac{\ell^{(j)}}{\varphi^{(j)}}-\rho_{k_l}^{(j)}\big)\|U^{(j)}a_{k_l}^{(j-1)}-a_{k_l}^{(j-1)}\|^2\\
&\qquad +\omega_{k_l}^{(j)}\big(2\frac{\tau^{(j)}}{\lambda^{(j)}}-\omega_{k_l}^{j}\big)\|V^{(j)}b_{k_l}^{(j-1)}-b_{k_l}^{(j-1)}\|^2\\
&\quad\le \limsup_{l\to\infty}\big((1-\delta^{k_l})\triangle^{k_l}-\triangle^{k_{l+1}}\big)\\
&\qquad +\limsup_{l\to\infty}\big(\delta^{k_l}(\|h_1(p^{k_l})-\overline{x}\|^2+\|h_2(q^{k_l})-\overline{y}\|^2)\big)\\
&\quad\le \limsup_{l\to\infty}\big(\triangle^{k_l}-\triangle^{k_{l+1}}\big)\\
&\quad=-\liminf_{l\in\infty}(\triangle^{k_{l+1}}-\triangle^{k_l})\le 0.
\end{aligned}
\tag{42}
$$

Thus,

$$\lim_{l\to\infty}\|f^{(j)}z_{k_l}^{(j-1)}\| = 0 = \lim_{l\to\infty}\|g^{(j)}m_{k_l}^{(j)}\|,\ j\in\{1,2,\dots,N\}, \tag{43}$$

and

$$\lim_{l\to\infty}\|U^{(j)}a_{k_l}^{(j-1)} - a_{k_l}^{(j-1)}\| = 0 = \lim_{l\to\infty}\|V^{(j)}b_{k_l}^{(j-1)} - b_{k_l}^{(j-1)}\|,\ j\in\{1,2,\dots,N\}. \tag{44}$$

This implies that

$$\lim_{l\to\infty}\|a_{k_l}^{(j)} - z_{k_l}^{(j)}\| = 0 = \lim_{l\to\infty}\|b_{k_l}^{(j)} - m_{k_l}^{(j)}\|,\ j\in\{1,2,\dots,N\}. \tag{45}$$

Also, from (44) and (45), we have

$$\begin{aligned}\|u^{k_l} - w^{k_l}\| &= \|a_{k_l}^{(N)} - w^{k_l}\|\\ &\le \|a_{k_l}^{(N)} - z_{k_l}^{(N)}\| + \|z_{k_l}^{(N)} - a_{k_l}^{(N-1)}\| + \|a_{k_l}^{(N-1)} - z_{k_l}^{(N-1)}\|\\ &+ \cdots + \|z_{k_l}^{(1)} - a_{k_l}^{(0)}\| \to 0,\ l\to\infty.\end{aligned} \tag{46}$$

Similarly,

$$\lim_{l\to\infty}\|y^{k_l} - r^{k_l}\| = \|b_{k_l}^{(N)} - m_{k_l}^{(N)}\| \to 0,\ l\to\infty. \tag{47}$$

Using (12) and (Q1), we have

$$\|p^{k_{l+1}} - u^{k_l}\| + \|q^{k_{l+1}} - y^{k_l}\| \le \delta^{k_l}\|h_1(p^{k_l}) - u^{k_l}\| + \delta^{k_l}\|h_2(q^{k_l}) - y^{k_l}\| \to 0, \tag{48}$$

when $l\to\infty$, and from (44) and (45), we obtain

$$\begin{aligned}\|z_{k_l} - w^{k_l}\| &= \|z_{k_l}^{(j)} - a_{k_l}^{(0)}\|\\ &\le \|z_{k_l}^{(j)} - a_{k_l}^{(j-1)}\| + \|a_{k_l}^{(j-1)} - z_{k_l}^{(j-1)}\| + \cdots + \|z_{k_l}^{(1)} - a_{k_l}^{(0)}\| \to 0,\end{aligned} \tag{49}$$

when $l\to\infty$.

Similarly,

$$\|m_{k_l}^{(j)} - r^{k_l}\| = \|m_{k_l}^{(j)} - b_{k_l}^{(0)}\| \to 0,\ l\to\infty. \tag{50}$$

From (12), (35) and (38), we have

$$\begin{aligned}&\|p^{k+1} - \overline{x}\|^2 + \|q^{k+1} - \overline{y}\|^2\\ &\le \delta^k\big[\|h_1(p^k) - \overline{x}\|^2 + \|h_2(q^k) - \overline{y}\|^2\big] + (1-\delta^k)\big[\|u^k - \overline{x}\|^2 + \|y^k - \overline{y}\|^2\big]\\ &\le \delta^k\big[\|h_1(p^k) - \overline{x}\|^2 + \|h_2(q^k) - \overline{y}\|^2\big] + (1-\delta^k)\big[\|w^k - \overline{x}\|^2 + \|r^k - \overline{y}\|^2\big]\\ &\le \delta^k\big[\|h_1(p^k) - \overline{x}\|^2 + \|h_2(q^k) - \overline{y}\|^2\big] + (1-\delta^k)\big[\|p^k - \overline{x}\|^2 + \|q^k - \overline{y}\|^2\\ &- \Phi_k\big(2\|Fp^k - Gq^k\|^2 - \Phi_k(\|G^*(Fp^k - Gq^k)\|^2 + \|F^*(Fp^k - Gq^k)\|^2)\big).\end{aligned} \tag{51}$$

By the assumption on $\Phi_k$, we obtain that

$$(\Phi_k + \epsilon)\|G^*(Fp^k - Gq^k)\|^2 + \|F^*(Fp^k - Gq^k)\|^2 \le 2\|Fp^k - Gq^k\|^2.$$

Thus, we obtain from (41) and (51) that

$$\begin{aligned}
&\limsup_{l\to\infty} \big[(1-\delta^{k_l})\Phi_{k_l}\epsilon\big(\|G^*(Fp^{k_l} - Gq^{k_l})\|^2 + \|F^*(Fp^k - Gq^{k_l})\|^2\big)\big] \\
&\le \limsup_{l\to\infty} \big[(1-\delta^k)\Phi_{k_l}\big(2\|Fp^{k_l} - Gq^{k_l}\|^2 - \Phi_{k_l}(\|G^*(Fp^{k_l} - Gq^{k_l})\|^2 \\
&\qquad + \|F^*(Fp^{k_l} - Gq^{k_l})\|^2)\big) \\
&\le \limsup_{l\to\infty} \big((1-\delta^{k_l})\,\triangle^{k_l} - \triangle^{k_{l+1}}\,\big) \\
&\qquad + \limsup_{l\to\infty} \big(\delta^{k_l}(\|h_1(p^{k_l}) - \overline{x}\|^2 + \|h_2(q^{k_l}) - \overline{y}\|^2)\big) \\
&\le \limsup_{l\to\infty}(\triangle^{k_l} - \triangle^{k_{l+1}}) \\
&= -\liminf_{l\in\infty}(\triangle^{k_{l+1}} - \triangle^{k_l}) \\
&\le 0.
\end{aligned} \tag{52}$$

Thus

$$\lim_{l\to\infty} \big(\|G^*(Fp^{k_l} - Gq^{k_l})\|^2 + \|F^*(Fp^{k_l} - Gq^{k_l})\|^2\big) = 0, \tag{53}$$

which implies that

$$\lim_{l\to\infty} \big(\|G^*(Fp^{k_l} - Gq^{k_l})\| = 0 = \lim_{l\to\infty} \|F^*(Fp^{k_l} - Gq^{k_l})\|. \tag{54}$$

Hence,

$$\lim_{l\to\infty} \|Fp^{k_l} - Gq^{k_l}\| = 0. \tag{55}$$

From (12) and (55), we get

$$\|w^{k_l} - p^{k_l}\| + \|r^{k_l} - q^{k_l}\| \le \Phi_{k_l}(\|G^*(Fp^k - Gq^{k-l})\| + \|F^*(Fp^{k_l} - Gq^{k_l})) \to 0, \tag{56}$$

when $l \to \infty$.

From (49)-(53), and (56), we have

$$\begin{cases}
\lim\limits_{l\to\infty} \|u^{k_l} - p^{k_l}\| = 0, \\
\lim\limits_{l\to\infty} \|p^{k_{l+1}} - p^{k_l}\| = 0, \\
\lim\limits_{l\to\infty} \|y^{k_l} - q^{k_l}\| = 0, \\
\lim\limits_{l\to\infty} \|q^{k_{l+1}} - q^{k_l}\| = 0, \\
\lim\limits_{l\to\infty} \|z^{(j)}_{k_l} - p^{k_l}\| = 0, \\
\lim\limits_{l\to\infty} \|z^{(j)}_{k_l} - p^{k_l}\| = 0, \\
\lim\limits_{l\to\infty} \|m^{(j)}_{k_l} - q^{k_l}\| = 0.
\end{cases} \tag{57}$$

By (12), we get

$$\begin{aligned}\|p^{k+1}-\overline{x}\|^2 &= \|\delta^k h_1(p^k)+(1-\delta^k)u^k-\overline{x}\|^2\\ &\le (\delta^k)^2\|h_1(p^k)-\overline{x}\|^2+2\delta^k(1-\delta^k)\langle h_1(p^k)-\overline{x},u^k-\overline{x}\rangle+(1-\delta^k)^2\|u^k-\overline{x}\|^2\\ &= (\delta^k)^2\|h_1(p^k)-\overline{x}\|^2+2\delta^k(1-\delta^k)\langle h_1(p^k)-h_1(\overline{x}),u^k-\overline{x}\rangle\\ &\quad +2\delta^k(1-\delta^k)\langle h_1(\overline{x})-\overline{x},u^k-\overline{x}\rangle+(1-\delta^k)^2\|u^k-\overline{x}\|^2\\ &\le (\delta^k)^2\|h_1(p^k)-\overline{x}\|^2+\delta^k(1-\delta^k)\big(\|h_1(p^k)-h_1(\overline{x})\|^2+\|u^k-\overline{x}\|^2\big)\\ &\quad +2\delta^k(1-\delta^k)\langle h_1(\overline{x})-\overline{x},u^k-\overline{x}\rangle+(1-\delta^k)\|u^k-\overline{x}\|^2\\ &\le (1-\delta^k)\|u^k-\overline{x}\|^2+\delta^k(1-\delta^k)\psi_1^2\|p^k-\overline{x}\|^2+\delta^k\big(\delta^k\|h_1(p^k)-\overline{x}\|^2\\ &\quad +2(1-\delta^k)\langle h_1(\overline{x})-\overline{x},u^{k-\overline{x}}\rangle\big). \end{aligned}\tag{58}$$

Similarly,

$$\begin{aligned}\|q^{k+1}-\overline{y}\|^2 &\le (1-\delta^k)\|y^k-\overline{y}\|^2+\delta^k(1-\delta^k)\psi_2^2\|q^k-\overline{y}\|^2\\ &\quad +\delta^k\big(\delta^k\|h_2(q^k)-\overline{y}\|^2+2(1-\delta^k)\langle h_2(\overline{y})-\overline{y},y^k-\overline{y}\rangle\big).\end{aligned}\tag{59}$$

By adding (58) and (59) and substituting (35) and (39), we get

$$\begin{aligned}\triangle^{k+1} &\le \big(1-\delta^k(1-(1-\delta^k)\psi^2)\big)\triangle^k+\delta^k\big(\delta^k\big(\|h_1(p^k)-\overline{x}\|^2+\|h_2(q^k)-\overline{y}\|^2\big)\\ &\quad +2(1-\delta^k)\big(\langle h_1(\overline{x})-\overline{x},u^k-\overline{x}\rangle+\langle h_2(\overline{y})-\overline{y},y^k-\overline{y}\rangle\big),\end{aligned}\tag{60}$$

where $\chi_k=\delta^k(1-(1-\delta^k)\psi^2)$ and

$$\Theta_k=\frac{\delta^k(\|h_1(p^k)-\overline{x}\|^2+\|h_2(q^k)-\overline{y}\|^2)+2(1-\delta^k)\big(\langle h_1(\overline{x})-\overline{x},u^k-\overline{x}\rangle+\langle h_2(\overline{y})-\overline{y},y^k-\overline{y}\rangle}{(1-(1-\delta^k)\psi^2)}.$$

We can re-write

$$\triangle^{k+1}\le(1-\chi_k)\triangle^k+\chi_k\Theta_k.\tag{61}$$

Since $\{(p^{k_l},q^{k_l})\}$ are bounded, there exists a subsequence $\{(p^{k_{l_m}},q^{k_{l_m}})\}$ which converge weakly to $(x^*,y^*)$. From (57), we have subsequence $\{(u^{k_{l_m}},y^{k_{l_m}})\}$ of $\{(u^{k_l},y^{k_l})\}$ which converge weakly to $(x^*,y^*)$. Similarly, from (56), we have subsequence $\{(w^{k_{l_m}},r^{k_{l_m}})\}$ of $\{(w^{k_l},r^{k_l})\}$ which converge weakly to $(x^*,y^*)$. Using (45), (49), (50) and (56), we get

$$\lim_{l\to\infty}\|a_{k_l}^{(j)}-p^{k_l}\|=0=\lim_{l\to\infty}\|b_{k_l}^{(j)}-q^{k_l}\|.\tag{62}$$

From (44) and the demiclosedness of $I-U^{(j)}$ and $I-V^{(j)}, j\in\{1,2,\ldots,N\}$, we have that $(x^*,y^*)\in\bigcap_{j=1}^{N}\big(F(U^{(j)},F(V^{(j)})\big)$, respectively. Also, from (43), assumptions on (Q2) and (Q3), and since $\{(a_{k_{l_m}},b_{k_{l_m}})\}$ of $\{(a_{k_l},b_{k_l})\}$ converge weakly to $(x^*,y^*)$, then we obtain

$$\lim_{l\to\infty}\|P_{C_j}(I-\gamma_{k_{l_m}}f^{(j)})z_{k_{l_m}}^{(j)}-z_{k_{l_m}}\|=0=\lim_{l\to\infty}\|P_{K_j}(I-\mu_{k_{l_m}}^{(j)}g^{(j)})m_{k_{l_m}}^{(j)}-m_{k_{l_m}}^{(j)}\|,\tag{63}$$

for $j \in \{1, 2, \ldots, N\}$.

Since for $j \in \{1, 2, \cdots, N\}$, $\gamma_{k_l}^{(m)}$ and $\mu_{k_l}^{(j)}$ are bounded, there exist subsequence $\gamma_{k_{l_m}}^{(j)}$ of $\gamma_{k_l}^{(j)}$ and $\mu_{k_{l_m}}^{(j)}$ of $\mu_{k_l}^{(j)}$ which converge weakly to $(x^*, y^*)$ such that $\lim_{m\to\infty} \gamma_{k_{l_m}} = \gamma^{(j)}$ and $\lim_{m\to\infty} \mu_{k_{l_m}} = \mu^{(j)}$ satisfying Assumptions (Q2) and (Q3) respectively. For any $j \in \{1, 2, \cdots, N\}$, we get

$$\begin{aligned}
&\|z_{k_l}^{(j)} - P_{C_j}(I - \gamma_k^{(j)} f^{(j)}) z_{k_l}^{(j)}\| \\
&\quad \le \|z_{k_l}^{(j)} - P_{C_j}(I - \gamma_{k_l} f^{(j)}) z_{k_l}^{(j)}\| + \|P_{C_j}(I - \gamma_{k_l}^{(j)} f^{(j)}) z_{k_l}^{(j)} - P_{C_j}(I - \gamma_k^{(j)} f^{(j)}) z_{k_l}^{(j)}\| \\
&\quad \le \|z_{k_l}^{(j)} - P_{C_j}(I - \gamma_{k_l} f^{(j)}) z_{k_l}^{(j)}\| + \|(I - \gamma_{k_l}^{(j)} f^{(j)}) z_{k_l}^{(j)} - (I - \gamma_k^{(j)} f^{(j)}) z_{k_l}^{(j)}\| \\
&\quad \le \|z_{k_l}^{(j)} - P_{C_j}(I - \gamma_{k_l} f^{(j)}) z_{k_l}^{(j)}\| + |\gamma_{k_l}^{(j)} - \gamma_k^{(j)}| \, \|f^{(j)} z_{k_l}^{(j)}\|. \qquad (64)
\end{aligned}$$

Thus,

$$\lim_{l\to\infty} \|P_{C_j}(I - \gamma^{(j)} f^{(j)}) z_{k_l}^{(j)} - z_{k_l}^{(j)}\| = 0, \; j \in \{1, 2, \ldots, N\}. \qquad (65)$$

In a similar way, we have

$$\lim_{l\to\infty} \|P_{K_j}(I - \mu^{(j)} g^{(j)}) m_{k_l}^{(j)} - m_{k_l}^{(j)}\| = 0, \qquad (66)$$

for $j \in \{1, 2, \ldots, N\}$.

Now, from (65) and (66), since $P_{C_j}(I - \gamma^{(j)} f^{(j)})$ and $P_{K_j}(I - \mu^{(j)} g^{(j)})$ are demiclosed at 0 for $j \in \{1, 2, \ldots, N\}$ we have that $(x^*, y^*) \in (VI(C_j, f^{(j)}), VI(K_j, g^{(j)}))$. On the other hand, since F and G are bounded linear operators, we obtain that that $Fp^k \rightharpoonup Fx^*$ and $Gq^k \rightharpoonup Gy^*$. Also by the weakly semicontinuity of the norm, we have

$$\|Fx^* - Gy^*\| \le \liminf_{k\to\infty} \|Fp^k - Gq^k\| = 0.$$

Therefore, we conclude that conclude that $(x^*, y^*) \in \Gamma$ as desired.

Next, we establish that

$$\limsup_{m\to\infty} \big(\langle h_1(x^*) - x^*, u^{k_l} - x^*\rangle + \langle h_2(y^*) - y^*, y^{k_l} - y^*\rangle\big) \le 0. \qquad (67)$$

From (57), we have $\{u^{k_l}, y^{k_l}\} \rightharpoonup (x^*, y^*)$. It follows that

$$\begin{aligned}
&\limsup_{l\to\infty} \big(\langle h_1(x^*) - x^*, u^{k_l} - x^*\rangle + \langle h_2(y^*) - y^*, y^{k_l} - y^*\rangle\big) \\
&\qquad = \limsup_{l\to\infty} \big(\langle h_1(x^*) - x^*, p^{k_l} - x^*\rangle + \langle h_2(y^*) - y^*, q^{k_l} - y^*\rangle\big) \\
&\qquad = \lim_{m\to\infty} \big(\langle h_1(x^*) - x^*, p^{k_{l_m}} - x^*\rangle + \langle h_2(y^*) - y^*, q^{k_{l_m}} - y^*\rangle\big) \\
&\qquad = \langle h_1(x^*) - x^*, \overline{x} - x^*\rangle + \langle h_2(y^*) - y^*, \overline{y} - y^*\rangle \\
&\qquad \le 0. \qquad (68)
\end{aligned}$$

Now, we establish that $\{(p^k, q^k)\}$ establish to $(x^*, y^*) = (P_\Gamma h_1(x^*), P_\Gamma(y^*))$. From (60), we have that

$$\Delta^{k+1} \le \big(1 - \delta^k(1-(1-\delta^k))\psi^2\big)\Delta^k + \delta^k\big(\delta^k(h_1(p^k) - x^*\|^2 + \|h_2(q^k) - y^*\|^2) \\ + 2(1-\delta^k)\langle h_1(x^*) - x^*, u^k - x^*\rangle + \langle h_2(y^*) - y^*, y^k - y^8\rangle\big).$$

By substituting (68) into the last inequality and applying Lemma 6, we obtain that $\{(p^k, q^k)\}$ converges strongly to $(x^*, y^*)$ as desired. □

Now, we state some of the consequences of our main result.

If $j = 1$, and Assumptions 3 hold, then (12) reduces to

**Corollary 1.**

$$\begin{cases} w^k = p^k - \Phi_k F^*(Fp^k - Gq^k) \\ u^k = H_k w^k \\ p^{k+1} = \delta^k h_1(p^k) + (1-\delta^k)u^k \\ r^k = q^k + \Phi_k G^*(Fp^k - Gq^k) \\ y^k = S_k r^k \\ q^{k+1} = \delta^k h_2(q^k) + (1-\delta^k)y^k, \end{cases} \tag{69}$$

*where* $H_k = P_C(I - \gamma_k f)U_k$, *and* $S_k = P_K(I - \mu_k g)V_k$.

Suppose $U^{(j)}$ and $V^{(j)}$ are finite families of demimetric mappings for $j \in \{1, 2, \ldots, N\}$ then (12) reduces to

**Corollary 2.**

$$\begin{cases} w^k = p^k - \Phi_k F^*(Fp^k - Gq^k) \\ u^k = H_k^{(N)} H_k^{(N-1)} \cdots H_k^{(1)} w^k \\ p^{k+1} = \delta^k h_1(p^k) + (1-\delta^k)u^k \\ r^k = q^k + \Phi_k G^*(Fp^k - Gq^k) \\ y^k = S_k^{(N)} S_k^{(N-1)} \cdots S_k^{(1)} r^k \\ q^{k+1} = \delta^k h_2(q^k) + (1-\delta^k)y^k. \end{cases} \tag{70}$$

# 4 Application

## 4.1 Common minimization problem

Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$, the constrained convex minimization problem is to find $\overline{x} \in C$ such that

$$\psi(\overline{x}) = \min_{x \in C} \psi(x), \tag{71}$$

where $\psi$ is a real-valued convex function. We denote by $argmin_{x\in C}\psi(x)$, the set of solution of (71).

Let $f : H \to \mathbb{R}$. Then, $f$ is convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for every $x, y \in H$ and $\lambda \in [0,1]$. The function $f$ is Fréchet differentiable at $x$ if there is $\nabla f(x) \in H$ such that

$$\lim_{\|y\|\to 0} \frac{f(x+y) - f(x) - \langle \nabla f(x), y\rangle}{\|y\|} = 0.$$

**Lemma 7.** *[37] Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $\psi : H \to \mathbb{R}$ be a convex function. If $\psi$ is differentiable, then $z$ is a solution of* (71) *if and only if $z \in VI(C, \nabla\psi)$.*

**Lemma 8.** *[4] Let $H$ be a real Hilbert space, and $\psi : H \to \mathbb{R}$ be a Fréchet differentiable function. Hence, $\psi$ is convex if and only if $\nabla$ is a monotone mappings if and only if $\psi$ is convex and $\nabla\psi$ is L-Lipschitz continuous, then $\nabla\psi$ is $\frac{1}{L}-$ inverse strongly monotone.*

By substituting $f^{(j)} = \nabla\psi^{(j)}$ and $g^{(j)} = \nabla\zeta^{(j)}$, then we have the following algorithm: Let $\{p^k\}$ and $\{q^k\}$ be sequences generated by $p^1 \in X_1,\ q^1 \in X_2$ and

$$\begin{cases} w^k = p^k - \Phi_k F^*(Fp^k - Gq^k) \\ u^k = H_k^{(N)} H_k^{(N-1)} \cdots H_k^{(1)} w^k \\ p^{k+1} = \delta^k h_1(p^k) + (1-\delta^k)u^k \\ r^k = q^k + \Phi_k G^*(Fp^k - Gq^k) \\ y^k = S_k^{(N)} S_k^{(N-1)} \cdots S_k^{(1)} r^k \\ q^{k+1} = \delta^k h_2(q^k) + (1-\delta^k)y^k, \end{cases} \tag{72}$$

where $H_k^{(j)} = P_{C_j}(I - \gamma_k^{(j)}\nabla\psi^{(j)})U_k^{(j)},\ U_k^{(j)} = I + \ell^j \rho_k^{(j)}(U^{(j)} - I)$ and $\ell^{(j)} = \frac{\varphi^{(j)}}{|\varphi^{(j)}|}$, and $S_k^{(j)} = P_{K_j}(I - \mu_k^{(j)}\nabla\zeta^{(j)})V_k^{(j)},\ V_k^{(j)} = I + \tau^{(j)}\omega_k^{(j)}(V^{(j)} - I)$ and $\tau^{(j)} = \frac{\lambda^{(j)}}{|\lambda^{(j)}|}$. Then the sequences $\{(p^k, q^k)\}$ generated iteratively by (70) strongly converges to $(\overline{x}, \overline{y}) \in \Gamma$, where

$$\Gamma := \{\overline{x} \in \bigcap_{j=1}^{N} (F(U^{(j)}) \bigcap VI(C_j, \nabla\psi^{((j))})) : \overline{y} \in \bigcap_{j=1}^{N} (F(V^{(j)}) \bigcap VI(K_j, \nabla\zeta^{(j)})) : F\overline{x} = G\overline{y}\}$$

is nonempty.

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