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SOME RESULTS ON α-COSYMPLETIC MANIFOLDS

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Abstract

In this paper, we deal with some geometric properties of an α-cosymplectic manifold. First, we give some classifications for an α-cosymplectic manifold endowed with some special vector fields such as projective, concircular and torse-forming. Then, we study α-cosymplectic manifold admitting η-Ricci solitons with projective, affine conformal vector fields. Finally, we obtain some characterizations for such a manifold to be Einstein, η-Einstein, cosymplectic.

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1 Introduction

Let (M, g) be a Riemannian manifold and S be the Ricci tensor field of such a manifold. If there exist a vector field Z and a real number δ satisfying the condition

$$(\pounds_Z g)(W, U) + 2S(W, U) + 2\delta g(W, U) = 0 \quad for\ all\ W, U \in \Gamma(TM) \tag{1}$$

then, Riemannian manifold M is called a Ricci soliton [14]. Here, $\pounds_Z$ stands for the Lie-derivative with respect to Z and g is the Riemannian metric. We stand for a Ricci soliton by (g, Z, δ). If $\pounds_Z g = 0$ and $\pounds_Z g = \mu g$, then Z is called Killing and conformal Killing, respectively, where μ is a function. Also, when Z is Killing or zero in (1), then the Ricci soliton reduces to Einstein manifold.

A Ricci soliton (g, Z, δ) on M becomes a gradient Ricci soliton if Z is the gradient of a function $-f$ (i.e., $Z = -Df$). In addition, the Ricci soliton is called expanding, steady or shrinking depending on $\delta > 0$, $\delta = 0$ or $\delta < 0$, respectively. Ricci solitons have many applications not only in Riemannian geometry but also in physics.

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Over the past few years, several various classes of Ricci solitons have been investigated and studied. One important class of them is the concept of η-Ricci soliton that Cho et al. introduced in 2009 [9]. In 2012, Calin and Crasmareanu studied this concept for Hopf hypersurfaces in complex space forms [5]. Following these studies, many mathematicians have studied $\eta-$Ricci solitons in some different kinds of almost contact metric manifolds [1], [4], [15], [18], [19], [23]-[25].

Let (M, g) be a Riemannian manifold with structure (Z, δ, β) such that Z is a vector field, δ and β are real numbers. For any vector fields W, U on M if the following

$$(\pounds_Z g)(W, U) + 2S(W, U) + 2\delta g(W, U) + 2\beta\eta(W)\eta(U) = 0 \tag{2}$$

is satisfied, then M is called an η-Ricci soliton. Here, $\pounds_Z$ denotes the Lie-derivative with respect to Z and S is the Ricci tensor of M. We denote an η-Ricci soliton on M by (g, Z, δ, β). If $\beta = 0$, then η-Ricci soliton reduces to Ricci soliton. The η-Ricci soliton is called expanding, steady or shrinking depending on $\delta > 0$, $\delta = 0$ or $\delta < 0$, respectively.

On the other hand, vector fields have been used for studying differential geometry of manifolds since they determine the most geometric properties of the related object. They have a significant role in the studies as regards Riemannian geometry. Also, they arise in many fields of physics and differential geometry. Therefore, in recent years, many mathematicians have investigated extensively the manifolds equipped with geometric vector fields in many context. We refer to ([3], [6]-[8], [10], [17], [22] and [28]).

A vector field ν on a Riemannian manifold (M, g) is called torse-forming if

$$\nabla_W \nu = fW + \phi(W)\nu \tag{3}$$

holds for any $W \in \Gamma(TM)$, where ϕ is a 1-form, f and ∇ represent a smooth function on M and the Levi-Civita connection of M, respectively. Depending on variables f and ϕ in (3), we have the following ([6], [8], [27]):

i) If (3) is satisfied together with $\phi(\nu) = 0$, then ν is said to be torqued vector field.

ii) If the 1-form ϕ vanishes identically in (3), then ν is said to be concircular vector field.

iii) If $f = 0$ in (3), then ν is said to be recurrent vector field.

On the other hand, a vector field Z is called affine conformal if it satisfies [11]

$$(\pounds_Z \nabla)(W, U) = W(\mu)U + U(\mu)W - g(W, U)D\mu, \tag{4}$$

or is said to be projective if it satisfies [26]

$$(\pounds_Z \nabla)(W, U) = p(W)U + p(U)W, \tag{5}$$

where p is an exact 1-form and $D\mu$ is the gradient of the smooth function μ on M. If variable μ in (4) is constant, then the vector field Z becomes affine. Also, if $p = 0$ in (5), then Z is called affine.

If Z is projective, then from (5) we have

$$\nabla_W \nabla_U Z - \nabla_{\nabla_W U} Z = R(W, Z)U + p(W)Y + p(U)W. \tag{6}$$

Motivated by these circumstances, we examine $\alpha-$cosymplectic manifolds equipped with some special vector fields, which prove to be rich in geometrical structures. Also, we study $\eta-$Ricci solitons on $\alpha-$cosymplectic manifolds. The present paper is organized in the following way. In section 2, we give some fundamental definitions, notations and formulas about $\alpha-$cosymplectic manifolds. In section 3, we deal with $\alpha-$cosymplectic manifolds endowed with projective, torse-forming vector fields. In last section, we focus on $\alpha-$cosymplectic manifolds admitting $\eta-$Ricci solitons and give some characterizations for such manifolds.

2 Preliminaries

In this section, we shall give some essential notions and formulas which are going to be needed for later [2] and [13].

Let M be a $(2n+1)$-dimensional differentiable manifold which admits an almost contact structure (φ, ξ, η, g). For any vector fields $W, U \in \Gamma(TM)$, such a structure on M satisfies

$$\eta(\xi) = 1, \quad \varphi^2 W = -W + \eta(W)\xi, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(W) = g(W, \xi) \tag{7}$$

and

$$g(\varphi W, \varphi U) = g(W, U) - \eta(W)\eta(U), \quad g(\varphi W, U) = -g(W, \varphi U) \tag{8}$$

then, M is said to define an almost contact metric manifold. Here, g is the Riemannian metric, φ is a tensor field on M, ξ is a vector field (called characteristic vector field) and η is a 1-form, which is g-dual of ξ.

On the other side, in [2], D.E. Blair introduced the fundamental 2-form Φ of $(M, \varphi, \xi, \eta, g)$ as follows:

$$\Phi(W, U) = g(W, \varphi U)$$

for any $U, W \in \Gamma(TM)$. Moreover, an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ becomes a contact metric manifold if the relation

$$\Phi(W, U) = d\eta(W, U)$$

holds for all $U, W \in \Gamma(TM)$, where

$$d\eta(W, U) = \frac{1}{2}\Big\{W\eta(U) - U\eta(W) - \eta([W, U])\Big\}.$$

The Nijenhuis tensor field of φ is defined by

$$N_\varphi(U, W) = [\varphi U, \varphi W] + \varphi^2[U, W] - \varphi[U, \varphi W] - \varphi[\varphi U, W]$$

for all $U, W \in \Gamma(TM)$. If $(M, \varphi, \xi, \eta, g)$ is an almost contact metric manifold and the Nijenhuis tensor of φ satisfies

$$N_\varphi + 2d\eta \otimes \xi = 0$$

then, this manifold is said to be normal.

An almost contact metric manifold is said to be almost cosymplectic and almost Kenmotsu if it satisfies $d\eta = 0, \quad d\Phi = 0$ and $d\eta = 0, \quad d\Phi = 2\eta \wedge \Phi$, respectively. As it is well known that if an almost cosymplectic (or Kenmotsu) manifold is normal, then it is called cosymplectic (or Kenmotsu) manifold.

As a generalization of almost Kenmotsu manifolds, an almost α-Kenmotsu manifold is an almost contact metric manifold M along with $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, where α is a non-zero real number. If we unify almost cosymplectic and almost α-Kenmotsu manifold, then we obtain a new class of almost contact metric manifolds, which is known as almost α-cosymplectic manifold. Such a manifold is defined by the following formula

$$d\eta = 0, \qquad d\Phi = 2\alpha\eta \wedge \Phi.$$

This is equivalent to

$$(\nabla_W \varphi)U = \alpha(g(\varphi W, U)\xi - \eta(U)\varphi W), \tag{9}$$
$$\nabla_W \xi = -\alpha\varphi^2 W \tag{10}$$

for any $W, U \in \Gamma(TM)$ and any real number α [16]. It is obvious that if an almost α-cosymplectic manifold is normal, then it is called an α-cosymplectic manifold. For $\alpha \in \mathbb{R}$, an α-cosymplectic manifold is cosymplectic manifold and α-Kenmotsu manifold accordingly as $\alpha = 0$ and $\alpha \neq 0$, respectively. For further readings, we refer to ([12], [16], [20], [21] and [29]).

For the Riemann curvature tensor R and the Ricci tensor field S of an α-cosymplectic manifold, the followings are satisfied:

$$R(W, U)\xi = \alpha^2(\eta(W)U - \eta(U)W), \tag{11}$$
$$R(W, \xi)U = \alpha^2(g(W, U)\xi - \eta(U)W), \tag{12}$$
$$R(\xi, W)U = \alpha^2(\eta(U)W - g(W, U)\xi), \tag{13}$$
$$R(W, \xi)\xi = \alpha^2(\eta(W)\xi - W), \tag{14}$$
$$S(W, \xi) = -2n\alpha^2\eta(W), \tag{15}$$
$$S(\xi, \xi) = -2n\alpha^2. \tag{16}$$

On the other hand, an α-cosymplectic manifold M is called η-Einstein if

$$S(W, U) = ag(W, U) + b\eta(W)\eta(U)$$

holds for some real constants a and b. If the constant b is equal to zero, then M becomes Einstein. The Ricci tensor field S of an α-cosymplectic manifold M is said to has η-parallel if it satisfies

$$(\nabla_W S)(\varphi U, \varphi F) = 0$$

such that

$$(\nabla_W S)(\varphi U, \varphi F) = \nabla_W S(\varphi U, \varphi F) - S(\nabla_W \varphi U, \varphi F) - S(\varphi U, \nabla_W \varphi F)$$

for any $W, U, F \in \Gamma(TM)$.

3 Some special vector fields on α-cosymplectic manifolds

In this section, we study some special vector fields such as torse-forming, concircular and projective on α-cosymplectic manifolds and obtain some important results which classify such manifolds.

Let us assume that ξ is a torse-forming vector field on an α-cosymplectic manifold M. Then, from (3) we can write

$$\nabla_W \xi = fW + \phi(W)\xi \tag{17}$$

for any $W \in \Gamma(TM)$. If we take inner product of (17) with ξ, one can see that

$$\phi(W) = -f\eta(W). \tag{18}$$

Putting into (18) equation (17) gives

$$\nabla_W \xi = f(W - \eta(W)\xi) \tag{19}$$

Also, making use of the equalities (10) and (18) we have

$$f(W - \eta(W)\xi) = \alpha(W - \eta(W)\xi). \tag{20}$$

Operating inner product of (20) with arbitrary vector field U, we get

$$(f - \alpha)g(\varphi W, \varphi U) = 0. \tag{21}$$

On the other hand, let $\{\xi = e_1, e_2, ..., e_{2n+1}\}$ be an orthonormal basis (which is called φ-basis) of the tangent space T_pM, $\forall p \in M$. If we set $W = U = e_j$ in (21) and sum over j ($j = 1, 2, ..., 2n + 1$), we obtain

$$(f - \alpha)(2n + 1) = 0$$

which means that $\alpha = f$.

Now, we have the following cases:

Case I: If ξ is a torqued vector field on M, then we get $\phi(\xi) = 0$. From (18), we get $f = 0$. In this case, M becomes cosymplectic and hence ξ becomes Killing.

Case II: If ξ is a recurrent vector field on M, then we have that $f = 0$. Then, M becomes cosymplectic and ξ is Killing.

Therefore, we can give:

Proposition 1. *Let ξ be a torse-forming vector field an α-cosymplectic manifold M. Then, the followings are satisfied:*
i) If ξ is a torqued vector field on M, then the manifold M is cosymplectic.
ii) If ξ is a recurrent vector field on M, then the manifold M is cosymplectic.

Let ν be a concircular vector field on α-cosymplectic manifold M. Then, we have

$$\nabla_W \nu = fW \tag{22}$$

for any $W \in \Gamma(TM)$. Differentiating (22) along vector field U provides

$$\nabla_U \nabla_W \nu = U(f)W + f\nabla_U W \tag{23}$$

for any $U \in \Gamma(TM)$. Symmetrizing (23) with respect to W and U gives

$$\nabla_W \nabla_U \nu = W(f)U + f\nabla_W U. \tag{24}$$

Taking $[W, U]$ instead of W in (22) we arrive at

$$\nabla_{[W,U]} \nu = f\nabla_W U - f\nabla_U W. \tag{25}$$

From (23), (24) and (25), we derive that

$$R(W, U)\nu = W(f)U - U(f)W. \tag{26}$$

If we take inner product of (26) with Y, one has

$$g(R(W, U)\nu, Y) = W(f)g(U, Y) - U(f)g(W, Y) \tag{27}$$

for any $Y \in \Gamma(TM)$. Taking into account φ-basis and putting $W = Y = e_j$ in (27), we obtain

$$S(U, \nu) = -2nU(f). \tag{28}$$

Also, putting $W = Y = \xi$ in (27) and using (13) yields

$$\alpha^2(\eta(\nu)\eta(U) - g(\nu, U)) = \xi(f)\eta(U) - U(f). \tag{29}$$

On the other hand, replacing U by ξ in (28) and keeping in mind (15) we get

$$\xi(f) = \alpha^2\eta(\nu). \tag{30}$$

With the help of (29) and (30), we find that

$$U(f) = \alpha^2 g(\nu, U) \tag{31}$$

and hence

$$Df = \alpha^2\nu. \tag{32}$$

Because of being ν concircular on M, differentiating (32) with respect to F, we obtain

$$\nabla_F Df = \alpha^2 fF$$

which implies that the gradient Df of f defined by (3) is concircular on M.

Hence, we state:

Theorem 1. *Let ν be a concircular vector field an α-cosymplectic manifold M. Then, the gradient Df of f defined by (3) is a concircular vector field on M.*

Suppose that ξ is a projective vector field on an α-cosymplectic manifold M. Then, from (6) we can write

$$\nabla_W \nabla_U \xi - \nabla_{\nabla_W U}\xi = R(W,\xi)U + p(W)U + p(U)W. \tag{33}$$

for any $W, U \in \Gamma(TM)$. Using (10) and (12) in (33), the equation (33) transforms into

$$\begin{aligned} -\alpha\nabla_W \varphi^2(U) + \alpha\varphi^2(\nabla_W U) &= \alpha^2(g(W,U)\xi - \eta(U)W) \\ &\quad + p(W)U + p(U)W. \end{aligned} \tag{34}$$

Making use of (7), (10) in (34) and after a straight forward calculation, we get

$$\begin{aligned} p(W)U + p(U)W &= -2\alpha^2 g(W,U)\xi + \alpha^2\eta(W)\eta(U)\xi \\ &\quad + \alpha\eta(W)\eta(U)\xi. \end{aligned} \tag{35}$$

Also, applying inner product of (35) with ξ yields

$$\begin{aligned} p(W)\eta(U) + p(U)\eta(W) &= -2\alpha^2 g(W,U) + \alpha^2\eta(W)\eta(U) \\ &\quad + \alpha\eta(W)\eta(U). \end{aligned} \tag{36}$$

Substituting W for ξ in (36) provides

$$p(U) = (\alpha - \alpha^2 - p(\xi))\eta(U). \tag{37}$$

Setting $U = \xi$ in (37), one immediately has

$$p(\xi) = \frac{1}{2}(\alpha - \alpha^2). \tag{38}$$

Combining (37) and (38), we have that

$$p(U) = \frac{1}{2}(\alpha - \alpha^2)\eta(U). \tag{39}$$

In view of (36) and (39), we obtain that

$$\alpha^2 g(\varphi W, \varphi U) = 0. \tag{40}$$

Taking into account φ-basis and putting $W = U = e_j$ in (40), we arrive at

$$\alpha^2(2n+1) = 0 \tag{41}$$

which gives the conclusion $\alpha = 0$. Using this fact in (39), we also have $p = 0$.

Therefore, we are ready to give the following:

Theorem 2. *If the characteristic vector field ξ is projective on an α-cosymplectic manifold M, then M is a cosymplectic manifold.*

As a result of the equality (41), we state the following:

Corollary 1. *The characteristic vector field ξ is never projective on an α--Kenmotsu manifold M.*

4 η-Ricci solitons on $\alpha-$cosymplectic manifolds

In this section, we deal with η-Ricci solitons on α-cosymplectic manifolds and obtain some important classifications for such manifolds.

The first result of this section is the following:

Theorem 3. *Let (g, ξ, δ, β) be an η-Ricci soliton on an α-cosymplectic manifold $(M, \varphi, \xi, \eta, g)$ satisfying the curvature condition $R.Q = 0$. In this case, M is either an Einstein manifold or a cosymplectic manifold.*

Proof. Consider that the data (g, ξ, δ, β) is η-Ricci soliton on an α-cosymplectic manifold M. Then, the equation (2) can be written as

$$(\pounds_\xi g)(W,U) + 2S(W,U) + 2\delta g(W,U) + 2\beta\eta(W)\eta(U) = 0 \tag{42}$$

for any $W, U \in \Gamma(TM)$. It follows from (10) that we get

$$(\pounds_\xi g)(W,U) = 2\alpha(g(W,U) - \eta(W)\eta(U)). \tag{43}$$

By virtue of (42) and (43), we find that

$$S(W,U) = -(\delta+\alpha)g(W,U) + (\alpha-\beta)\eta(W)\eta(U) \tag{44}$$

which yields

$$QW = -(\delta+\alpha)W + (\alpha-\beta)\eta(W)\xi. \tag{45}$$

On the other hand, suppose that M satisfies $(R.Q)(W,U)E = 0$. This implies that

$$R(W,U)QE - Q(R(W,U)E) = 0. \tag{46}$$

Using (11) and (45), we obtain

$$\begin{aligned} R(W,U)QE &= (\alpha-\beta)\alpha^2\Big\{\eta(W)\eta(E)U - \eta(U)\eta(E)W\Big\} \qquad (47)\\ &\quad -(\delta+\alpha)R(W,U)E,\\ Q(R(W,U)E) &= -(\alpha-\beta)\alpha^2\Big\{\eta(W)g(U,E)\xi - \eta(U)g(W,E)\xi\Big\} \qquad (48)\\ &\quad -(\delta+\alpha)R(W,U)E, \end{aligned}$$

Applying the equalities (47) and (48) in (46), we infer that

$$\begin{aligned} &(\alpha-\beta)\alpha^2\Big\{\eta(W)\eta(E)U - \eta(U)\eta(E)W \\ &-\eta(W)g(U,E)\xi + \eta(U)g(W,E)\xi\Big\} = 0. \end{aligned} \tag{49}$$

Also, setting $E = \varphi E$ in (49) gives

$$(\alpha-\beta)\alpha^2\Big\{-\eta(W)g(U,\varphi E)\xi + \eta(U)g(W,\varphi E)\xi\Big\} = 0. \tag{50}$$

Performing inner product of (50) with ξ, we get

$$(\alpha-\beta)\alpha^2\Big\{-\eta(W)g(U,\varphi E)+\eta(U)g(W,\varphi E)\Big\}=0. \tag{51}$$

Putting $W=\varphi W$ and $U=\xi$ in (51), we obtain

$$(\alpha-\beta)\alpha^2 g(\varphi W,\varphi E)=0. \tag{52}$$

Considering φ-basis and setting $W=E=e_j$ in (52) yields

$$(\alpha-\beta)\alpha^2(2n+1)=0. \tag{53}$$

Therefore, we get either $\alpha=0$ or $\alpha=\beta$. If $\alpha=0$, then the manifold M is cosymplectic. If $\alpha=\beta$, then from (44), the manifold M is Einstein. Thus, the proof is completed. □

Corollary 2. *Let (g,ξ,δ,β) be an η-Ricci soliton on an α-Kenmotsu manifold (M,φ,ξ,η,g) satisfying the curvature condition $R.Q=0$. Then, M is an Einstein manifold and the η-Ricci soliton is steady with $\alpha=\dfrac{1}{2n}$.*

Proof. Since M is an α-Kenmotsu manifold, from (53) we have that $\alpha=\beta$. Using this equality in (44), we get

$$S(W,U)=-(\delta+\alpha)g(W,U) \tag{54}$$

for any $W,U\in\Gamma(TM)$. Putting $W=U=\xi$ in (54) and keeping in mind (16), one can see that

$$\delta=\alpha(2n\alpha-1). \tag{55}$$

If we choose $\alpha=\dfrac{1}{2n}$ in (55), we find that $\delta=0$. This is the desired result. □

Now, we have an important theorem which classifies an α-cosymplectic manifold.

Theorem 4. *Let (g,Z,δ,β) be an η-Ricci soliton on an α-cosymplectic manifold (M,φ,ξ,η,g) such that Z is the gradient Dk of a smooth function k on M. Then, either Z is pointwise collinear with the structure ξ or M is a cosymplectic manifold.*

Proof. Consider that M is an α-cosymplectic manifold admitting an η-Ricci soliton (g,Z,δ,β) such that Z is the gradient Dk of a smooth function k on M, that is, $Z=Dk$. Then, from (2) we can write

$$\nabla_U Dk=-QU-\delta U-\beta\eta(U)\xi \tag{56}$$

for any $U\in\Gamma(TM)$. Differentiating (56) with respect to W and using (10) provides

$$\begin{aligned}\nabla_W\nabla_U Dk &= -\nabla_W QU-\delta\nabla_W U-\beta\eta(\nabla_W U)\xi\\ &\quad -\alpha\beta g(\varphi U,\varphi W)\xi+\alpha\beta\eta(U)\varphi^2(W).\end{aligned} \tag{57}$$

for any $W \in \Gamma(TM)$. Interchanging the roles of T and U in (57) gives

$$\begin{aligned}\nabla_U \nabla_W Dk &= -\nabla_U QW - \delta\nabla_U W - \beta\eta(\nabla_U W)\xi \\ &\quad -\alpha\beta g(\varphi W, \varphi U)\xi + \alpha\beta\eta(W)\varphi^2(U).\end{aligned} \tag{58}$$

Moreover, taking $[W,U]$ instead of U in (56) one has

$$\nabla_{[W,U]} Dk = -Q[W,U] - \delta[W,U] - \beta\eta([W,U])\xi. \tag{59}$$

Making use of the equalities (58), (57) and (59) gives

$$R(W,U)Dk = (\nabla_U Q)W - (\nabla_W Q)U + \alpha\beta(\eta(W)U - \eta(U)W). \tag{60}$$

Substituting W for ξ in (60) and using (13), we arrrive at

$$\alpha^2(\xi(k)U - U(k)\xi) = (\nabla_U Q)\xi - (\nabla_\xi Q)U + \alpha\beta(U - \eta(U)\xi)). \tag{61}$$

Taking inner product on both sides of (61) by ξ yields

$$\alpha^2(U(k) - \xi(k)\eta(U)) = g((\nabla_\xi Q)U, \xi) - g(\nabla_U Q)\xi, \xi). \tag{62}$$

On the other hand, using the equation (15) it can be easily seen that

$$g((\nabla_\xi Q)U, \xi) = 0 \quad and \quad g((\nabla_\xi Q)U, \xi) = 0. \tag{63}$$

From (62) and (63), the equation (62) takes the form

$$\alpha^2(U(k) - \xi(k)\eta(U)) = 0. \tag{64}$$

If $\alpha = 0$ in (64), then M becomes cosymplectic manifold. In case of $\alpha \neq 0$ in (64), then we have

$$U(k) = \xi(k)\eta(U). \tag{65}$$

Removing U from both sides in (65) we get

$$Dk = \xi(k)\xi \tag{66}$$

and hence

$$Z = \xi(k)\xi$$

which gives conclusion. □

An immediate consequence of Theorem 4 is the following:

Corollary 3. *Let (g, Z, δ, β) be an η-Ricci soliton on an α-cosymplectic manifold $(M, \varphi, \xi, \eta, g)$ such that Z is the gradient Dk of a smooth function k on M. Then, Z is also pointwise collinear with the structure ξ.*

The next theorem provides a characterization for an $\eta-$Ricci soliton with affine conformal vector field.

Theorem 5. *Let (g, Z, δ, β) be an η-Ricci soliton on an α-cosymplectic manifold $(M, \varphi, \xi, \eta, g)$ such that Z is affine conformal vector field. Then, the followings are satisfied:*
i) Z is an affine vector field.
ii) M is either an η-Einstein manifold or a cosymplectic manifold.
iii) M has η-parallel Ricci tensor field.

Proof. Let the data (g, Z, δ, β) be an $\eta-$Ricci soliton on $\alpha-$cosymplectic manifold M. Then, equation (2) takes the form

$$(\pounds_Z g)(U, F) = -2S(U, F) - 2\delta g(U, F) - 2\beta\eta(U)\eta(F) \tag{67}$$

for any $U, F \in \Gamma(TM)$. It is well known that the following formula is satisfied (see Yano [26], p.23):

$$\begin{aligned}(\pounds_Z \nabla_W g - \nabla_W \pounds_Z g - \nabla_{[Z,W]} g)(U, F) \;&=\; -g((\pounds_Z \nabla)(W, U), F) \\ &\quad -g((\pounds_Z \nabla)(W, F), U).\end{aligned}$$

Since the Riemannian metric g is parallel, namely $\nabla g = 0$, the above formula turns into

$$(\nabla_W \pounds_Z g)(U, F) = g((\pounds_Z \nabla)(W, U), F) + g((\pounds_Z \nabla)(W, F), U). \tag{68}$$

Due to the fact that Z is an affine conformal vector field on M, and using the equalities (10), (4), (67) in (68), we deduce that

$$\begin{aligned}2W(\mu)g(U, F) \;&=\; (\nabla_W S)(U, F) + 2\alpha\beta\Big\{g(\varphi W, \varphi U)\eta(F) \\ &\quad +g(\varphi W, \varphi F)\eta(U)\Big\}.\end{aligned} \tag{69}$$

Also, plugging $U = F = \xi$ in (69) and making use of (7), (10), (16) gives $W(\mu) = 0$. This means that μ is constant. Thus, Z becomes an affine vector field and also (69) reduces to

$$(\nabla_W S)(U, F) = -2\alpha\beta\Big\{g(\varphi W, \varphi U)\eta(F) + g(\varphi W, \varphi F)\eta(U)\Big\}. \tag{70}$$

Taking ξ instead of F in (70) and from (8), (10), (15) we get

$$\alpha S(W, U) + 2n\alpha^3 g(W, U) = 2\alpha\beta(g(W, U) - \eta(W)\eta(U).$$

If we rearrange the last equation, we write

$$\alpha(S(W, U) + (2n\alpha^2 - 2\beta)g(W, U) + 2\beta\eta(W)\eta(U)) = 0.$$

Hence, one has

$$\alpha = 0$$

or

$$S(W,U) = -(2n\alpha^2 - 2\beta)g(W,U) - 2\beta\eta(W)\eta(U)$$

which implies that M is either cosymplectic or η-Einstein manifold.

On the other hand, if we take $U = \varphi U$ and $F = \varphi F$ in (70), we obtain

$$(\nabla_T S)(\varphi U, \varphi F) = 0$$

which gives conclusion that M has η-parallel Ricci tensor field. Therefore, we get the requested result. □

The proof of the next theorem can be done similar to that of Theorem 5.

Theorem 6. *Let (g, Z, δ, β) be an η-Ricci soliton on an α-cosymplectic manifold $(M, \varphi, \xi, \eta, g)$ such that Z is a projective vector field. Then, the followings are satisfied:*
i) The vector field Z is affine.
ii) M is either an η-Einstein manifold or a cosymplectic manifold.
iii) M has η-parallel Ricci tensor.

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