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THE SCHOUTEN-VAN KAMPEN CONNECTION ON QUASI-SASAKIAN MANIFOLDS

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Abstract

In the present paper, we study three-dimensional quasi-Sasakian manifolds admitting the Schouten-van Kampen connection. We characterize quasi-Sasakian manifolds and find certain curvature properties with respect to the Schouten-van Kampen connection. Finally, we construct an example of a three-dimensional quasi-Sasakian manifold admitting the Schouten-van Kampen connection which verifies the results discussed in the present paper.

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1 Introduction

In [3], the notion of quasi-Sasakian manifold was introduced by D. E. Blair to unify Sasakian and cosymplectic structure. S. Tanno [15] also added some remarks on quasi-Sasakian structures. Also, the properties of quasi-Sasakian manifolds have been studied by several authors in papers [7, 8, 9]. The Schouten-van Kampen connection have been introduced for non-holomorphic manifolds in papers [13, 17]. The Schouten-van Kampen connection on foliated manifolds have been studied by A. Bejancu [1]. Recently, Z. Olszak studied the Schouten-van Kampen connection on almost contact metric structure [11]. A. Yildiz studied three-dimensional f-Kenmotsu manifolds with respect to the Schouten-van Kampen connection [18]. Also, G. Ghosh studied Sasakian manifolds with respect to the Schouten-van Kampen connection [6].

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be an n-dimensional Riemannian manifold. If there exist a one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidean space such that any geodesic of Riemannian

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manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 3$, M is locally projectively flat if and only if the well known projective curvature tensor vanishes. Here projective curvature tensor $\tilde{P}$ with respect to the Schouten-van Kampen connection is defined by [14]

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-1}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\},$$

for $X, Y, Z \in T(M)$, where $\tilde{R}$ and $\tilde{S}$ are curvature tensor and Ricci tensor of M with respect to the Schouten-van Kampen connection, respectively.

The present paper is organized as follows: After the introduction, we give some required preliminaries in Section 2. In Section 3, we consider projectively flat and ϕ-projectively flat quasi-Sasakian manifolds of dimension three with respect to the Schouten-van Kampen connection. In the next section we study locally ϕ-symmetric three-dimensional quasi-Sasakian manifolds with respect to the Schouten-van Kampen connection. In the last section, we cited an example of a three-dimensional quasi-Sasakian manifold admitting the Schouten-van Kampen connection to verify some results.

2 Preliminaries

Let M be an $n(= 2m+1)$-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is an 1-form and g is compatible Riemannian metric such that [2, 3, 4]

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0, \tag{1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{3}$$

for all $X, Y \in T(M)$. The fundamental 2-form Φ of the manifold is defined by

$$\Phi(X,Y) = g(X, \phi Y),$$

for $X, Y \in T(M)$. M is said to be *quasi-Sasakian* if the almost contact structure (ϕ, ξ, η, g) is normal and the fundamental 2-form is closed ($d\Phi = 0$), which was first introduced by Blair [3]. An almost complex structure J can be defined on the product $M \times \mathbb{R}$ of M and the real line $\mathbb{R}$ by $J(X, t\frac{d}{dt}) = (\phi X - t\xi, \eta(X)\frac{d}{dt})$, where t is a scalar field on $M \times \mathbb{R}$. If the structure J is complex analytic, the almost contact metric structure (ϕ, ξ, η, g) is said to be normal. A necessary and sufficient condition of an almost contact metric manifold to be normal is that the Nijehaus tensor field $N[\phi, \phi] + 2\xi \otimes d\eta$ vanishes on M [2]. The rank of a quasi-Sasakian structure is always odd [3], it is equal to 1 if the structure is cosymplectic and it is equal to $(2m+1)$ if the structure is Sasakian.

An almost contact metric manifold M of dimension three is quasi-Sasakian if and only if [10]

$$\nabla_X \xi = -\beta\phi X, \tag{4}$$

where $X \in T(M)$ and β is some function on M, such that $\xi\beta = 0$, ∇ being the operator of covariant differentiation with respect to the Levi-Civita connection on M. Hence a three-dimensional quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$. For $\beta =$ constant, the manifold reduces to a β-Sasakian manifold and $\beta = 1$ gives the Sasakian structure.

From (4) we have [10]

$$(\nabla_X \phi)Y = \beta(g(X,Y)\xi - \eta(Y)X), \tag{5}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y) = -\beta g(\phi X, Y), \tag{6}$$

for $X, Y \in T(M)$.

From (4) and (5) we get

$$\nabla_X(\nabla_Y \xi) = -(X\beta)\phi Y - \beta^2\{g(X,Y)\xi - \eta(Y)X\} - \beta\phi\nabla_X Y,$$

which implies that

$$R(X,Y)\xi = -(X\beta)\phi Y + (Y\beta)\phi X + \beta^2[\eta(Y)X - \eta(X)Y], \tag{7}$$

$$R(X,\xi)\xi = \beta^2[X - \eta(X)\xi] \tag{8}$$

and

$$R(X,\xi)Y = -(X\beta)\phi Y + \beta^2[g(X,Y)\xi - \eta(Y)X]. \tag{9}$$

In a three-dimensional Riemannian manifold we have

$$\begin{aligned} R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{10}$$

where Q is the Ricci operator, i.e., $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold. The Ricci tensor S of M is given by [11]

$$S(Y,Z) = (\frac{r}{2} - \beta^2)g(Y,Z) + (3\beta^2 - \frac{r}{2})\eta(Y)\eta(Z) - \eta(Y)d\beta(\phi Z) - \eta(Z)d\beta(\phi Y) \tag{11}$$

where r is the scalar curvature of M. Now from (10) and (11) we get

$$QY = (\frac{r}{2} - \beta^2)Y + (3\beta^2 - \frac{r}{2})\eta(Y)\xi - \eta(Y)(\phi grad\beta) - d\beta(\phi Y)\xi, \tag{12}$$

where the gradient of a function f is related to the exterior derivative df by the formula $df(X) = g(gradf, X)$.

From (11) we have

$$S(Y,\xi) = 2\beta^2\eta(Y) - d\beta(\phi Y), \tag{13}$$

$$S(\phi Y, \phi Z) = S(Y, Z) - 2\beta^2\eta(Y)\eta(Z). \tag{14}$$

For an almost contact metric manifold the *Schouten-van Kampen connection* $\tilde{\nabla}$ is given by [12]

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X\xi + (\nabla_X\eta)(Y)\xi. \tag{15}$$

Let M be a three-dimensional quasi-Sasakian manifold. Then from the above equation we have

$$\tilde{\nabla}_X Y = \nabla_X Y + \beta\eta(Y)\phi X + \beta g(X, \phi Y)\xi. \tag{16}$$

The curvature tensor and Ricci tensor of a three-dimensional quasi-Sasakian manifold with respect to the Levi-Civita connection (∇) and Schouten-van Kampen connection ($\tilde{\nabla}$) is given by [12]

$$\begin{aligned}\tilde{R}(X, Y)Z = R(X, Y)Z &+ (X\beta)\{g(Y, \phi Z)\xi + \eta(Z)\phi Y\}\\ &- (Y\beta)\{g(X, \phi Z)\xi + \eta(Z)\phi X\}\\ &+ \beta^2\{g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + \eta(X)\eta(Z)Y\\ &- \eta(Y)\eta(Z)X + g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X\},\end{aligned} \tag{17}$$

$$\tilde{S}(Y, Z) = S(Y, Z) + (\phi Y)(\beta)\eta(Z) - 2\beta^2\eta(Y)\eta(Z), \tag{18}$$

$$\tilde{Q}Y = QY + (\phi Y)(\beta)\xi - 2\beta^2\eta(Y)\xi, \tag{19}$$

$$\tilde{r} = r - 2\beta^2, \tag{20}$$

where $\tilde{R}$, $\tilde{Q}$ and $\tilde{r}$ are curvature tensor, Ricci tensor and scalar curvature of the Schouten-van Kampen connection ($\tilde{\nabla}$).

3 Projective curvature tensor and ϕ-projectively flat on quasi-Sasakian manifolds with respect to the Schouten-van Kampen connection

In this section, we study projectively flat three-dimensional quasi-Sasakian manifold M with respect to the Schouten-van Kampen connection. In a three-dimensional quasi-Sasakian manifold, the projective curvature tensor with respect to the Schouten-van Kampen connection is given by

$$\tilde{P}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{2}\{\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y\}. \tag{21}$$

If $\tilde{P} = 0$, then the manifold M is called *projectively flat* with respect to the Schouten-van Kampen connection.

Theorem 1. *Let M be a three-dimensional quasi-Sasakian manifold with respect to the Schouten-van Kampen connection. Then the following statements are equivalent:*

(i) M is projectively flat with respect to the Schouten-van Kampen connection,

(ii) M is Ricci flat with respect to the Schouten-van Kampen connection,

(iii) β is a constant.

Proof. Let M be a projectively flat manifold with respect to the Schouten-van Kampen connection. Then from (21) we have

$$\tilde{R}(X,Y)Z = \frac{1}{2}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\}, \tag{22}$$

i.e.,

$$g(\tilde{R}(X,Y)Z,W) = \frac{1}{2}\{\tilde{S}(Y,Z)g(X,W) - \tilde{S}(X,Z)g(Y,W)\}. \tag{23}$$

Using (17) and (19) in (23) we get

$$\begin{aligned}
&R(X,Y,Z,W) + (X\beta)\{g(Y,\phi Z)\eta(W) + \eta(Z)g(\phi Y,W)\}\\
&- (Y\beta)\{g(X,\phi Z)\eta(W) + \eta(Z)g(\phi X,W)\}\\
&+ \beta^2\{g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W) + \eta(X)\eta(Z)g(Y,W)\\
&- \eta(Y)\eta(Z)g(X,W) + g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W)\}\\
&= \frac{1}{2}[S(Y,Z) + (\phi Y)\beta\eta(Z) - 2\beta^2\eta(Y)\eta(Z)]g(X,W)\\
&- \frac{1}{2}[S(X,Z) + (\phi X)\beta\eta(Z) - 2\beta^2\eta(X)\eta(Z)]g(Y,W).
\end{aligned} \tag{24}$$

Taking $X = W = \xi$ in (24), we get

$$S(Y,Z) = S(\xi,Z)\eta(Y) - (d\beta)(\phi Z)\eta(Y) - (\phi Y)\beta\eta(Z). \tag{25}$$

Putting this value in (19), we have

$$\tilde{S}(Y,Z) = -(d\beta)(\phi Z)\eta(Y). \tag{26}$$

Clearly, if β is constant, then from (26) we have $\tilde{S}(Y,Z) = 0$; then from (22) we have $\tilde{R}(X,Y)Z=0$.

Conversely, if $\tilde{R}(X,Y)Z = 0$, then using (13) and (18) in (22) we have $\tilde{S}(Y,Z) = 0$, provided β is constant.

Hence the theorem is proved. □

Definition 1. *A quasi-Sasakian manifold M with respect to the Schouten-van Kampen connection is said to be ϕ-projectively flat if*

$$\phi^2\tilde{P}(\phi X,\phi Y)\phi Z = 0.$$

It can be easily seen that $\phi^2\tilde{P}(\phi X,\phi Y)\phi Z = 0$ holds if and only if

$$g(\tilde{P}(\phi X,\phi Y)\phi Z,\phi W) = 0, \tag{27}$$

for $X,Y,Z,W \in T(M)$.

Theorem 2. *Let M be a three-dimensional quasi-Sasakian manifold with constant structure function β is ϕ-projectively flat with respect to the the Schouten-van Kampen connection. Then the manifold is an η-Einstein manifold.*

Proof. Using (21) and (27), ϕ-projectively flat means

$$g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{2}\{\tilde{S}(\phi Y, \phi Z)g(\phi X, \phi W) - \tilde{S}(\phi X, \phi Z)g(\phi Y, \phi W)\}. \tag{28}$$

Let $\{e_1, e_2, \xi\}$ be a local orthonormal basis of the vector fields in M and using the fact that $\{\phi e_1, \phi e_2, \xi\}$ is also a local orthonormal basis, putting $X = W = e_i$ in (28) and summing up with respect to i, we have

$$\sum_{i=1}^{2} g(\tilde{R}(\phi e_i, \phi Y)\phi Z, \phi e_i) = \frac{1}{2}\sum_{i=1}^{2}\{\tilde{S}(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - \tilde{S}(\phi e_i, \phi Z)g(\phi Y, \phi e_i)\}. \tag{29}$$

Using (17) and (19) it can be easily verified that

$$\begin{aligned}\sum_{i=1}^{2} g(\tilde{R}(\phi e_i, \phi Y)\phi Z, \phi e_i) &= \sum_{i=1}^{2} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) + \beta^2 g(\phi Z, \phi Y)\\ &= S(\phi Y, \phi Z) + \beta^2 g(\phi Y, \phi Z)\\ &= \tilde{S}(\phi Y, \phi Z) + \beta^2 g(\phi Y, \phi Z).\end{aligned} \tag{30}$$

$$\sum_{i=1}^{2} g(\phi e_i, \phi e_i) = 2. \tag{31}$$

$$\sum_{i=1}^{2} \tilde{S}(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = \tilde{S}(\phi Y, \phi Z). \tag{32}$$

Using (30), (31) and (32), the equation (29) becomes

$$\tilde{S}(\phi Y, \phi Z) = -\beta^2 g(\phi Y, \phi Z). \tag{33}$$

Putting $Y = \phi Y$ and $Z = \phi Z$ in (33) and using (1) and (18) with β =constant, we get

$$S(Y, Z) = -\beta^2 g(Y, Z) + 2\beta^2 \eta(Y)\eta(Z). \tag{34}$$

Hence the proof. □

4 Locally ϕ-symmetry with respect to the Schouten-van Kampen connection

Definition 2. *A quasi-Sasakian manifold M with respect to the Schouten-van Kampen connection is called to be locally ϕ-symmetric if*

$$\phi^2(\tilde{\nabla}_W \tilde{R})(X, Y)Z = 0, \tag{35}$$

for all vector fields X, Y, Z, W orthogonal to ξ on M.

This notion was introduced by Takahashi [16], for Sasakian manifold.

Theorem 3. *A three-dimensional non-cosymplectic quasi-Sasakian manifold is locally ϕ-symmetry with respect to the Schouten-van Kampen connection $\tilde{\nabla}$ if and only if it is locally ϕ-symmetry with respect the Levi-Civita connection ∇ provided β is constant.*

Proof. Using (4), (6), (16) and (17) we have

$$\begin{aligned}(\tilde{\nabla}_W \tilde{R})(X,Y)Z = (\nabla_W \tilde{R})(X,Y)Z &+ \{R(X,Y,Z,\xi) + (X\beta)g(Y,\phi Z)\\ &- (Y\beta)g(X,\phi Z) + \beta^2(g(X,Z)\eta(Y) - g(Y,Z)\eta(X))\}\beta(\phi W)\\ &- \beta g(\phi W, \tilde{R}(X,Y)Z)\xi. \end{aligned} \tag{36}$$

Now differentiating (17) with respect to W, we obtain

$$\begin{aligned}(\nabla_W \tilde{R})(X,Y)Z = &(\nabla_W R)(X,Y)Z\\ &+ (X\beta)\{g(Y,\phi Z)\nabla_W \xi + \eta(Z)(\nabla_W \phi)Y + (\phi Y)(\nabla_W \eta)(Z)\}\\ &- (Y\beta)\{g(X,\phi Z)\nabla_W \xi + \eta(Z)(\nabla_W \phi)X + (\phi X)(\nabla_W \eta)(Z)\}\\ &+ \beta^2\{g(X,Z)(\nabla_W \eta)Y\xi + g(X,Z)\eta(Y)\nabla_W \xi\\ &- g(Y,Z)(\nabla_W \eta)(X)\xi - g(Y,Z)\eta(X)\nabla_W \xi + (\nabla_W \eta)(X)\eta(Z)Y\\ &+ \eta(Z)(\nabla_W \eta)(Z)Y - (\nabla_W \eta)(Y)\eta(Z)X - \eta(Y)(\nabla_W \eta)(Z)X\\ &+ g(X,\phi Z)(\nabla_W \phi)Y - g(Y,\phi Z)(\nabla_W \phi)(X)\}\\ &+ 2\beta(W\beta)\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)\xi\\ &- \eta(Y)\eta(Z)X + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X\}. \end{aligned} \tag{37}$$

Using (4), (5) and (6) in (37) we get

$$\begin{aligned}(\nabla_W \tilde{R})(X,Y)Z = &(\nabla_W R)(X,Y)Z - \beta(X\beta)\{g(Y,\phi Z)\phi W - g(W,Y)\eta(Z)\xi\\ &+ (\phi Y)g(\phi W, Z) + \eta(Z)\eta(Y)W\}\\ &+ \beta(Y\beta)\{g(X,\phi Z)\phi W - g(W,X)\eta(Z)\xi + (\phi X)g(\phi W, Z)\\ &+ \eta(Z)\eta(X)W\}\\ &+ \beta^3\{-g(X,Z)g(\phi W, Y)\xi - (\phi W)g(X,Z)\eta(Y)\\ &+ g(Y,Z)g(\phi W, X)\xi + (\phi W)g(Y,Z)\eta(X) - g(\phi W, X)\eta(Z)Y\\ &- g(\phi W, Z)\eta(X)Y + g(\phi W, Y)\eta(Z)X + g(\phi W, Z)\eta(Y)X\\ &- g(W,Y)g(X,\phi Z)\xi + \eta(Y)g(X,\phi Z)W + g(W,X)g(Y,\phi Z)\xi\\ &- \eta(X)g(Y,\phi Z)W\}\\ &+ 2\beta(W\beta)\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)\xi\\ &- \eta(Y)\eta(Z)X + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X\}. \end{aligned} \tag{38}$$

Using (38) in (36) we have

$$\begin{aligned}(\tilde\nabla_W \tilde R)(X,Y)Z &= (\nabla_W R)(X,Y)Z + R(X,Y,Z,\xi)\\ &+ \beta(X\beta)\{g(W,Y)\eta(Z)\xi - \eta(Z)\eta(Y)W - (\phi Y)g(\phi W,Z)\}\\ &+ \beta(Y\beta)\{-g(W,X)\eta(Z)\xi + (\phi X)g(\phi W,Z) + \eta(Z)\eta(X)W\}\\ &+ \beta^3\{-g(X,Z)g(\phi W,Y)\xi + g(Y,Z)g(\phi W,X)\xi\\ &- g(\phi W,X)\eta(Z)Y - g(\phi W,Z)\eta(X)Y\\ &+ g(\phi W,Z)\eta(Y)X + g(\phi W,Y)\eta(Z)X\\ &+ \eta(Y)g(X,\phi Z)W - \eta(X)g(Y,\phi Z)W\\ &+ g(W,X)g(Y,\phi Z)\xi - g(W,Y)g(X,\phi Z)\xi\}\\ &+ 2\beta(W\beta)\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)\xi\\ &- \eta(Y)\eta(Z)X + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X\}\\ &- \beta g(\phi W, \tilde R(X,Y)Z)\xi. \end{aligned} \tag{39}$$

Using (1) we get

$$\begin{aligned}\phi^2(\tilde\nabla_W \tilde R)(X,Y)Z &= \phi^2(\nabla_W R)(X,Y)Z + \phi^2(R(X,Y,Z,\xi))\\ &+ \beta(X\beta)\{-\eta(Z)\eta(Y)\phi^2 W - (\phi^3 Y)g(\phi W,Z)\}\\ &+ \beta(Y\beta)\{(\phi^3 X)g(\phi W,Z) + \eta(Z)\eta(X)\phi^2 W\}\\ &+ \beta^3\{g(\phi W,X)\eta(Z)Y - g(\phi W,X)\eta(Z)\eta(Y)\xi\\ &+ g(\phi W,Z)\eta(X)Y - g(\phi W,Z)\eta(X)\eta(Y)\xi\\ &- g(\phi W,Y)\eta(Z)X + g(\phi W,Y)\eta(X)\eta(Z)\xi\\ &- g(\phi W,Z)\eta(Y)X + g(\phi W,Z)\eta(X)\eta(Y)\xi\\ &- g(\phi Z,X)\eta(Y)W + g(\phi Z,X)\eta(Y)\eta(W)\xi\\ &+ g(\phi Z,Y)\eta(X)W - g(\phi Z,Y)\eta(X)\eta(W)\xi\}\\ &+ 2\beta(W\beta)\{\eta(Y)\eta(Z)X - \eta(Y)\eta(Z)\eta(X)\xi\\ &+ g(X,\phi Z)\phi^3 Y - g(Y,\phi Z)\phi^3 X\}. \end{aligned} \tag{40}$$

Taking X, Y, Z, W orthogonal to ξ and using (1), we get from above equation

$$\begin{aligned}\phi^2(\tilde\nabla_W \tilde R)(X,Y)Z &= \phi^2(\nabla_W R)(X,Y)Z\\ &+ \beta(X\beta)g(\phi W,Z)(\phi Y) - \beta(Y\beta)g(\phi W,Z)(\phi X)\\ &- 2\beta(W\beta)\{g(X,\phi Z)(\phi Y) + g(Y,\phi Z)(\phi X)\}. \end{aligned} \tag{41}$$

If β is constant, then $(X\beta) = (Y\beta) = (W\beta) = 0$ for all X, Y, W. Then from (41) we have

$$\phi^2(\tilde\nabla_W \tilde R)(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z.$$

It completes the proof of the theorem.

□

5 Example

In this section we have cited an example [5] of a three-dimensional quasi-Sasakian manifold with respect to the Schouten-Van Kampen connection.

We consider the three-dimensional manifold $M = \{(x, y, z) \in R^3, z \neq 0\}$, where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z},$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any $Z \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. Now, by direct computations we obtain

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = 0, \quad [e_1, e_3] = 0.$$

The Riemannian connection ∇ of the metric tensor g, given by the Koszul's formula is

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) \\ &\quad + g(Z, [X, Y]). \end{aligned}$$

By Koszul formula

$$\begin{array}{lll} \nabla_{e_1} e_3 = -\frac{1}{2}e_2, & \nabla_{e_1} e_2 = \frac{1}{2}e_3, & \nabla_{e_1} e_1 = 0, \\ \nabla_{e_2} e_3 = \frac{1}{2}e_1, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_1 = -\frac{1}{2}e_3, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = \frac{1}{2}e_1, & \nabla_{e_3} e_1 = -\frac{1}{2}e_2. \end{array}$$

From above we see that the manifold satisfies (4) for $\beta = -\frac{1}{2}$, and $e_3 = \xi$. Hence the manifold is a quasi-Sasakian three-manifold.

With the help of the above results it can be verified that

$$\begin{array}{lll} R(e_1, e_2)e_3 = 0, & R(e_2, e_3)e_3 = \frac{1}{4}e_2, & R(e_1, e_3)e_3 = \frac{1}{4}e_1, \\ R(e_1, e_2)e_2 = -\frac{3}{4}e_1, & R(e_2, e_3)e_2 = \frac{1}{4}e_3, & R(e_1, e_3)e_2 = 0, \\ R(e_1, e_2)e_1 = \frac{3}{4}e_2, & R(e_2, e_3)e_1 = 0, & R(e_1, e_3)e_1 = \frac{1}{4}e_3. \end{array}$$

Now we consider the Schouten-Van Kampen connection adapted to this example.

Using (16) and above result we have

$$\begin{array}{lll} \tilde{\nabla}_{e_1}e_3 = -(\beta+\frac{1}{2})e_2, & \tilde{\nabla}_{e_1}e_2 = (\beta+\frac{1}{2})e_3, & \tilde{\nabla}_{e_1}e_1 = 0, \\ \tilde{\nabla}_{e_2}e_3 = (\beta+\frac{1}{2})e_1, & \tilde{\nabla}_{e_2}e_2 = 0, & \tilde{\nabla}_{e_2}e_1 = -(\beta+\frac{1}{2})e_3, \\ \tilde{\nabla}_{e_3}e_3 = 0, & \tilde{\nabla}_{e_3}e_2 = \frac{1}{2}e_1, & \tilde{\nabla}_{e_3}e_1 = -\frac{1}{2}e_2. \end{array}$$

By the above results, we can easily obtain the non-vanishing components of the curvature tensor with respect to the Schouten-van Kampen connection as follows:

$$\begin{array}{ll} \tilde{R}(e_1,e_2)e_3 = -(\beta+\frac{1}{2})^2e_3, & \tilde{R}(e_2,e_3)e_3 = \frac{1}{2}(\beta+\frac{1}{2})e_2, \\ \tilde{R}(e_1,e_2)e_2 = -\{(\beta+\frac{1}{2})^2+\frac{1}{2}\}e_1, & \tilde{R}(e_2,e_3)e_2 = -\frac{1}{2}(\beta+\frac{1}{2})e_3, \\ \tilde{R}(e_1,e_2)e_1 = \{\frac{1}{2}+(\beta+\frac{1}{2})^2)\}e_2, & \tilde{R}(e_2,e_3)e_1 = 0, \\ . & \end{array}$$

$$\begin{array}{l} \tilde{R}(e_1,e_3)e_3 = \frac{1}{2}(\beta+\frac{1}{2})e_1, \\ \tilde{R}(e_1,e_3)e_2 = 0, \\ \tilde{R}(e_1,e_3)e_1 = -\frac{1}{2}(\beta+\frac{1}{2})e_3 \end{array}$$

For $\beta = -\frac{1}{2}$, with the help of above results we get Ricci tensor as follows:

$$S(e_1,e_1) = -\frac{1}{2}, \quad S(e_2,e_2) = -\frac{1}{2}, \quad S(e_3,e_3) = \frac{1}{2}.$$

$$\tilde{S}(e_1,e_1) = \frac{1}{2}, \quad \tilde{S}(e_2,e_2) = -\frac{1}{2}, \quad \tilde{S}(e_3,e_3) = 0.$$

Therefore $r = \sum_{i=1}^{3} S(e_i,e_i) = -\frac{1}{2}$ and $\tilde{r} = \sum_{i=1}^{3} \tilde{S}(e_i,e_i) = 0$. Thus the manifold M is Ricci flat with respect to the Schouten-van Kampen connection. Therefore Theorem 1 is verified.

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