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A NEW CLASS OF HARMONIC FUNCTIONS ASSOCIATED WITH A (p, q)-RUSCHEWEYH OPERATOR

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Abstract

With the use of post-quantum or (p, q)-calculus, in this paper we define a new class $S_H^0(n, p, q, \alpha)$ of certain harmonic functions $f \in S_H^0$ associated with a (p, q)-Ruscheweyh operator $\mathcal{R}_{p,q}^n$. For functions in this class, we obtain a necessary and sufficient convolution condition. A sufficient coefficient inequality is given for functions $f \in S_H^0(n, p, q, \alpha)$. It is proved that this coefficient inequality is necessary for functions in its subclass $\mathcal{T}S_H^0(n, p, q, \alpha)$. Certain properties such as convexity, compactness and results on bounds, extreme points are also derived for functions in the subclass $\mathcal{T}S_H^0(n, p, q, \alpha)$.

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1 Introduction

Jackson [9, 10] was the first to give some applications of quantum calculus known as the q-calculus by introducing the q-analogues of derivative and integral. Research work in connection with function theory and q-calculus was first introduced by Ismail *et al.* [8]. Recently, the q-calculus and its generalization called post-quantum calculus also known as the (p, q)-calculus has been involved in the theory of analytic and harmonic functions in the work [1, 2, 4, 6, 7, 11, 13, 15, 16, 18] (see also [3]). Some definitions related to the (p, q)-calculus are as follows:

Definition 1. *For $0 < q < p \leq 1$, a (p, q)-derivative operator $\partial_{p,q}$ on an analytic function h is defined by*

$$\partial_{p,q} h(z) = \begin{cases} \frac{h(pz)-h(qz)}{(p-q)z} & (z \neq 0), \\ h'(0) & (z = 0). \end{cases} \tag{1}$$

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Note that $\lim\limits_{q\to p}\partial_{p,q}h(z) = h'(pz)$.

Definition 2. *For* $0 < q < p \leq 1$, *a* (p,q)*-number* $[k]_{p,q}$ *is defined by*

$$[k]_{p,q} = \begin{cases} \frac{p^k-q^k}{p-q}, \; k \in \mathbb{C} \\ p^{k-1} + p^{k-2}q + \cdots + pq^{k-2} + q^{k-1}, \; k = 2, 3, \ldots. \end{cases}$$

Definition 3. *For any non-negative integer* k, *a* (p,q)*-number factorial* $[k]_{p,q}!$ *is defined by*

$$[k]_{p,q}! = [1]_{p,q}[2]_{p,q}[3]_{p,q}\ldots[k]_{p,q}, \quad k \neq 0 \quad and \quad [0]_{p,q}! = 1.$$

Definition 4. *For* $k > 0$, *a* (p,q)*-gamma function is defined by*

$$\Gamma_{p,q}(k+1) = [k]_{p,q}\Gamma_{p,q}(k) \qquad and \qquad \Gamma_{p,q}(1) = 1.$$

For $k \in \mathbb{N} \cup \{0\}$,

$$\Gamma_{p,q}(k+1) = [k]_{p,q}!.$$

Definition 5. *For* $k > 0$ *and for* $n \in \mathbb{N} \cup \{0\}$, *a* (p,q)*-shifted factorial* $([k]_{p,q})_n$ *is defined by*

$$([k]_{p,q})_n = \frac{\Gamma_{p,q}(k+n)}{\Gamma_{p,q}(k)} = \begin{cases} [k]_{p,q}[k+1]_{p,q}[k+2]_{p,q}\ldots[k+n-1]_{p,q}, & if\ n \in \mathbb{N}, \\ 1, & if\ n = 0. \end{cases}$$

For a function $h(z) = z^k, \quad k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, we have

$$\partial_{p,q}(z^k) = [k]_{p,q}\; z^{k-1},$$

where $[k]_{p,q}$ is defined by (1).

Let $\mathcal{H}$ denote the class of complex-valued functions $f = u + iv$ which are harmonic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, where u and v are real-valued harmonic functions in $\mathbb{D}$. Functions $f \in \mathcal{H}$ can also be expressed as $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$, called the analytic and co-analytic parts of the function f, respectively. The Jacobian of the function $f = h + \overline{g}$ is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. According to the Lewy [14], every harmonic function $f = h + \overline{g} \in \mathcal{H}$ is locally univalent and sense preserving in $\mathbb{D}$ if and only if $J_f(z) > 0$ in $\mathbb{D}$. By requiring harmonic function to be sense-preserving, we retain some basic properties exhibited by analytic functions, such as the open mapping property, the argument principal, and zeros being isolated (see for detail [5]). The class of all univalent, sense preserving harmonic functions $f = h + \overline{g} \in \mathcal{H}$, with the normalized conditions $h(0) = 0 = g(0)$ and $h'(0) = 1$ is denoted by $S_{\mathcal{H}}$. If $f = h + \overline{g} \in S_{\mathcal{H}}$, then h and g are of the form:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad \text{and} \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k. \tag{2}$$

A subclass of functions $f = h + \overline{g} \in S_{\mathcal{H}}$ with the condition $g'(0) = 0$ is denoted by $S^0_{\mathcal{H}}$. If $f = h + \overline{g} \in S^0_{\mathcal{H}}$, then h and g are of the form:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad \text{and} \qquad g(z) = \sum_{k=2}^{\infty} b_k z^k. \tag{3}$$

Further, if $g(z) \equiv 0$, the class $S_{\mathcal{H}}$ reduces to the class $\mathcal{A}$ of normalized univalent functions.

The convolution of two analytic functions $f(z) = \sum_{k=1}^{\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k$ is defined by $(f * g)(z) = \sum_{k=1}^{\infty} a_k b_k z^k$. The convolution of two harmonic functions $f = h + \overline{g}$ and $F = H + \overline{G}$ is defined by $(f * F)(z) = g * G + \overline{h * H}$. A function $f \in S_{\mathcal{H}}$ is said to be starlike of order α if

$$\Re e \left\{ \frac{\mathcal{D} f(z)}{f(z)} \right\} > \alpha \quad (0 \le \alpha < 1)$$

and the class of all harmonic functions which are starlike of order α is denoted by $S^*_H(\alpha)$, where $\mathcal{D} f(z) = z h'(z) - \overline{z g'(z)}$. A function $f \in S_{\mathcal{H}}$ is said to be convex of order α if $\mathcal{D} f \in S^*_H(\alpha)$ and the class of all harmonic functions which are convex of order α is denoted by $S^c_H(\alpha)$. Classes $S^*_H(\alpha)$ and $S^c_H(\alpha)$ were investigated by Jahangiri [12]. Recently, a q-analogue of the Ruscheweyh operator was introduced by Kanas and Raducanu [13] (see also [15]). Motivated with this q-analogue of Ruscheweyh operator, we define a (p,q)-analogue of the Ruscheweyh operator $R^n_{p,q} : \mathcal{A} \to \mathcal{A}$ of order n $(n > -1)$ by

$$R^n_{p,q} h(z) = h(z) * \phi(p, q, n+1; z), \tag{4}$$

where

$$\phi(p, q, n+1; z) = z + \sum_{k=2}^{\infty} \frac{([k]_{p,q})_n}{\Gamma_{p,q}(n+1)} z^k$$

which converges absolutely in the unit disk $\mathbb{D}$. For simplicity, we denote

$$\psi_k = \frac{([k]_{p,q})_n}{\Gamma_{p,q}(n+1)}. \tag{5}$$

The operator $R^n_{p,q}$ for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ may also be defined by

$$\begin{aligned} R^0_{p,q} h(z) &= h(z), \quad R^1_{p,q} h(z) = R_{p,q} h(z) = z \partial_{p,q} h(z), \\ R^2_{p,q} h(z) &= \frac{z \partial_{p,q} \partial_{p,q}(z h(z))}{[2]_{p,q}!} = \frac{z \partial^2_{p,q}(z h(z))}{[2]_{p,q}!}, \ldots, \\ R^n_{p,q} h(z) &= \frac{z \partial^n_{p,q}(z^{n-1} h(z))}{[n]_{p,q}!}. \end{aligned} \tag{6}$$

Observe that

$$R_{p,q} h(z) = h(z) * R_{p,q} \left(\frac{z}{1-z} \right), \tag{7}$$

and

$$
\begin{aligned}
R_{p,q}\left(\frac{z}{1-z}\right) &= z+\sum_{k=2}^{\infty}[k]_{p,q}z^k \\
&= \frac{z}{(1-pz)(1-qz)}. \qquad (8)
\end{aligned}
$$

If $p=1$, the operator $R^n_{p,q}$ is called a q-analogue of the Ruscheweyh operator denoted by R^n_q (see in [13][15]). As (in case $p=1$) $q\to 1$, the operator $R^n_{p,q}$ reduces to the Ruscheweyh operator R^n, defined by Ruscheweyh in [19].

A (p,q)-Salagean operator $D^m_{p,q}:\mathcal{A}\to\mathcal{A}$ is defined [1] by

$$
\begin{aligned}
D^0_{p,q}h(z) &= h(z), D^1_{p,q}h(z)=z\partial_{p,q}h(z) \\
D^m_{p,q}h(z) &= z\partial_{p,q}(D^{m-1}_{p,q}h((z)), m\in\mathbb{N}
\end{aligned}
$$

and a modified (p,q)-Salagean operator $\mathcal{D}^m_{p,q}$ for harmonic function $f=h+\overline{g}$ is defined for any $m\in\mathbb{N}_0$ by ([1]):

$$\mathcal{D}^m_{p,q}f(z)=D^m_{p,q}h(z)+(-1)^m\overline{D^m_{p,q}g(z)}.$$

Involving the Ruscheweyh operator $R^n_{p,q}$ defined by (6), a (p,q)-modified Ruscheweyh operator $\mathcal{R}^n_{p,q}$ for a harmonic function $f=h+\overline{g}$ is defined for any $n\in\mathbb{N}_0$ by

$$\mathcal{R}^n_{p,q}f(z)=R^n_{p,q}h(z)+(-1)^n\overline{R^n_{p,q}g(z)}, \qquad (9)$$

If $p=1$, the operator $\mathcal{R}^n_{p,q}$ reduces to the operator $\mathcal{R}^n_q$ defined in [16] and (in case $p=1$) as $q\to 1$ the operator $\mathcal{R}^n_{p,q}$ reduces to the modified Ruscheweyh operator $\mathcal{R}^n$ for $f=h+\overline{g}$ see in [17]. We denote $\mathcal{R}^1_{p,q}f(z)=\mathcal{R}_{p,q}f(z)$ and observe that for $f=h+\overline{g}$, $\mathcal{R}_{p,q}f(z)=\mathcal{D}_{p,q}f(z)=z\partial_{p,q}h(z)-\overline{z\partial_{p,q}g(z)}$ and $\mathcal{R}_{p,q}\mathcal{R}_{p,q}f(z)=\mathcal{D}^2_{p,q}f(z)=D^2_{p,q}h(z)+\overline{D^2_{p,q}g(z)}$.

We now define a class $S^0_H(n,p,q,\alpha)$ of the functions $f\in S^0_H$ that satisfy the condition

$$\Re e\left\{\frac{\mathcal{R}_{p,q}(\mathcal{R}^n_{p,q}f)(z)}{\mathcal{R}^n_{p,q}f(z)}\right\}>\alpha,\quad 0\le\alpha<1. \qquad (10)$$

In particular, we denote the classes $S^0_H(0,p,q,\alpha)$ and $S^0_H(1,p,q,\alpha)$, respectively, by (p,q)-$S^*_H(\alpha)$ and (p,q)-$S^c_H(\alpha)$ and are called the classes of (p,q)-harmonic starlike and (p,q)-harmonic convex functions of order α.

As (in case $p=1$) $q\to 1$, the classes (p,q)-$S^*_H(\alpha)$ and (p,q)-$S^c_H(\alpha)$, respectively, reduce to the classes $S^*_H(\alpha)$ and $S^c_H(\alpha)$ of functions $f\in S^0_H$. For $p=1$, the class $S^0_H(n,p,q,\alpha)$ was studied by Murugusundaramoorthy and Jahangiri [16, (vi), p.82] for the functions $f\in S_H$.

In this paper, we introduce a (p,q)-analogue of Ruscheweyh operator for analytic and for harmonic functions and study a new class $S^0_H(n,p,q,\alpha)$ of harmonic functions associated with the (p,q)-Ruscheweyh operator. For functions in this class, we obtain a necessary and sufficient convolution condition. A sufficient coefficient inequality is given for functions $f\in S^0_H(n,p,q,\alpha)$. It is proved that this coefficient inequality is necessary for functions in its subclass $\mathcal{T}S_H^0(n,p,q,\alpha)$. Certain properties such as convexity, compactness and results on extreme points, bounds are also derived for functions in the subclass $\mathcal{T}S_H^0(n,p,q,\alpha)$. Throughout the work we consider the values of p,q $(0<q<p\le 1)$ such that for any $k\in\mathbb{N}$,

$$p+\frac{q^k}{[k]_{p,q}}\ge 1. \tag{11}$$

2 Main results

Theorem 1. *Let $f\in S_{\mathcal{H}}^0$. Then the function $f\in S_H^0(n,p,q,\alpha)$ if and only if*

$$\mathcal{R}_{p,q}^n f(z)*\Phi(z;\zeta)\neq 0 \qquad (\zeta\in\mathbb{C},|\zeta|=1,z\in\mathbb{D}\backslash\{0\}), \tag{12}$$

where

$$\Phi(z;\zeta)=\frac{z\{(1-z)(1+\zeta)-(1-pz)(1-qz)(2\alpha+\zeta-1)\}}{(1-pz)(1-qz)(1-z)} - \frac{\bar{z}\{(1+\zeta)(1-\bar{z})+(1-p\bar{z})(1-q\bar{z})(2\alpha+\zeta-1)\}}{(1-p\bar{z})(1-q\bar{z})(1-\bar{z})}. \tag{13}$$

Proof. Since at $z=0$

$$\frac{\mathcal{R}_{p,q}(\mathcal{R}_{p,q}^n f)(z)}{\mathcal{R}_{p,q}^n f(z)}=1,$$

which proves by (10) that $f\in S_H^0(n,p,q,\alpha)$. Hence, for $z\in\mathbb{D}\backslash\{0\}$, $f=h+\overline{g}\in S_H^0(n,p,q,\alpha)$ if and only if

$$\frac{1}{1-\alpha}\left(\frac{\mathcal{R}_{p,q}(\mathcal{R}_{p,q}^n f)(z)}{\mathcal{R}_{p,q}^n f(z)}-\alpha\right)\neq\frac{\zeta-1}{\zeta+1} \ \ (-1\neq\zeta\in\mathbb{C},|\zeta|=1)$$

or

$$(1+\zeta)\left\{\mathcal{R}_{p,q}(\mathcal{R}_{p,q}^n f)(z)-\alpha\mathcal{R}_{p,q}^n f(z)\right\}-(\zeta-1)\left\{\mathcal{R}_{p,q}^n f(z)-\alpha\mathcal{R}_{p,q}^n f(z)\right\}\neq 0. \tag{14}$$

On using (9), $\mathcal{R}_{p,q}(\mathcal{R}_{p,q}^n f)(z)=R_{p,q}R_{p,q}^n h(z)-(-1)^n\overline{R_{p,q}R_{p,q}^n g(z)}$, $R_{p,q}R_{p,q}^n h(z)=R_{p,q}^n h(z)*\frac{z}{(1-pz)(1-qz)}$ and $R_{p,q}^n h(z)=R_{p,q}^n h(z)*\frac{z}{1-z}$, we express the condition (14) as

$$\begin{aligned}&R_{p,q}^n h(z)*\left[(1+\zeta)\frac{z}{(1-pz)(1-qz)}-\alpha(1+\zeta)\frac{z}{1-z}-(\zeta-1)\frac{z}{1-z}\right.\\&\left.+\alpha(\zeta-1)\frac{z}{1-z}\right]\\&-(-1)^n\overline{R_{p,q}^n g(z)}*\left[(1+\zeta)\frac{\bar{z}}{(1-p\bar{z})(1-q\bar{z})}+\alpha(1+\zeta)\frac{\bar{z}}{1-\bar{z}}+(\zeta-1)\frac{\bar{z}}{1-\bar{z}}\right.\\&\left.-\alpha(\zeta-1)\frac{\bar{z}}{1-\bar{z}}\right]\\&\neq 0\end{aligned}$$

which proves the result (12). $\square$

Taking $n = 0, 1$, respectively we get following results for the classes (p,q)-$S_H^*(\alpha)$ and (p,q)-$S_H^c(\alpha)$:

Corollary 1. *Let $f \in S_{\mathcal{H}}^0$. Then the function $f \in (p,q)$-$S_H^*(\alpha)$ if and only if*

$$f(z) * \Phi(z;\zeta) \neq 0 \qquad (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{D}\backslash\{0\}),$$

where $\Phi(z;\zeta)$ is given by (13).

Corollary 2. *Let $f \in S_{\mathcal{H}}^0$. Then the function $f \in (p,q)$-$S_H^c(\alpha)$ if and only if*

$$\mathcal{R}_{p,q} f(z) * \Phi(z;\zeta) \neq 0 \qquad (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{D}\backslash\{0\}),$$

where $\Phi(z;\zeta)$ is given by (13).

Theorem 2. *Let $f = h + \overline{g} \in \mathcal{H}$, where h and g are given by* (3). *Then $f \in S_H^0(n,p,q,\alpha)$ if*

$$\sum_{k=2}^{\infty} \psi_k \left\{([k]_{p,q} - \alpha)|a_k| + ([k]_{p,q} + \alpha)\,|b_k|\right\} \leq 1 - \alpha, \tag{15}$$

where ψ_k is given by (5).

Proof. It is clear that the theorem is true for the function $f(z) \equiv z$. Let $f = h + \overline{g}$, where h and g of the form (3) and assume that there exist $k \in \{2, 3, \dots\}$ such that $a_k \neq 0$ or $b_k \neq 0$. The condition (11) ensures that $[k]_{p,q}$ is an increasing function of k and hence the condition (15) implies

$$\sum_{k=2}^{\infty} [k]_{p,q}\,(|a_k| + |b_k|) \leq 1. \tag{16}$$

Hence, we have

$$\begin{aligned} |\partial_{p,q} h(z)| - |\partial_{p,q} g(z)| &\geq 1 - \sum_{k=2}^{\infty} [k]_{p,q} |a_k|\,|z|^{k-1} - \sum_{k=2}^{\infty} [k]_{p,q} |b_k|\,|z|^{k-1} \\ &> 1 - |z| \sum_{k=2}^{\infty} \left([k]_{p,q}|a_k| + [k]_{p,q}|b_k|\right) \\ &\geq 1 - |z| > 0 \end{aligned}$$

which proves as $q \to p$ that the function f is locally univalent and sense-preserving in $\mathbb{D}$. Moreover, if $z_1, z_2 \in \mathbb{D}$ and for some p, q $(0 < q < p \leq 1)$ such that $pz_1 \neq qz_2$,

$$\left| \frac{(pz_1)^k - (qz_2)^k}{(pz_1) - (qz_2)} \right| = \left| \sum_{l=1}^{k} (pz_1)^{l-1} (qz_2)^{k-l} \right| \leq \sum_{l=1}^{k} |z_1|^{l-1} p^{l-1} q^{k-l} |z_2|^{k-l} < [k]_{p,q}$$
$$(k = 2, 3, \dots).$$

Hence, by (16), we have

$$
\begin{aligned}
|f(pz_1) - f(qz_2)| &\geq |h(pz_1) - h(qz_2)| - |g(pz_1) - g(qz_2)| \\
&\geq \left| pz_1 - qz_2 - \sum_{k=2}^{\infty} ((pz_1)^k - (qz_2)^k) a_k \right| - \left| \sum_{k=2}^{\infty} \overline{((pz_1)^k - (qz_2)^k) b_k} \right| \\
&\geq |pz_1 - qz_2| \left(1 - \sum_{k=2}^{\infty} \left| \frac{(pz_1)^k - (qz_2)^k}{pz_1 - qz_2} \right| |a_k| - \sum_{k=2}^{\infty} \left| \frac{(pz_1)^k - (qz_2)^k}{pz_1 - qz_2} \right| |b_k| \right) \\
&> |pz_1 - qz_2| \left(1 - \sum_{k=2}^{\infty} [k]_{p,q} |a_k| - \sum_{k=2}^{\infty} [k]_{p,q} |b_k| \right) \geq 0
\end{aligned}
$$

which proves that f is univalent in $\mathbb{D}$. Now using the fact $\Re e(w) > \alpha \Leftrightarrow |1 - \alpha + w| > |1 + \alpha - w|$, to show $f \in S_H^0(n, p, q, \alpha)$, we prove that

$$
\left| \frac{\mathcal{R}_{p,q}(\mathcal{R}_{p,q}^n) f(z) - (1 + \alpha)(\mathcal{R}_{p,q}^n) f(z)}{\mathcal{R}_{p,q}(\mathcal{R}_{p,q}^n) f(z) + (1 - \alpha)(\mathcal{R}_{p,q}^n) f(z)} \right| < 1, \ z \in \mathbb{D} \tag{17}
$$

or,

$$
\left| \mathcal{R}_{p,q}(\mathcal{R}_{p,q}^n) f(z) + (1 - \alpha)(\mathcal{R}_{p,q}^n) f(z) \right| - \left| \mathcal{R}_{p,q}(\mathcal{R}_{p,q}^n) f(z) - (1 + \alpha)(\mathcal{R}_{p,q}^n) f(z) \right| > 0,
$$

where the left-hand-side is

$$
\begin{aligned}
&\left| (2 - \alpha) z + \sum_{k=2}^{\infty} ([k]_{p,q} + 1 - \alpha) \psi_k a_k z^k - (-1)^n \sum_{k=2}^{\infty} ([k]_{p,q} - 1 + \alpha) \psi_k \overline{b_k z^k} \right| \\
&\quad - \left| -\alpha z + \sum_{k=2}^{\infty} ([k]_{p,q} - 1 - \alpha) \psi_k a_k z^k \right. \\
&\qquad \left. -(-1)^n \sum_{k=2}^{\infty} ([k]_{p,q} + 1 + \alpha) \psi_k \overline{b_k z^k} \right| \\
&\geq (2 - \alpha)|z| - \sum_{k=2}^{\infty} ([k]_{p,q} + 1 - \alpha) \psi_k |a_k| |z|^k - \sum_{k=2}^{\infty} ([k]_{p,q} - 1 + \alpha) \psi_k |b_k| |z|^k \\
&\quad - \alpha |z| - \sum_{k=2}^{\infty} ([k]_{p,q} - 1 - \alpha) \psi_k |a_k| |z|^k \\
&\qquad - \sum_{k=2}^{\infty} ([k]_{p,q} + 1 + \alpha) \psi_k |b_k| |z|^k \\
&\geq 2(1 - \alpha)|z| \left\{ 1 - \sum_{k=2}^{\infty} \frac{[k]_{p,q} - \alpha}{1 - \alpha} \psi_k |a_k| |z|^{k-1} - \frac{[k]_{p,q} + \alpha}{1 - \alpha} \psi_k |b_k| |z|^{k-1} \right\} > 0,
\end{aligned}
$$

if(15) holds. This completes the proof of Theorem 2.

□

Definition 6. *Let $\mathcal{T}S_H^0(n,p,q,\alpha)$ be the family of harmonic functions $f = h+\bar{g} \in S_H^0(n,p,q,\alpha)$ such that for that value of n, functions h and g are of the form*

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \qquad \text{and} \qquad g(z) = (-1)^n \sum_{k=2}^{\infty} |b_k| z^k, \qquad z \in \mathbb{D} \tag{18}$$

Theorem 3. *Let $f = h + \overline{g}$, where h and g are of the form (18). Then $f \in \mathcal{T}S_H^0(n,p,q,\alpha)$ if and only if the condition (15) holds.*

Proof. The "if part" follows from Theorem 2. For the "only if" part, assume that $f \in \mathcal{T}S_H^0(n,p,q,\alpha)$, then from (17) we have

$$\left| \frac{-\alpha z - \sum_{k=2}^{\infty}([k]_{p,q} - 1 - \alpha)\psi_k\, |a_k|\, z^k - \sum_{k=2}^{\infty}([k]_{p,q} + 1 + \alpha)\psi_k |b_k| \overline{z}^k}{(2-\alpha)z - \sum_{k=2}^{\infty}\left([k]_{p,q} + 1 - \alpha\right)\psi_k\, |a_k|\, z^k - \sum_{k=2}^{\infty}([k]_{p,q} - 1 + \alpha)\psi_k |b_k| \overline{z}^k} \right| < 1$$

for any $z \in \mathbb{D}$, where ψ_k is given by (5). Since, $\pm\Re e\,(w) \leq |w|$, we have for real value of $z \to 1^-$,

$$\Re e \left(\frac{\alpha + \sum_{k=2}^{\infty}([k]_{p,q} - 1 - \alpha)\psi_k |a_k| + \sum_{k=2}^{\infty}([k]_{p,q} + 1 + \alpha)\psi_k |b_k|}{2 - \alpha - \sum_{k=2}^{\infty}\left([k]_{p,q} + 1 - \alpha\right)\psi_k |a_k| - \sum_{k=2}^{\infty}\left([k]_{p,q} - 1 + \alpha\right)\psi_k |b_k|} \right) \leq 1$$

which proves the inequality (15). □

In particular, we get the following results for the classes (p,q)-$S_H^*(\alpha)$ and (p,q)-$S_H^c(\alpha)$:

Corollary 3. *Let $f = h+\overline{g}$, where h and g are of the form (18). Then $f \in (p,q)$-$S_H^*(\alpha)$ if and only if the condition*

$$\sum_{k=2}^{\infty} \left\{([k]_{p,q} - \alpha)|a_k| + ([k]_{p,q} + \alpha)\,|b_k|\right\} \leq 1 - \alpha,$$

holds.

Corollary 4. *Let $f = h+\overline{g}$, where h and g are of the form (18). Then $f \in (p,q)$-$S_H^c(\alpha)$ if and only if the condition*

$$\sum_{k=2}^{\infty} [k]_{p,q} \left\{([k]_{p,q} - \alpha)|a_k| + ([k]_{p,q} + \alpha)\,|b_k|\right\} \leq 1 - \alpha,$$

holds.

Remark 1. *(in case $p = 1$) as $q \to 1^-$, Corollaries 3 and 4 coincide with the results proved by Jahangiri [12] for the classes $S_H^*(\alpha)$ and $S_H^c(\alpha)$.*

Theorem 4. *The class $\mathcal{T}S_H^0(n,p,q,\alpha)$ forms a convex and compact set.*

Proof. Let for $t = 1, 2,\ f_t \in \mathcal{T}S_H^0(n, p, q, \alpha)$ be of the form

$$f_t(z) = z - \sum_{k=2}^{\infty} |a_{t,k}| z^k + (-1)^n \sum_{k=2}^{\infty} |b_{t,k}| \overline{z}^k. \tag{19}$$

Then for $0 \le \rho \le 1$,

$$\begin{aligned} f(t) \quad : \quad &= \rho f_1(z) + (1-\rho) f_2(z) \\ &= z - \sum_{k=2}^{\infty} \left(\rho|a_{1,k}| + (1-\rho)|a_{2,k}|\right) z^k + (-1)^n \sum_{k=2}^{\infty} (\rho|b_{1,k}| + (1-\rho)|b_{2,k}|) \overline{z}^k \end{aligned}$$

and by Theorem (3), we have

$$\begin{aligned} &\sum_{k=2}^{\infty} \psi_k \left[([k]_{p,q} - \alpha) \{\rho|a_{1,k}| + (1-\rho)|a_{2,k}|\} + ([k]_{p,q} + \alpha) \{\rho|b_{1,k}| + (1-\rho)|b_{2,k}|\}\right] \\ &= \rho \sum_{k=2}^{\infty} \psi_k \{([k]_{p,q} - \alpha)|a_{1,k}| + ([k]_{p,q} + \alpha)|b_{1,k}|\} \\ &+ (1-\rho) \sum_{k=2}^{\infty} \psi_k \{([k]_{p,q} - \alpha)|a_{2,k}| + ([k]_{p,q} + \alpha)|b_{2,k}|\} \\ &\le \rho(1-\alpha) + (1-\rho)(1-\alpha) = 1-\alpha, \end{aligned}$$

where ψ_k is given by (5). Thus the function $f \in \mathcal{T}S_H^0(n, p, q, \alpha)$. Hence, the class $\mathcal{T}S_H^0(n, p, q, \alpha)$ is convex.
On the other hand, let for $t = 1, 2, 3, ...,\ f_t \in \mathcal{T}S_H^0(n, p, q, \alpha)$ be of the form (19). Then by Theorem (3) for $|z| \le r\ (0 < r < 1)$, we get

$$\begin{aligned} |f_t(z)| &\le r + \sum_{k=2}^{\infty} \left(|a_{t,k}| + |b_{t,k}|\right) r^k \\ &\le r + (1-\alpha) \sum_{k=2}^{\infty} \frac{\psi_k}{1-\alpha} \{([k]_{p,q} - \alpha)|a_{t,k}| + ([k]_{p,q} + \alpha)|b_{t,k}|\} r^k \\ &\le r + (1-\alpha) r^2, \end{aligned}$$

where ψ_k is given by (5) and this proves that the class $\mathcal{T}S_H^0(n, p, q, \alpha)$ is locally uniformly bounded. Let $f = h + \overline{g}$, where h and g are given by (18). Further, since for $t = 1, 2, 3, ...,\ f_t \in \mathcal{T}S_H^0(n, p, q, \alpha)$ be of the form (19), by Theorem (3), we have

$$\sum_{k=2}^{\infty} \psi_k \{([k]_{p,q} - \alpha)|a_{t,k}| + ([k]_{p,q} + \alpha)|b_{t,k}|\} \le 1 - \alpha, \tag{20}$$

where ψ_k is given by (5). If we assume that $f_t \to f$, then we conclude that $|a_{t,k}| \to |a_k|$ and $|b_{t,k}| \to |b_k|$ as $t \to \infty$ $(k \in \mathbb{N})$. Let σ_k be the sequence of partial sums of the series $\sum_{k=2}^{\infty} \psi_k \{([k]_{p,q} - \alpha)|a_k| + ([k]_{p,q} + \alpha)|b_k|\}$. Then σ_k is a non decreasing sequence and by (20) it is bounded above by $(1-\alpha)$. Thus, it is convergent and

$$\sum_{k=2}^{\infty} \psi_k\{([k]_{p,q}-\alpha)|a_k| + ([k]_{p,q}+\alpha)|b_k|\} = \lim_{k\to\infty} \sigma_k \leq 1-\alpha.$$

Thus the function $f \in \mathcal{T}S_H^0(n,p,q,\alpha)$ and therefore the class $\mathcal{T}S_H^0(n,p,q,\alpha)$ is compact . □

We can easily find the following result on bounds.

Theorem 5. *Let $f \in \mathcal{T}S_H^0(n,p,q,\alpha)$. Then for $|z| = r$ $(r < 1)$,*

$$r - \frac{1-\alpha}{\psi_2\left([2]_{p,q}-\alpha\right)} r^2 < |f(z)| < r + \frac{1-\alpha}{\psi_2\left([2]_{p,q}-\alpha\right)} r^2$$

where ψ_2 is given by (5) for $k = 2$.

Theorem 6. *A function $f \in clco\mathcal{T}S_H^0(n,p,q,\alpha)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty} \left(x_k \; h_k(z) + y_k \; g_k(z)\right), \tag{21}$$

where

$$\begin{aligned} h_1(z) &= z, \qquad h_k(z) = z - \frac{1-\alpha}{\psi_k([k]_{p,q}-\alpha)} z^k, \qquad (22)\\ g_k(z) &= z + (-1)^n \frac{1-\alpha}{\psi_k([k]_{p,q}+\alpha)} \overline{z}^k \quad (k = 2,3,..., z \in \mathbb{D}) \end{aligned}$$

and ψ_k is given by (5), $x_k,\ y_k \geq 0,\ k = 1,2,3,...,\ x_1 = 1 - \sum_{k=2}^{\infty}(x_k+y_k)$. In particular, the extreme points of the class $\mathcal{T}S_H^0(n,p,q,\alpha)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. Let a function f satisfy (21). Then, we have

$$f(z) = z - (1-\alpha)\sum_{k=2}^{\infty}\left(\frac{x_k}{\psi_k([k]_{p,q}-\alpha)}\; z^k - (-1)^n \frac{y_k}{\psi_k([k]_{p,q}+\alpha)}\; \overline{z}^k\right)$$

such that

$$\begin{aligned} &(1-\alpha)\sum_{k=2}^{\infty}\left(\psi_k\left[([k]_{p,q}-\alpha)\frac{x_k}{\psi_k([k]_{p,q}-\alpha)} + ([k]_{p,q}-\alpha)\frac{y_k}{\psi_k([k]_{p,q}+\alpha)}\right]\right)\\ = \;&(1-\alpha)\,(1-x_1) \leq 1-\alpha \end{aligned}$$

which proves that $f \in clco\mathcal{T}S_H^0(n,p,q,\alpha)$. Conversely, let $f = h + \bar{g} \in clco\mathcal{T}S_H^0(n,p,q,\alpha)$, where h and g are of the form (18) and $x_1 = 1 - \sum_{k=2}^{\infty}(x_k+y_k)$. Set $x_k =$

$\frac{\psi_k([k]_{p,q}-\alpha)}{1-\alpha}|a_k|$ and $y_k = \frac{\psi_k([k]_{p,q}+\alpha)}{1-\alpha}|b_k|$, $k = 2, 3, ...,$ we obtain the representation (21). Since,

$$\begin{aligned} f(z) &= z - \sum_{k=2}^{\infty}|a_k|z^k + (-1)^n\sum_{k=2}^{\infty}|b_k|\overline{z}^k \\ &= z - \sum_{k=2}^{\infty}\frac{(1-\alpha)x_k}{\psi_k([k]_{p,q}-\alpha)}z^k + (-1)^n\sum_{k=2}^{\infty}\frac{(1-\alpha)y_k}{\psi_k([k]_{p,q}+\alpha)}\overline{z}^k \\ &= z - \sum_{k=2}^{\infty}(z-h_k(z))\,x_k + \sum_{k=2}^{\infty}(g_k(z)-z)\,y_k \\ &= \sum_{k=1}^{\infty}(x_k\ h_k(z) + y_k\ g_k(z)). \end{aligned}$$

□

From Theorem 3, we get the following result:

Corollary 5. *Let $f = h + \overline{g} \in \mathcal{T}S_H^0(n,p,q,\alpha)$, where h and g are of the form (18). Then*

$$|a_k| \le \frac{1-\alpha}{\psi_k([k]_{p,q}-\alpha)} \quad \textit{and} \quad |b_k| \le \frac{1-\alpha}{\psi_k([k]_{p,q}+\alpha)}, \quad k = 2,3,4,...,$$

where ψ_k is given by (5). Equality occurs in these inequalities for the extremal functions h_k and g_k given by (22). Furthermore,

$$|a_k| \le 1 \quad \textit{and} \quad |b_k| \le 1, \quad k = 2,3,4,..., \tag{23}$$

Theorem 7. *Let $0 \le \beta < \alpha < 1$ and $f, F \in \mathcal{T}S_H^0(n,p,q,\alpha)$. Then $f * F \in S_H^0(n,p,q,\alpha) \subset S_H^0(n,p,q,\beta)$.*

Proof. Let $f(z) = z - \sum_{k=2}^{\infty}|a_k|z^k + (-1)^n\sum_{k=2}^{\infty}|b_k|\overline{z}^k$ and $F(z) = z - \sum_{k=2}^{\infty}|A_k|z^k + (-1)^n\sum_{k=2}^{\infty}|B_k|\overline{z}^k$. Then

$$(f * F) = z + \sum_{k=2}^{\infty}|a_k||A_k|z^n + \sum_{k=2}^{\infty}|b_k||B_k|\overline{z}^k.$$

Since $F \in \mathcal{T}S_H^0(n,p,q,\alpha)$, with the use of (23) and Theorem 3, we get

$$\begin{aligned} &\sum_{k=2}^{\infty}\psi_k\left(\frac{[k]_{p,q}-\beta}{1-\beta}|a_k||A_k| + \frac{[k]_{p,q}+\beta}{1-\beta}|b_k||B_k|\right) \\ &\le \sum_{k=2}^{\infty}\psi_k\left(\frac{[k]_{p,q}-\alpha}{1-\alpha}|a_k||A_k| + \frac{[k]_{p,q}+\alpha}{1-\alpha}|b_k||B_k|\right) \\ &\le \sum_{k=2}^{\infty}\psi_k\left(\frac{[k]_{p,q}-\alpha}{1-\alpha}|a_k| + \frac{[k]_{p,q}+\alpha}{1-\alpha}|b_k|\right) \le 1 \end{aligned}$$

which proves the result. □

References

[1] Ahuja, Om P., Cetinkaya, A., Ravichandran, V., *Harmonic univalent functions defined by post quantum calculas operators*, Acta Univ. Sapientiae, Mathematica, **11** (2019), no. 1, 5–17.

[2] Altinkaya and Yalçin, S., *Faber polynomial coefficient estimates for bi-univalent functions of complex order based on subordinate conditions involving of the Jackson (p, q)-derivative*, Miskolc Math. Notes **18** (2017), no. 2, 555–572.

[3] Araci, S., Duran, Uğur., Acikgoz, M. and Srivastava, H., *A certain (p, q)-derivative operator and associated divided differences*, Journal of Inequalities and Applications (2016) 2016:301 DOI 10.1186/s13660-016-1240-8.

[4] Burban., M.I. and Klimyk, A., *P, Q-differentiation, P, Q-integration, and P, Q-hypergeometric functions related to quantum groups*, Integral Transform. Spec. Funct. **2** (1994), no. 1, 15–36.

[5] Duren, P., Hengartner, W. and Laugesen, S.R., *The argument principle for harmonic functions*, Amer. Math. Monthly **103** (1996), no. 5, 411–415.

[6] Govindaraj, M. and Sivasubramanian, S., *On a class of analytic functions related to conic domains involving q-calculus*, Anal. Math. **43** (2017), no. 3, 475–487.

[7] Hussain, S., Khan, S., Zaighum A.M. and Darus, M., *Certain subclass of analytic functions related with conic domains and associated with Salagean q-differential operator*, AIMS Mathematics, **2** (2017), no. 4, 622–634, 2017. DOI: 10.3934/Math.2017.4.622

[8] Ismail Merkes E. and Styer, D., *A generalization of starlike functions*, Complex Variables Theory Appl. **14** (1990), no. 1-4, 77–84.

[9] Jackson, F., *On q-functions and a certain difference operator*, Earth Environ. Sci. Trans. R. Soc. of Edinb., **46** (1909), no. 2, 253–281.

[10] Jackson, H.F., *On q-definite integrals*, Q. J. Pure Appl. Math., 41(15), 193–203 (1910).

[11] Jahangiri, M.J., *Harmonic univalent functions defined by q-calculus operator*, International journal of Mathematical analysis and applications **5** (2018), no. 2, 39–43.

[12] Jahangiri, M.J., *Harmonic functions starlike in the unit disk*, J. Math. Anal. Appl. **235** (1999), no. 2, 470–477.

[13] Kanas, S. and Răducanu, D., *Some class of analytic functions related to conic domains*, Math. Slovaca **64** (2014), no. 5, 1183–1196.

[14] Lewy, H., *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. **42** (1936), no. 10, 689–692.

[15] Mahmood, S. and Sokół, J., *New subclass of analytic functions in conical domain associated with Ruscheweyh q-differential operator*, Results Math. **71** (2017), no. 3-4, 1345–1357.

[16] Murugusundaramoorthy, G. and Jahangiri, M.J., *Ruscheweyth-type harmonic functions defined by q-differential operators*, Khayyam J. Math. **5** (2019), no. 1, 79–88.

[17] Murugusundaramoorthy, G. and Vijaya, K., *On certain classes of harmonic functions involving Ruscheweyh derivatives*, Bull. Calcutta Math. Soc. **96** (2004), no. 2, 99–108.

[18] Polatoğlu, Y., Uçar, F. and Yilmaz, B., *q-starlike functions of order alpha*, TWMS J. App. Eng. Math. **8** (2018), no. 1, 186-192. DOI: 10.26837/jaem.428337

[19] Ruscheweyh, S., *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109–115.