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ON D-HOMOTHETICALLY DEFORMED $N(\kappa)$-CONTACT METRIC MANIFOLDS

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Abstract

In the present paper, we have studied generalized weakly symmetric and generalized weakly Ricci symmetric D-homothetically deformed $N(\kappa)$-contact metric manifolds. Also we have studied Ricci solitons on deformed $N(\kappa)$-contact metric manifold and obtained several results if the manifold has generalized weakly symmetric and generalized weakly Ricci symmetric restrictions. We have also proved that there does not exist a Ricci soliton in a D-homothetically deformed $N(\kappa)$-contact metric manifold. Finally, we give an example.

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1 Introduction

In 1988, the notion of κ-nullity distribution of a Riemannian manifold was introduced by S. Tanno in the paper [19]. In κ-nullity distribution the characteristic vector field ξ of the manifold belongs to the distribution. The κ-nullity distribution of a Riemannian manifold M of dimension $(2n+1)$ is given by

$$N(\kappa) : p \longrightarrow N_p(\kappa) = \{Z \in T_pM : R(X,Y)Z = \kappa[g(Y,Z)X - g(X,Z)Y]\}, \quad (1)$$

for all $X, Y \in T_pM$, where κ is a real number and T_pM is the Lie algebra of all vector fields at p. Since the characteristic vector field ξ belongs to the κ-nullity distribution, thus

$$R(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\}. \quad (2)$$

A contact metric manifold of dimension $(2n+1)$ satisfying (2) is said to be an $N(\kappa)$-contact metric manifold. If $\kappa = 1$, then the manifold is reduced to Sasakian

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manifold and for $\kappa = 0$, the manifold is locally isometric to the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold with scalar curvature 4 when $n > 1$ and flat when $n = 1$ [1]. Contact metric manifolds and $N(\kappa)$-contact metric manifolds have been studied by several authors such as Blair {[1], [2]}, Blair, Koufogiorgos and Papantoniou [3], De, Yildiz and Ghosh [10], Kar, Majhi and De [12], and Mandal [14].

The notion of D-homothetic deformation was introduced by Tanno [18] in 1968. In [7], authors have studied D_a-homothetic deformation on generalized (κ, μ)-space forms. In paper [15], H. G. Nagaraja, D. L. Kiran Kumar and D. G. Prakasha have studied D_a-homothetic deformation of (κ, μ)-contact metric manifolds. Nagaraja and Premalatha have studied D_a-homothetic deformation of K-contact manifolds in the paper [16].

The notion of generalized weakly symmetric manifolds was introduced by K. K. Baishya [5]. A Riemannian manifold is said to be generalized weakly symmetric if the Riemann curvature tensor of type $(0,4)$ satisfies the condition

$$\begin{aligned}(\nabla_X \bar{R})(Y,U,V,W) =& \alpha_1(X)\bar{R}(Y,U,V,W) + \beta_1(Y)\bar{R}(X,U,V,W)\\ &+\beta_1(U)\bar{R}(Y,X,V,W) + \delta_1(V)\bar{R}(Y,U,X,W)\\ &+\delta_1(W)\bar{R}(Y,U,V,X) + \alpha_2(X)\bar{G}(Y,U,V,W)\\ &+\beta_2(Y)\bar{G}(X,U,V,W) + \beta_2(U)\bar{G}(Y,X,V,W)\\ &+\delta_2(V)\bar{G}(Y,U,X,W) + \delta_2(W)\bar{G}(Y,U,V,X),\end{aligned} \tag{3}$$

where

$$\bar{G}(Y,U,V,W) = g(U,V)g(Y,W) - g(Y,V)g(U,W) \tag{4}$$

and α_i, β_i, δ_i are non-zero 1-forms defined by $\alpha_i(X) = g(X,\sigma_i)$, $\beta_i(X) = g(X,\rho_i)$ and $\delta_i(X) = g(X,\pi_i)$, where σ_i, ρ_i and π_i are associated vector fields, for $i = 1,2$.

A Riemannian manifold is said to be generalized weakly Ricci symmetric [5] if it satisfies the condition

$$\begin{aligned}(\nabla_X S)(Y,Z) =& A_1(X)S(Y,Z) + B_1(Y)S(X,Z) + D_1(Z)S(Y,X)\\ &+A_2(X)g(Y,Z) + B_2(Y)g(X,Z) + D_2(Z)g(Y,X),\end{aligned} \tag{5}$$

where A_i, B_i and D_i are non-zero 1-forms defined by $A_i(X) = g(X,\theta_i)$, $B_i(X) = g(X,\nu_i)$ and $D_i(X) = g(X,\omega_i)$, θ_i, ν_i and ω_i being associated vector fields, for $i = 1,2$. In [6], authors have studied Ricci solitons in a generalized weakly (Ricci) symmetric D-homothetically deformed Kenmotsu manifold.

The notion of Ricci soliton was introduced by Hamilton [11] which is the generalization of the Einstein metrics and is defined by

$$(L_X g)(Y,Z) + 2S(Y,Z) + 2\lambda g(Y,Z) = 0, \tag{6}$$

where $L_X g$ denotes the Lie-derivatives of Riemannian metric g along the vector field X, λ is a constant, S the Ricci tensor of type $(0,2)$ and Y, Z are arbitrary vector fields on the manifold. Here X is called the potential vector field. A Ricci soliton is called shrinking or steady or expanding according as λ is negative or zero or positive. A Ricci soliton is the limit of the solutions of Ricci flow equation given by

$$\frac{\partial g}{\partial t} = -2S. \tag{7}$$

Ricci solitons on different kinds of manifolds have been studied in the papers [4], [8], [9] and [17] by several authors.

In this paper we would like to study some properties of D-homothetically deformed $N(\kappa)$-contact metric manifolds.

The paper is organized as follows: After the introduction, we give some preliminaries in Section 2. In Section 3, we studied D-homothetic deformation in $N(\kappa)$-contact metric manifolds. Section 4 is devoted to studying generalized weakly symmetric deformed $N(\kappa)$-contact metric manifolds . In Section 5, we deduced some results on generalized weakly Ricci symmetric deformed $N(\kappa)$-contact metric manifolds. In Section 6, we derived Ricci soliton on homothetically deformed $N(\kappa)$-contact metric manifolds. In Section 7, we proved that there does not exist Ricci solitons on homothetically deformed $N(\kappa)$-contact metric manifolds unless $\kappa = 1$. In the last Section, we give an example.

2 Preliminaries

Let M be a $(2n+1)$-dimensional smooth manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is the Riemannian metric on M such that [10], [12]

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1. \tag{8}$$

As a consequence, we get the following:

$$\phi\xi = 0, \quad g(X,\xi) = \eta(X), \quad \eta(\phi X) = 0, \tag{9}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{10}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0, \tag{11}$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y), \tag{12}$$

for all vector fields X, $Y \in \chi(M)$. A differentiable manifold M of dimension $(2n+1)$ with almost contact metric structure is called an almost contact metric manifold.

An almost contact metric manifold is called contact metric manifold if the almost contact metric structure (ϕ, ξ, η, g) satisfies the following condition

$$g(X, \phi Y) = d\eta(X,Y), \tag{13}$$

for all vector fields X, Y. For a contact metric manifold M, we define a symmetric $(1,1)$-tensor field h as $h = \frac{1}{2}L_\xi\phi$, where L_ξ denotes the Lie differentiation in the direction ξ and satisfies the following relations

$$h\xi = 0, \quad h\phi + \phi h = 0, \quad tr(h) = tr(h\phi) = 0, \tag{14}$$

$$\nabla_X \xi = -\phi X - \phi h X. \tag{15}$$

For a $N(\kappa)$-contact metric manifold of dimension $(2n+1), n \geq 1$, we have [10], [12]

$$h^2 = (\kappa - 1)\phi^2, \tag{16}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{17}$$

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\}, \tag{18}$$

$$R(\xi, X)Y = \kappa\{g(X, Y)\xi - \eta(Y)X\}, \tag{19}$$

$$\begin{aligned} S(X, Y) =& 2(n-1)\{g(X, Y) + g(hX, Y)\} \\ &+ \{2n\kappa - 2(n-1)\}\eta(X)\eta(Y), \end{aligned} \tag{20}$$

$$S(X, \xi) = 2n\kappa\eta(X), \tag{21}$$

$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y), \tag{22}$$

$$\begin{aligned} (\nabla_X h)(Y) =& \{(1-\kappa)g(X, \phi Y) + g(X, h\phi Y)\}\xi \\ &+ \eta(Y)[h(\phi X + \phi h X)], \end{aligned} \tag{23}$$

$$r = 2n(2n - 2 + \kappa), \tag{24}$$

for all vector fields X, Y, Z, where R, S and r are the Riemannian curvature, Ricci tensor and scalar curvature, respectively.

3 D-Homothetic deformation in $N(\kappa)$-contact metric manifolds

If the contact metric structure (ϕ, ξ, η, g) of a contact metric manifold M of dimension $(2n + 1)$ is transformed into $(\phi^*, \xi^*, \eta^*, g^*)$, where

$$\phi^* = \phi, \quad \xi^* = \frac{1}{a}\xi, \quad \eta^* = a\eta, \quad g^* = ag + a(a-1)\eta \otimes \eta \tag{25}$$

and a is a positive constant, then the transformation is called a D-homothetic deformation [7].

The relation between the Levi-Civita connections ∇ of g and ∇^* of g^* is given by

$$\nabla^*_X Y = \nabla_X Y + \frac{a-1}{a} g(\phi h X, Y)\xi + (1-a)[\eta(Y)\phi X + \eta(X)\phi Y]. \tag{26}$$

The Riemannian curvature tensor R^* of a D-homothetically deformed $N(\kappa)$-contact metric manifold $(M, \phi^*, \xi^*, \eta^*, g^*)$ is given by [15]

$$\begin{aligned} R^*(X, Y)Z =& R(X, Y)Z + (1-a)[g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &+ 2\eta(X)\eta(Z)hY - 2\eta(Y)\eta(Z)hX - 2g(\phi X, Y)\phi Z \\ &+ \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi] + \frac{1-a}{a}[2\eta(Y)g(hX, Z)\xi \\ &- 2\eta(X)g(hY, Z)\xi + (1-\kappa)\{\eta(Y)g(X, Z)\xi \\ &- \eta(X)g(Y, Z)\xi\} + g(\phi h X, Z)\phi h Y - g(\phi h Y, Z)\phi h X] \\ &+ (a^2 - 1)[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y], \end{aligned} \tag{27}$$

for any vector fields X, Y, Z on M.

Contracting (27), we get the Ricci tensor S^* of D-homothetically deformed $N(\kappa)$-contact metric manifolds as

$$\begin{aligned} S^*(Y,Z) =& S(Y,Z) + \frac{a-1}{a}[(a^2-2a-\kappa+1)g(Y,Z) \\ &+(2na^2+2na+2a-a^2+\kappa-1)\eta(Y)\eta(Z)+2ag(hY,Z)]. \end{aligned} \tag{28}$$

If a $N(\kappa)$-contact metric structure (ϕ,ξ,η,g) on M is transformed into $(\phi^*,\xi^*,\eta^*,g^*)$, then for any vector fields X, Y, Z on M, we have

$$\phi^{*2} = -I + \eta^* \otimes \xi^*, \tag{29}$$

$$\eta^*(\xi^*) = 1, \tag{30}$$

$$\phi^*\xi^* = 0, \tag{31}$$

$$\eta^* \circ \phi^* = 0, \tag{32}$$

$$g^*(\phi^*X, \phi^*Y) = g^*(X,Y) - \eta^*(X)\eta^*(Y), \tag{33}$$

$$g^*(X,\xi^*) = \eta^*(X), \tag{34}$$

$$\nabla^*_X\xi^* = -\phi^*X - \phi^*h^*X, \tag{35}$$

$$(\nabla^*_X\eta^*)(Y) = -g^*(\phi^*X,Y) - g^*(\phi^*h^*X,Y), \tag{36}$$

$$\begin{aligned} R^*(X,Y)\xi^* =& \frac{\kappa+a^2-1}{a^2}[\eta^*(Y)X - \eta^*(X)Y] \\ &+\frac{2a-2}{a}[\eta^*(Y)h^*X - \eta^*(X)h^*Y], \end{aligned} \tag{37}$$

$$\begin{aligned} R^*(\xi^*,X)Y =& \frac{\kappa+a^2-1}{a^2}[g^*(X,Y)\xi^* - \eta^*(Y)X] \\ &+\frac{2a-2}{a}[g^*(h^*X,Y)\xi^* - \eta^*(Y)h^*X], \end{aligned} \tag{38}$$

$$S^*(X,\xi^*) = 2n\frac{\kappa+a^2-1}{a^2}\eta^*(X), \tag{39}$$

where $h^* = \frac{1}{2}L_{\xi^*}\phi^* = \frac{1}{a}h$.
From (36) and (39), we obtain

$$\begin{aligned} (\nabla^*_X S^*)(Y,\xi^*) =& -2n\frac{\kappa+a^2-1}{a^2}[g^*(\phi^*X,Y) + g^*(\phi^*h^*X,Y)] \\ &+S^*(\phi^*X,Y) + S^*(\phi^*h^*X,Y). \end{aligned} \tag{40}$$

4 Generalized weakly symmetric D-homothetically deformed $N(\kappa)$-contact metric manifolds

A D-homothetically deformed $N(\kappa)$-contact metric manifold M of dimension $(2n+1)$ is said to be generalized weakly symmetric if it satisfies the condition

$$\begin{aligned}(\nabla^*_X\bar{R}^*)(Y,U,V,W) =&\alpha^*_1(X)\bar{R}^*(Y,U,V,W)+\beta^*_1(Y)\bar{R}^*(X,U,V,W)\\&+\beta^*_1(U)\bar{R}^*(Y,X,V,W)+\delta^*_1(V)\bar{R}^*(Y,U,X,W)\\&+\delta^*_1(W)\bar{R}^*(Y,U,V,X)+\alpha^*_2(X)\bar{G}^*(Y,U,V,W)\\&+\beta^*_2(Y)\bar{G}^*(X,U,V,W)+\beta^*_2(U)\bar{G}^*(Y,X,V,W)\\&+\delta^*_2(V)\bar{G}^*(Y,U,X,W)+\delta^*_2(W)\bar{G}^*(Y,U,V,X),\end{aligned} \tag{41}$$

where

$$\bar{G}^*(Y,U,V,W)=g^*(U,V)g^*(Y,W)-g^*(Y,V)g^*(U,W) \tag{42}$$

and α^*_i, β^*_i δ^*_i are non-zero 1-forms defined by $\alpha^*_i(X)=g^*(X,\sigma_i)$, $\beta^*_i(X)=g^*(X,\rho_i)$ and $\delta^*_i(X)=g^*(X,\pi_i)$, σ_i, ρ_i and π_i being associated vector fields, for $i=1,2$.
Contracting Y over W in both sides of (41) and using (42), we get

$$\begin{aligned}(\nabla^*_XS^*)(U,V) =&\alpha^*_1(X)S^*(U,V)+\beta^*_1(U)S^*(X,V)+\beta^*_1(R^*(X,U)V)\\&+\delta^*_1(R^*(X,V)U)+\delta^*_1(V)S^*(X,U)+\beta^*_2(X)g^*(U,V)\\&-\beta^*_2(U)g^*(X,V)+\delta^*_2(X)g^*(U,V)-\delta^*_2(V)g^*(X,U)\\&+2n[\alpha^*_2(X)g^*(U,V)\beta^*_2(U)g^*(X,V)+\delta^*_2(V)g^*(X,U)].\end{aligned} \tag{43}$$

Putting $V=\xi^*$ in (43), and using (37), (38), (39) and (40), we obtain

$$\begin{aligned}&-2n\frac{\kappa+a^2-1}{a^2}[g^*(\phi^*X,U)+g^*(\phi^*h^*X,U)]+S^*(\phi^*X,U)\\&+S^*(\phi^*h^*X,U)=[(2n-1)\delta^*_2(\xi^*)-\frac{\kappa+a^2-1}{a^2}\delta^*_1(\xi^*)]g^*(X,U)\\&+[2n\frac{\kappa+a^2-1}{a^2}\alpha^*_1(x)+2n\alpha^*_2(X)+\frac{\kappa+a^2-1}{a^2}\beta^*_1(X)\\&+\frac{2a-2}{a}\beta^*_1(h^*X)+\frac{\kappa+a^2-1}{a^2}\delta^*_1(X)+\frac{2a-2}{a}\delta^*_1(h^*X)\\&+\beta^*_2(X)+\delta^*_2(X)]\eta^*(U)+[(2n-1)\frac{\kappa+a^2-1}{a^2}\beta^*_1(U)\\&-\frac{2a-2}{a}\beta^*_1(h^*U)+(2n-1)\beta^*_2(U)]\eta^*(X)\\&-\frac{2a-2}{a}\delta^*_1(\xi^*)g^*(h^*X,U)+\delta^*_1(\xi^*)S^*(X,U).\end{aligned} \tag{44}$$

Putting successively $X=U=\xi^*$, $X=\xi^*$ and $U=\xi^*$ in (44), we get

$$\frac{\kappa+a^2-1}{a^2}[\alpha^*_1(\xi^*)+\beta^*_1(\xi^*)+\delta^*_1(\xi^*)]+[\alpha^*_2(\xi^*)+\beta^*_2(\xi^*)+\delta^*_2(\xi^*)]=0, \tag{45}$$

$$\begin{aligned}&[2n\frac{\kappa+a^2-1}{a^2}\alpha_1^*(\xi^*)+\frac{\kappa+a^2-1}{a^2}\beta_1^*(\xi^*)+2n\alpha_2^*(\xi^*)+\beta_2^*(\xi^*)\\&+2n\frac{\kappa+a^2-1}{a^2}\delta_1^*(\xi^*)+2n\delta_2^*(\xi^*)]\eta^*(U)+[(2n-1)\frac{\kappa+a^2-1}{a^2}\beta_1^*(U)\\&-\frac{2a-2}{a}\beta_1^*(h^*U)+(2n-1)\beta_2^*(U)]=0\end{aligned} \tag{46}$$

and

$$\begin{aligned}&(2n-1)[\frac{\kappa+a^2-1}{a^2}\delta_1^*(\xi^*)+\delta_2^*(\xi^*)+\frac{\kappa+a^2-1}{a^2}\beta_1^*(\xi^*)\\&+\beta_2^*(\xi^*)]\eta^*(X)+[2n\frac{\kappa+a^2-1}{a^2}\alpha_1^*(X)+2n\alpha_2^*(X)\\&+\frac{\kappa+a^2-1}{a^2}\beta_1^*(X)+\frac{2a-2}{a}\beta_1^*(h^*X)+\frac{\kappa+a^2-1}{a^2}\delta_1^*(X)\\&+\frac{2a-2}{a}\delta_1^*(h^*X)+\beta_2^*(X)+\delta_2^*(X)]=0.\end{aligned} \tag{47}$$

Using (45) in (46) and (47), we get

$$\begin{aligned}&(2n-1)[\frac{\kappa+a^2-1}{a^2}\beta_1^*(\xi^*)+\beta_2^*(\xi^*)]\eta^*(U)\\&=(2n-1)\frac{\kappa+a^2-1}{a^2}\beta_1^*(U)-\frac{2a-2}{a}\beta_1^*(h^*U)+(2n-1)\beta_2^*(U)\end{aligned} \tag{48}$$

and

$$\begin{aligned}&(2n-1)[\frac{\kappa+a^2-1}{a^2}\alpha_1^*(\xi^*)+\alpha_2^*(\xi^*)]\eta^*(X)\\&=2n\frac{\kappa+a^2-1}{a^2}\alpha_1^*(X)+2n\alpha_2^*(X)+\frac{\kappa+a^2-1}{a^2}\beta_1^*(X)\\&+\frac{2a-2}{a}\beta_1^*(h^*X)+\frac{\kappa+a^2-1}{a^2}\delta_1^*(X)\\&+\frac{2a-2}{a}\delta_1^*(h^*X)+\beta_2^*(X)+\delta_2^*(X).\end{aligned} \tag{49}$$

Using (45), (48) and (49) in (44), we obtain

$$\begin{aligned}&-2n\frac{\kappa+a^2-1}{a^2}[g^*(\phi^*X,U)+g^*(\phi^*h^*X,U)]+S^*(\phi^*X,U)\\&+S^*(\phi^*h^*X,U)=[(2n-1)\delta_2^*(\xi^*)-\frac{\kappa+a^2-1}{a^2}\delta_1^*(\xi^*)]g^*(X,U)\\&-(2n-1)[\frac{\kappa+a^2-1}{a^2}\delta_1^*(\xi^*)+\delta_2^*(\xi^*)]\eta^*(X)\eta^*(U)\\&-\frac{2a-2}{a}\delta_1^*(\xi^*)g^*(h^*X,U)+\delta_2^*(\xi^*)S^*(X,U).\end{aligned} \tag{50}$$

Interchanging X and U in (50), we obtain

$$\begin{aligned}&-2n\frac{\kappa+a^2-1}{a^2}[-g^*(\phi^*X,U)+g^*(\phi^*h^*X,U)]-S^*(\phi^*X,U)\\&+S^*(\phi^*h^*X,U)=[(2n-1)\delta_2^*(\xi^*)-\frac{\kappa+a^2-1}{a^2}\delta_1^*(\xi^*)]g^*(X,U)\\&-(2n-1)[\frac{\kappa+a^2-1}{a^2}\delta_1^*(\xi^*)+\delta_2^*(\xi^*)]\eta^*(X)\eta^*(U)\\&-\frac{2a-2}{a}\delta_1^*(\xi^*)g^*(h^*X,U)+\delta_2^*(\xi^*)S^*(X,U).\end{aligned} \tag{51}$$

Adding (50) and (51), we obtain

$$\begin{aligned}&-2n\frac{\kappa+a^2-1}{a^2}g^*(\phi^*h^*X,U)+S^*(\phi^*h^*X,U)\\=&[(2n-1)\delta_2^*(\xi^*)-\frac{\kappa+a^2-1}{a^2}\delta_1^*(\xi^*)]g^*(X,U)\\&-(2n-1)[\frac{\kappa+a^2-1}{a^2}\delta_1^*(\xi^*)+\delta_2^*(\xi^*)]\eta^*(X)\eta^*(U)\\&-\frac{2a-2}{a}\delta_1^*(\xi^*)g^*(h^*X,U)+\delta_2^*(\xi^*)S^*(X,U).\end{aligned} \tag{52}$$

Again, subtracting (51) from (50), we get

$$-2n\frac{\kappa+a^2-1}{a^2}g^*(\phi^*X,U)+S^*(\phi^*X,U)=0. \tag{53}$$

Replacing X by h^*X in (53), we get

$$-2n\frac{\kappa+a^2-1}{a^2}g^*(\phi^*h^*X,U)+S^*(\phi^*h^*X,U)=0. \tag{54}$$

Therefore, from(52) and (54), we obtain

$$\begin{aligned}\delta_2^*(\xi^*)S^*(X,U)=&[\frac{\kappa+a^2-1}{a^2}\delta_1^*(\xi^*)-(2n-1)\delta_2^*(\xi^*)]g^*(X,U)\\&+(2n-1)[\frac{\kappa+a^2-1}{a^2}\delta_1^*(\xi^*)+\delta_2^*(\xi^*)]\eta^*(X)\eta^*(U)\\&+\frac{2a-2}{a}\delta_1^*(\xi^*)g^*(h^*X,U).\end{aligned} \tag{55}$$

Let $\{e_i^*\}$ be an orthonormal basis of the tangent space of the manifold M. Putting $X=U=e_i^*$ in (55), we obtain

$$r*=\frac{4n(\kappa+a^2-1)\delta_1^*(\xi^*)-2n(2n-1)a^2\delta_2^*(\xi^*)}{a^2\delta_2^*(\xi^*)}. \tag{56}$$

Thus we can state the following

Theorem 1. *Let $(\phi^*,\xi^*,\eta^*,g^*)$ be a D-homothetically deformed structure of (ϕ,ξ,η,g) of a $N(\kappa)$-contact metric manifold of dimension $(2n+1)$. If the deformed structure is generalized weakly symmetric, then the Ricci tensor and scalar curvature are respectively given by (55) and (56), provided $\delta_2^*(\xi^*)\neq 0$.*

5 Generalized weakly Ricci symmetric D-homothetically deformed $N(\kappa)$-contact metric manifolds

A D-homothetically deformed $N(\kappa)$-contact metric structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M of dimension $(2n+1)$ is said to be generalized weakly Ricci symmetric if it satisfies the condition

$$\begin{aligned}(\nabla^*_X S^*)(Y,Z) =& A^*_1(X)S^*(Y,Z) + B^*_1(Y)S^*(X,Z)\\ &+D^*_1(Z)S^*(Y,X) + A^*_2(X)g^*(Y,Z)\\ &+B^*_2(Y)g^*(X,Z) + D^*_2(Z)g^*(Y,X),\end{aligned} \tag{57}$$

where A^*_i, B^*_i and D^*_i are non-zero 1-forms defined by $A^*_i(X) = g^*(X,\theta_i)$, $B^*_i(X) = g^*(X,\nu_i)$ and $D^*_i(X) = g^*(X,\omega_i)$, θ_i, ν_i and ω_i being associated vector fields, for $i = 1, 2$.
Putting $Z = \xi^*$ in (57) and using (39), (40), we obtain

$$\begin{aligned}&-2n\frac{\kappa+a^2-1}{a^2}[g^*(\phi^*X,Y) + g^*(\phi^*h^*X,Y)] + S^*(\phi^*X,Y)\\ &+S^*(\phi^*h^*X,Y) = [2n\frac{\kappa+a^2-1}{a^2}B^*_1(Y) + B^*_2(Y)]\eta^*(X)\\ &+[2n\frac{\kappa+a^2-1}{a^2}A^*_1(X) + A^*_2(X)]\eta^*(Y)\\ &+D^*_1(\xi^*)S^*(X,Y) + D^*_2(\xi^*)g^*(X,Y).\end{aligned} \tag{58}$$

Putting Successively $X = Y = \xi^*$, $X = \xi^*$ and $Y = \xi^*$ in (58), we get

$$\begin{aligned}&2n\frac{\kappa+a^2-1}{a^2}A^*_1(\xi^*) + A^*_2(\xi^*) + 2n\frac{\kappa+a^2-1}{a^2}B^*_1(\xi^*)\\ &+B^*_2(\xi^*) + 2n\frac{\kappa+a^2-1}{a^2}D^*_1(\xi^*) + D^*_2(\xi^*) = 0,\end{aligned} \tag{59}$$

$$\begin{aligned}&2n\frac{\kappa+a^2-1}{a^2}B^*_1(Y) + B^*_2(Y) + [2n\frac{\kappa+a^2-1}{a^2}A^*_1(\xi^*)\\ &+A^*_2(\xi^*) + 2n\frac{\kappa+a^2-1}{a^2}D^*_1(\xi^*) + D^*_2(\xi^*)]\eta^*(Y) = 0\end{aligned} \tag{60}$$

and

$$\begin{aligned}&2n\frac{\kappa+a^2-1}{a^2}A^*_1(X) + A^*_2(X) + [2n\frac{\kappa+a^2-1}{a^2}B^*_1(\xi^*)\\ &+B^*_2(\xi^*) + 2n\frac{\kappa+a^2-1}{a^2}D^*_1(\xi^*) + D^*_2(\xi^*)]\eta^*(X) = 0.\end{aligned} \tag{61}$$

Using (59) in (60) and (61), we get respectively

$$2n\frac{\kappa+a^2-1}{a^2}B^*_1(Y) + B^*_2(Y) = [2n\frac{\kappa+a^2-1}{a^2}B^*_1(\xi^*) + B^*_2(\xi^*)]\eta^*(Y) \tag{62}$$

and

$$2n\frac{\kappa+a^2-1}{a^2}A^*_1(X) + A^*_2(X) = [2n\frac{\kappa+a^2-1}{a^2}A^*_1(\xi^*) + A^*_2(\xi^*)]\eta^*(X). \tag{63}$$

Using (59), (62) and (63) in (58), we get

$$\begin{aligned} & -2n\frac{\kappa+a^2-1}{a^2}[g^*(\phi^*X,Y)+g^*(\phi^*h^*X,Y)]+S^*(\phi^*X,Y) \\ & +S^*(\phi^*h^*X,Y)=-[2n\frac{\kappa+a^2-1}{a^2}D_1^*(\xi^*)+D_2^*(\xi^*)]\eta^*(X)\eta^*(Y) \\ & +D_1^*(\xi^*)S^*(X,Y)+D_2^*(\xi^*)g^*(X,Y). \end{aligned} \tag{64}$$

Interchanging X and Y in (64), we obtain

$$\begin{aligned} & -2n\frac{\kappa+a^2-1}{a^2}[-g^*(\phi^*X,Y)+g^*(\phi^*h^*X,Y)]-S^*(\phi^*X,Y) \\ & +S^*(\phi^*h^*X,Y)=-[2n\frac{\kappa+a^2-1}{a^2}D_1^*(\xi^*)+D_2^*(\xi^*)]\eta^*(X)\eta^*(Y) \\ & +D_1^*(\xi^*)S^*(X,Y)+D_2^*(\xi^*)g^*(X,Y). \end{aligned} \tag{65}$$

Adding (64) and (65), we get

$$\begin{aligned} & -2n\frac{\kappa+a^2-1}{a^2}g^*(\phi^*h^*X,Y)+S^*(\phi^*h^*X,Y) \\ & =-[2n\frac{\kappa+a^2-1}{a^2}D_1^*(\xi^*)+D_2^*(\xi^*)]\eta^*(X)\eta^*(Y) \\ & +D_1^*(\xi^*)S^*(X,Y)+D_2^*(\xi^*)g^*(X,Y). \end{aligned} \tag{66}$$

Again, subtracting (65) from (64), we obtain

$$-2n\frac{\kappa+a^2-1}{a^2}g^*(\phi^*X,Y)+S^*(\phi^*X,Y)=0. \tag{67}$$

Replacing X by h^*X in (67), we get

$$-2n\frac{\kappa+a^2-1}{a^2}g^*(\phi^*h^*X,Y)]+S^*(\phi^*h^*X,Y)=0. \tag{68}$$

Therefore, from (66) and (68), we obtain

$$\begin{aligned} D_1^*(\xi^*)S^*(X,Y)= & -D_2^*(\xi^*)g^*(X,Y)+[2n\frac{\kappa+a^2-1}{a^2}D_1^*(\xi^*) \\ & +D_2^*(\xi^*)]\eta^*(X)\eta^*(Y). \end{aligned} \tag{69}$$

Let $\{e_i^*\}$ be an orthonormal basis of the tangent space of the manifold. Putting $X=Y=e_i^*$ in (69) and taking summation over $i=1,2,...(2n+1)$, we obtain

$$r^*=\frac{2n(\kappa+a^2-1)D_1^*(\xi^*)-2na^2D_2^*(\xi^*)}{a^2D_1^*(\xi^*)}. \tag{70}$$

Thus we can state the following

Theorem 2. *Let $(\phi^*,\xi^*,\eta^*,g^*)$ be a D-homothetically deformed structure of (ϕ,ξ,η,g) of a $N(\kappa)$-contact metric manifold of dimension $(2n+1)$. If deformed structure is generalized weakly Ricci symmetric, then the Ricci tensor and scalar curvature are respectively given by (69) and (70), provided $D_1^*(\xi^*)\neq 0$.*

6 Ricci solitons on D-homothetically deformed $N(\kappa)$-contact metric manifolds

Let $(M, \phi^*, \xi^*, \eta^*, g^*)$ be a D-homothetically deformed $N(\kappa)$-contact metric manifold. A Ricci soliton (g^*, V, λ) is defined on $(M, \phi^*, \xi^*, \eta^*, g^*)$ as

$$(L^*_V g^*)(X,Y) + 2S^*(X,Y) + 2\lambda g^*(X,Y) = 0, \tag{71}$$

where $L^*_V g^*$ denotes the Lie-derivative of Riemannian metric g^* along the vector field V, S^* is the Ricci tensor on $(M, \phi^*, \xi^*, \eta^*, g^*)$.

Let the potential vector field V be the Reeb vector field ξ^*. Then we have

$$(L^*_{\xi^*} g^*)(X,Y) + 2S^*(X,Y) + 2\lambda g^*(X,Y) = 0. \tag{72}$$

Using (35), we obtain

$$(L^*_{\xi^*} g^*)(X,Y) = -2g^*(\phi^* h^* X, Y). \tag{73}$$

Combining (72) and (73), we obtain

$$S^*(X,Y) = -\lambda g^*(X,Y) + 2g^*(\phi^* h^* X, Y). \tag{74}$$

Replacing Y by ξ^* in (74) and using (39), we get

$$\lambda = -2n\frac{\kappa + a^2 - 1}{a^2}. \tag{75}$$

Therefore, equation (74) can be written as

$$S^*(X,Y) = 2n\frac{\kappa + a^2 - 1}{a^2} g^*(X,Y) + 2g^*(\phi^* h^* X, Y). \tag{76}$$

Thus we can state

Theorem 3. *If a D-homothetically deformed $N(\kappa)$-contact metric manifold admits Ricci soliton, then the Ricci soliton is shrinking or steady or expanding according as $\kappa > 1 - a^2$ or $\kappa = 1 - a^2$ or $\kappa < 1 - a^2$.*

From (55) and (76), we obtain

$$\begin{aligned}
&\delta_2^*(\xi^*)[2n\frac{\kappa + a^2 - 1}{a^2} g^*(X,Y) + 2g^*(\phi^* h^* X, Y)] \\
=&[\frac{\kappa + a^2 - 1}{a^2}\delta_1^*(\xi^*) - (2n-1)\delta_2^*(\xi^*)] g^*(X,Y) \\
+&(2n-1)[\frac{\kappa + a^2 - 1}{a^2}\delta_1^*(\xi^*) + \delta_2^*(\xi^*)]\eta^*(X)\eta^*(Y) \\
+&\frac{2a-2}{a}\delta_1^*(\xi^*) g^*(h^* X, Y).
\end{aligned} \tag{77}$$

Putting $X = Y = \xi^*$ in (77), we obtain

$$\kappa = 1 - a^2$$

or,

$$\delta_1^*(\xi^*) = \delta_2^*(\xi^*).$$

Thus we can state the following

Corollary 1. *If a generalized weakly symmetric D-homothetically deformed $N(\kappa)$-contact metric manifold admits Ricci soliton, then the soliton is steady or the 1-forms δ_1^* and δ_2^* satisfy the relation $\delta_1^*(\xi^*) = \delta_2^*(\xi^*)$.*

From (69) and (76), we obtain

$$\begin{aligned} &D_1^*(\xi^*)[2n\frac{\kappa + a^2 - 1}{a^2}g^*(X,Y) + 2g^*(\phi^* h^* X, Y)] \\ =& -D_2^*(\xi^*)g^*(X,Y) + [2n\frac{\kappa + a^2 - 1}{a^2}D_1^*(\xi^*) \\ &+ D_2^*(\xi^*)]\eta^*(X)\eta^*(Y). \end{aligned} \tag{78}$$

Contracting, we obtain

$$2n\frac{\kappa + a^2 - 1}{a^2}D_1^*(\xi^*) + D_2^*(\xi^*) = 0. \tag{79}$$

Thus we can state the following

Corollary 2. *If a generalized weakly Ricci symmetric D-homothetically deformed $N(\kappa)$-contact metric manifold admits Ricci soliton, then the 1-forms D_1^* and D_2^* satisfy the equation (79).*

7 Non-existence of Ricci solitons on D-homothetically deformed $N(\kappa)$-contact metric manifolds

Using (25) in (76), we obtain

$$S^*(X,Y) = 2n\frac{\kappa + a^2 - 1}{a^2}[ag(X,Y) + a(a-1)\eta(X)\eta(Y)] + 2g(\phi h X, Y). \tag{80}$$

From (20) and (28), we obtain

$$\begin{aligned} S^*(X,Y) =& 2(n-1)\{g(X,Y) + g(hX,Y)\} \\ &+ \{2n\kappa - 2(n-1)\}\eta(X)\eta(Y) \\ &+ \frac{a-1}{a}[(a^2 - 2a - \kappa + 1)g(X,Y) \\ &+ (2na^2 + 2na + 2a - a^2 + \kappa - 1)\eta(X)\eta(Y) \\ &+ 2ag(hX,Y)]. \end{aligned} \tag{81}$$

Comparing (80) and (81), we obtain

$$\begin{aligned}
&2n\frac{\kappa+a^2-1}{a^2}[ag(X,Y)+a(a-1)\eta(X)\eta(Y)]\\
&+2g(\phi hX,Y)=2(n-1)\{g(X,Y)+g(hX,Y)\}\\
&+\{2n\kappa-2(n-1)\}\eta(X)\eta(Y)\\
&+\frac{a-1}{a}[(a^2-2a-\kappa+1)g(X,Y)\\
&+(2na^2+2na+2a-a^2+\kappa-1)\eta(X)\eta(Y)\\
&+2ag(hX,Y)].
\end{aligned} \tag{82}$$

Replacing X by ϕX in (82), we obtain

$$\begin{aligned}
&2n\frac{\kappa+a^2-1}{a}g(\phi X,Y)+2g(hX,Y)\\
=&2(n-1)\{g(\phi X,Y)-g(\phi hX,Y)\}\\
&+\frac{a-1}{a}[(a^2-2a-\kappa+1)g(\phi X,Y)\\
&-2ag(hX,Y)].
\end{aligned} \tag{83}$$

Interchanging X and Y in (83), we get

$$\begin{aligned}
&2n\frac{\kappa+a^2-1}{a}g(\phi X,Y)-2g(hX,Y)\\
=&2(n-1)\{g(\phi X,Y)+g(\phi hX,Y)\}\\
&+\frac{a-1}{a}[(a^2-2a-\kappa+1)g(\phi X,Y)\\
&+2ag(hX,Y)].
\end{aligned} \tag{84}$$

Subtracting (84) from (83), we obtain

$$g(hX,Y)=-(n+a-2)g(\phi hX,Y). \tag{85}$$

Replacing X by ϕX in (85), we get

$$g(\phi hX,Y)=(n+a-2)g(hX,Y). \tag{86}$$

Therefore, from (85) and (86), we obtain

$$[(n+a-2)^2+1]g(hX,Y)=0. \tag{87}$$

But the equation $(n+a-2)^2+1=0$ has no real root. Thus, it follows from (87) that $g(hX,Y)=0$, that is, $h=0$. Applying $h=0$ in (16), we get $\kappa=1$. Hence the manifold reduces to Sasakian manifold.

Thus, we can state the following

Theorem 4. *There does not exist Ricci solitons in a homothetically deformed $N(\kappa)$-contact metric manifold of dimension $(2n+1)$, $n\geq 1$, whose potential vector field is the Reeb vector field ξ^*, unless $\kappa=1$.*

8 Examples

In [10], De, Yildiz and Ghosh gave an example of $N(\kappa)$-contact metric manifold. From their example, we get an example of $N(\kappa)$-contact metric manifold by taking $\lambda = 2$.

Let us consider the manifold $M = \{x, y, z \in \mathbb{R}^3 : z \neq 0\}$ of dimension 3, where $\{x, y, z\}$ are standard co-ordinates in $\mathbb{R}^3$. We choose the vector fields e_1, e_2 and e_3 which satisfy

$$[e_1, e_2] = 3e_3, \quad [e_1, e_3] = e_2, \quad [e_2, e_3] = 2e_1.$$

Let the metric tensor g be defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$$

and

$$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

The 1-form η is defined by

$$\eta(X) = g(X, e_1),$$

for all X on M. Let ϕ be the $(1, 1)$-tensor field defined by

$$\phi(e_1) = 0, \quad \phi(e_2) = e_3, \quad \phi(e_3) = -e_2.$$

Then we find that

$$\eta(e_1) = 1, \quad \phi^2 X = -X + \eta(X)e_1,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad d\eta(X, Y) = g(X, \phi Y),$$

for any vector fields X, Y on M. Thus (ϕ, e_1, η, g) defines a contact structure.

Let ∇ be the Levi-Civita connection on M, then by Koszul's formula, we obtain

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0,$$

$$\nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_1 = -3e_3, \quad \nabla_{e_2} e_3 = 3e_1,$$

$$\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_1 = -e_2, \quad \nabla_{e_3} e_2 = e_1.$$

From the above expressions of ∇, we obtain

$$he_1 = 0, \quad he_2 = 2e_2, \quad he_3 = -2e_3.$$

Using the formula $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, we get

$$R(e_1, e_2)e_2 = -3e_1, \quad R(e_2, e_1)e_1 = -3e_2, \quad R(e_2, e_3)e_3 = 3e_2,$$

$$R(e_3, e_2)e_2 = 3e_3, \quad R(e_1, e_3)e_3 = -3e_1, \quad R(e_3, e_1)e_1 = -3e_3,$$

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_2 = 0.$$

Thus the given manifold is an $N(\kappa)$-contact manifold with $\kappa = -3$.

From the expressions of curvature tensor, we get

$$S(e_1, e_1) = -6, \quad S(e_2, e_2) = 0, \quad S(e_3, e_3) = 0.$$

Let r be the scalar curvature, then from the above

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -6.$$

Let ∇^* be the Levi-Civita connection with respect to g^*, then we obtain

$$\nabla^*_{e_1^*} e_1^* = 0, \quad \nabla^*_{e_1^*} e_2 = \frac{1-a}{a} e_3, \quad \nabla^*_{e_1^*} e_3 = \frac{a-1}{a} e_2,$$

$$\nabla^*_{e_2} e_2 = 0, \quad \nabla^*_{e_2} e_1^* = -\frac{a+2}{a} e_3, \quad \nabla^*_{e_2} e_3 = \frac{5a-2}{a} e_1,$$

$$\nabla^*_{e_3} e_3 = 0, \quad \nabla^*_{e_3} e_1^* = \frac{a-2}{a} e_2, \quad \nabla^*_{e_3} e_2 = \frac{3a-2}{a} e_1.$$

The components of homothetically deformed Riemann curvature tensor are given by

$$R^*(e_1^*, e_2)e_2 = \frac{5a^2 - 16a + 8}{a^2} e_1, \quad R^*(e_2, e_1^*)e_1^* = \frac{a^2 + 4a - 8}{a^2} e_3,$$

$$R^*(e_2, e_3)e_3 = \frac{-7a^2 + 14a - 4}{a} e_2, \quad R^*(e_3, e_2)e_2 = \frac{a^2 + 6a - 4}{a} e_3,$$

$$R^*(e_1^*, e_3)e_3 = -3e_1, \quad R^*(e_3, e_1^*)e_1^* = \frac{a-4}{a} e_3,$$

$$R^*(e_1^*, e_2)e_3 = 0, \quad R^*(e_2, e_3)e_1^* = 0, \quad R^*(e_1^*, e_3)e_2 = 0.$$

Thus the components of homothetically deformed Ricci tensor are

$$S^*(e_1^*, e_1^*) = \frac{2a^2 - 16a + 8}{a},$$

$$S^*(e_2, e_2) = \frac{-7a^3 + 15a^2 - 8}{a^2},$$

$$S^*(e_3, e_3) = \frac{a^2 + 7a - 8}{a}.$$

Therefore, the corresponding scalar curvature is given by

$$r^* = \frac{-4a^3 + 6a^2 - 8}{a^2}.$$

In particular, if we take $a = 2$, then the components of deformed Ricci tensor are

$$S^*(e_1^*, e_1^*) = -8, \quad S^*(e_2, e_2) = -1, \quad S^*(e_3, e_3) = 5.$$

Putting $n = 1$, $\kappa = -3$ and $a = 2$ in equation (80) and then comparing with the above three expressions of Ricci tensor, we conclude that there does not exist Ricci soliton of a homothetically deformed $N(-3)$-contact metric manifold and that verifies theorem 4.

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References

[1] Blair, D.E., *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math., **509**, Springer-Verlag 1976.

[2] Blair, D.E., *Two remarks on contact metric structure*, Tohoku Math. J., **29** (1977), 319-324.

[3] Blair, D.E., Koufogiorgos, T. and Papantoniou, B.J., *Contact metric manifolds satisfying a nullity condition*, Israel J. Math., **19** (1995), 189-214.

[4] Bejan, C.L. and Crasmareanu, M., *Ricci solitons in manifolds with quasi-constant curvature*, Publ. Math. Debrecen **78** (2011), no. 1, 235-243.

[5] Baishya, K.K., *On generalized weakly symmetric manifolds*, Bull. Transilv. Univ. Braşov Sec. III **10**(59) (2017), no. 1, 31-38.

[6] Blaga, A.M., Baishya, K.K. and Sarkar, N., *Ricci solitons in a generalized weakly (Ricci) symmetric D-homothetically deformed Kenmotsu manifold*, Ann. Univ. Paedagog. Crac. Stud. Math. **18** (2019), 123-136.

[7] Carriazo, A. and Martín-Molina, V., *Generalized* (κ, μ)*-space forms and* D_a*-homothetic deformations*, Balkan J. Geom. Appl. **16** (2011), no. 1, 37-47.

[8] De, U.C. and Mondal, A.K., 3*-dimensional quasi-Sasakian manifolds and Ricci solitons*, SUT J. Math. **48** (2012), no. 1, 71-81.

[9] De, U.C. and Mandal, K., *Ricci solitons and gradient Ricci solitions on* $N(\kappa)$*-paracontact manifolds*, J. of Math. Phy., Ana.,Geom., 2019, Vol. **15**, no. 3, 307-320.

[10] De, U.C., Yildiz, A. and Ghosh, S., *On a class of* $N(\kappa)$*-contact metric manifolds*, Math. Reports **16**(66) (2014), no. 2, 207-217.

[11] Hamilton, R. S., *The Ricci flow on surfaces*, Contemporary Mathematics, **71** (1988), 237-261.

[12] Kar, D., Majhi, P. and De, U.C., η*-Ricci solitons on* 3*-dimensional* $N(\kappa)$*-contact metric manifolds*, Acta Uni. Apu., **54** (2018), 71-88.

[13] Kenmotsu, K., *A class of almost contact Riemannian manifolds*, Tohoku Math. J., **24** (1972), 93-103.

[14] Mandal, T., *Certain results on* $N(\kappa)$*-contact metric manifolds with conformal Ricci solitons*, J. Adv. Math. Stud. **14** (2021), no. 3, 343-351.

[15] Nagaraja, H.G., Kiran Kumar, D.L. and Prakasha, D.G., D_a*-Homothetic deformation and Ricci solitions in* (κ, μ)*-contact metric manifolds*, Konuralp J. of Math., **7** (2019), no. 1, 122-127.

[16] Nagaraja, H.G. and Premalatha, C.R., *D_a-Homothetic deformation of K-contact manifolds*, International Scholarly Research Notice, 2013.

[17] Sarkar, A. and Biswas, G.G., *Ricci solitons on three dimensional generalized Sasakian space forms with quasi Sasakian metric*, Afr. Mat., **31** (2020), 455-463.

[18] Tanno, S., *The topology of contact Riemannian manifolds*, Illinois Journals of Mathematics, Vol **12**, pp. 700-717, 1968.

[19] Tanno, S., *Ricci curvatures of contact Riemannian manifolds*, Tohoku Math. J., **40** (1988), 441-448.