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A COMPREHENSIVE CLASS OF BI-UNIVALENT FUNCTIONS SUBORDINATE TO SHELL-LIKE CURVES

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Abstract

In the current work, we discuss certain stirring results of coefficient estimates of a unified class which is bridge between bi-starlike and bi-convex functions related to shell-like curves by means of subordination. Further, appropriate connections are discussed.

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1 Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$\mathfrak{s}(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathscr{S}$ we shall denote the class of all functions in $\mathscr{A}$ which are univalent in $\mathbb{D}$.

Let $\mathscr{P}$ denote the class of Caratheodory functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \qquad (z \in \mathbb{D})$$

which are analytic with $\Re\{p(z)\} > 0$ (see [10]). It is well known that the following correspondence between the class $\mathscr{P}$ and the class of Schwarz functions w exists:

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$p \in \mathscr{P}$ if and only if $p(z) = 1 + w(z) \,/\, 1 - w(z)$. Let $\mathscr{P}(\beta)$, $0 \leq \beta < 1$, denote the class of analytic functions p in $\mathbb{D}$ with $p(0) = 1$ and $\Re\{p(z)\} > \beta$.

For analytic functions $\mathfrak{s}$ and $\mathfrak{j}$ in $\mathbb{D}$, $\mathfrak{s}$ is said to be subordinate to $\mathfrak{j}$ if $\exists$ an analytic function w such that

$$w(0) = 0, \qquad |w(z)| < 1 \quad \text{and} \quad \mathfrak{s}(z) = \mathfrak{j}(w(z)) \qquad (z \in \mathbb{D}).$$

This subordination will be denoted here by

$$\mathfrak{s} \prec \mathfrak{j} \qquad (z \in \mathbb{D})$$

or, conventionally, by

$$\mathfrak{s}(z) \prec \mathfrak{j}(z) \qquad (z \in \mathbb{D}).$$

In particular, when $\mathfrak{j}$ is univalent in $\mathbb{D}$,

$$\mathfrak{s} \prec \mathfrak{j} \qquad (z \in \mathbb{D}) \;\Leftrightarrow\; \mathfrak{s}(0) = \mathfrak{j}(0) \quad \text{and} \quad \mathfrak{s}(\mathbb{D}) \subset \mathfrak{j}(\mathbb{D}).$$

Some of the important and well-investigated subclasses of the univalent function class $\mathscr{S}$ include (for example) the class $\mathscr{S}^*(\alpha)$ of starlike functions of order α $(0 \leqq \alpha < 1)$ in $\mathbb{D}$ and the class $\mathscr{K}(\alpha)$ of convex functions of order α $(0 \leqq \alpha < 1)$ in $\mathbb{D}$, the class $\mathscr{S}^*(\varphi)$ of Ma-Minda starlike functions and the class $\mathscr{K}(\varphi)$ of Ma-Minda convex functions (φ is an analytic function with positive real part in $\mathbb{D}$, $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps $\mathbb{D}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis) (see [10]).

It is well known that every function $\mathfrak{s} \in \mathscr{S}$ has an inverse $\mathfrak{s}^{-1}$, defined by

$$\mathfrak{s}^{-1}(\mathfrak{s}(z)) = z \qquad (z \in \mathbb{D})$$

and

$$\mathfrak{s}(\mathfrak{s}^{-1}(w)) = w \qquad (|w| < r_0(f);\; r_0(f) \geqq \frac{1}{4}),$$

where

$$\mathfrak{s}^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots.$$

A function $\mathfrak{s} \in \mathscr{A}$ is said to be bi-univalent in $\mathbb{D}$ if both $\mathfrak{s}(z)$ and $\mathfrak{s}^{-1}(z)$ are univalent in $\mathbb{D}$. Let Σ denote the class of bi-univalent functions in $\mathbb{D}$ given by (1). Recently, in their pioneering work on the subject of bi-univalent functions, Srivastava et al. [23] actually revived the study of the coefficient problems involving bi-univalent functions. Various subclasses of the bi-univalent function class Σ were introduced and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1) were found in several recent investigations (see, for example, [1, 2, 3, 4, 5, 7, 11, 12, 13, 14, 15, 18, 19, 21, 22, 24, 25] and references therein). The afore-cited papers on the subject were actually motivated by the pioneering work of Srivastava et al. [23]. However, the problem to find the coefficient bounds on $|a_n|$ $(n = 3, 4, \dots)$ for functions $\mathfrak{s} \in \Sigma$ is still an open problem.

The classes $\mathscr{SL}(\tilde{\phi})$ and $\mathscr{KS}(\tilde{\phi})$ of shell-like functions and convex shell-like functions are respectively, characterized by $z\mathfrak{s}' / \mathfrak{s}(z) \prec \tilde{\phi}(z)$ or $1 + z^2\mathfrak{s}'' / \mathfrak{s}'(z) \prec \tilde{\phi}(z)$, where $\tilde{\phi}(z) = (1 + \varsigma^2 z^2) / (1 - \varsigma z - \varsigma^2 z^2)$, $\varsigma = (1 - \sqrt{5}) / 2 \approx -0.618$. The classes $\mathscr{SL}(\tilde{\phi})$ and $\mathscr{KS}(\tilde{\phi})$ were introduced and studied by Sokół [20] and Dziok et al. [8] respectively (see also [9, 17]). The function $\tilde{\phi}$ is not univalent in $\mathbb{D}$, but it is univalent in the disc $|z| < (3 - \sqrt{5}) / 2 \approx 0.38$. For example, $\tilde{\phi}(0) = \tilde{\phi}(-1 / 2\varsigma) = 1$ and $\tilde{\phi}\,(e^{\mp} \arccos(1/4)) = \sqrt{5} / 5$ and it may also be noticed that $1 / |\varsigma| = |\varsigma| / 1 - |\varsigma|$ which shows that the number $|\varsigma|$ divides $(0,\, 1)$ such that it fulfills the golden section. The image of the unit circle $|z| = 1$ under $\tilde{\phi}$ is a curve described by the equation given by $\left(10x - \sqrt{5}\right) y^2 = \left(\sqrt{5} - 2x\right)\left(\sqrt{5}x - 1\right)^2$, which is translated and revolved trisectrix of Maclaurin. The curve $\tilde{\phi}\left(re^{it}\right)$ is a closed curve without any loops for $0 < r \leq r_0 = (3 - \sqrt{5}) / 2 \approx 0.38$. For $r_0 < r < 1$, it has a loop and for $r = 1$, it has a vertical asymptote. Since ς satisfies the equation $\varsigma^2 = 1 + \varsigma$, this expression can be used to obtain higher powers ς^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of ς and 1. The resulting recurrence relationships yield Fibonacci numbers ϑ_n

$$\varsigma^n = \vartheta_n \varsigma + \vartheta_{n-1}.$$

Recently Raina and Sokół [17], taking $\varsigma z = t$, showed that

$$\tilde{\phi}(z) \quad = \quad \frac{1 + \varsigma^2 z^2}{1 - \varsigma z - \varsigma^2 z^2} = 1 + \sum_{n=2}^{\infty} \left(\vartheta_{n-1} + \vartheta_{n+1}\right) \varsigma^n z^n, \tag{2}$$

where

$$\vartheta_n = \frac{(1 - \varsigma)^n - \varsigma^n}{\sqrt{5}}, \qquad \varsigma = \frac{1 - \sqrt{5}}{2}; \qquad n = 1\ 2,\ \cdots. \tag{3}$$

This shows that the relevant connection of $\tilde{\phi}$ with the sequence of Fibonacci numbers ϑ_n, such that

$$\vartheta_0 = 0, \quad \vartheta_1 = 1, \quad \vartheta_{n+2} = \vartheta_n + \vartheta_{n+1}$$

for $n = 0,\ 1,\ 2,\ 3,\ \cdots$. Hence

$$\tilde{\phi}(z) \quad = \quad 1 + \varsigma z + 3\varsigma^2 z^2 + 4\varsigma^3 z^3 + 7\varsigma^4 z^4 + 11\varsigma^5 z^5 + \cdots. \tag{4}$$

We note that the function $\tilde{\phi}$ belongs to the class $\mathscr{P}(\beta)$ with $\beta = \dfrac{\sqrt{5}}{10} \approx 0.2236$ (see [17]). For more details one could refer recent works in this line from [1, 3, 4, 5, 11, 13, 14, 18, 19] and references therein.

Definition 1. *A function $\mathfrak{s} \in \Sigma$ of the form (1) belongs to the class $\mathscr{MS}_{\Sigma}^{\kappa,\, \upsilon,\, \gamma}(\tilde{\phi})$ with $0 \leqq \kappa \leqq 1$, $0 \leqq \upsilon \leqq 1$ and $1 \leqq \gamma \leqq 2$, if the following conditions are satisfied:*

$$\kappa \left(1 + \frac{z\mathfrak{s}''(z)}{\mathfrak{s}'(z)}\right)^{1-\upsilon} \left(\frac{z\mathfrak{s}'(z)}{\mathfrak{s}(z)}\right)^{\upsilon} + (1 - \kappa)\left(\frac{z\mathfrak{s}'(z)}{\mathfrak{s}(z)}\right)^{\gamma} \prec \widetilde{\phi(z)}, \qquad z \in \mathbb{D} \tag{5}$$

and for $\mathfrak{j}(w) = \mathfrak{s}^{-1}(w)$

$$\kappa\left(1+\frac{w\mathfrak{j}''(w)}{\mathfrak{j}'(w)}\right)^{1-\upsilon}\left(\frac{w\mathfrak{j}'(w)}{\mathfrak{j}(w)}\right)^{\upsilon}+(1-\kappa)\left(\frac{w\mathfrak{j}'(w)}{\mathfrak{j}(w)}\right)^{\gamma}\prec\widetilde{\phi(w)},\qquad w\in\mathbb{D},\quad (6)$$

where $\varsigma = \dfrac{1-\sqrt{5}}{2} \approx -0.618.$

Also, we note that the class $\mathscr{M}\mathscr{S}_{\Sigma}^{\kappa,\,\upsilon,\,\gamma}(\tilde{\phi})$ deduces to known classes introduced in [11] as given below:

1. $\mathscr{M}\mathscr{S}_{\Sigma}^{\kappa,\,0,\,1}(\tilde{\phi}) \equiv \mathscr{M}\mathscr{S}_{\Sigma}^{\kappa}(\tilde{\phi})$
2. $\mathscr{M}\mathscr{S}_{\Sigma}^{1,\,\upsilon,\,1}(\tilde{\phi}) \equiv \mathscr{L}\mathscr{S}_{\Sigma}^{\upsilon}(\tilde{\phi})$
3. $\mathscr{M}\mathscr{S}_{\Sigma}^{1,\,1,\,1}(\tilde{\phi}) \equiv \mathscr{S}\mathscr{L}_{\Sigma}(\tilde{\phi})$
4. $\mathscr{M}\mathscr{S}_{\Sigma}^{1,\,0,\,1}(\tilde{\phi}) \equiv \mathscr{K}\mathscr{S}_{\Sigma}(\tilde{\phi})$.

In view of a lemma given below, we prove the results of the current paper.

Lemma 1. *[16] If* $p \in \mathscr{P}$, *then* $|p_i| \leqq 2$ *for each* $i \in \mathbb{N}$.

In this investigation, we find the estimates for the coefficients $|a_2|$ and $|a_3|$ for functions in the subclass $\mathscr{M}\mathscr{S}_{\Sigma}^{\kappa,\,\upsilon,\,\gamma}(\tilde{\phi})$ Also, we obtain the Fekete-Szegö functional $\left|a_3 - \nu a_2^2\right|$ for $\nu \in \mathbb{R}$.

2 A Set of coefficient estimates

In the following theorem, we obtain coefficient estimates for functions in the class $\mathfrak{s} \in \mathscr{M}\mathscr{S}_{\Sigma}^{\kappa,\,\upsilon,\,\gamma}(\tilde{\phi})$.

Theorem 1. *Let* $\mathfrak{s}(z)$ *of the form* (1) *be in the class* $\mathscr{M}\mathscr{S}_{\Sigma}^{\kappa,\,\upsilon,\,\gamma}(\tilde{\phi})$. *Then*

$$|a_2| \le \sqrt{\frac{2}{\mathscr{R}}}\,|\varsigma|\,, \qquad |a_3| \le \frac{|\varsigma|\,\mathscr{P}}{2\mathscr{Q}\mathscr{R}} \tag{7}$$

and for $\nu \in \mathbb{R}$,

$$\left|a_3 - \nu a_2^2\right| \le \begin{cases} \dfrac{|\varsigma|}{2\mathscr{Q}} & ;0 \le |\nu - 1| \le \dfrac{\mathscr{R}}{4\,|\varsigma|\,\mathscr{Q}} \\[2ex] \dfrac{2\,|1-\nu|\,|\varsigma|^2}{\mathscr{R}} & ;|\nu - 1| \ge \dfrac{\mathscr{R}}{4\,|\varsigma|\,\mathscr{Q}}, \end{cases} \tag{8}$$

where

$$\mathscr{P} = \varsigma\left((1-\kappa)\gamma(\gamma-3) + \kappa(\upsilon^2+5\upsilon-8)\right) + 2(1-3\varsigma)[(1-\kappa)\gamma + \kappa(2-\upsilon)]^2 \tag{9}$$

$$\mathscr{Q} = (1-\kappa)\gamma + \kappa(3-2\upsilon) \tag{10}$$

and

$$\mathscr{R} = \varsigma\left((1-\kappa)\gamma(1+\gamma) + \kappa(\upsilon^2-3\upsilon+4)\right) + 2(1-3\varsigma)[(1-\kappa)\gamma + \kappa(2-\upsilon)]^2. \tag{11}$$

Proof. Since $\mathfrak{s} \in \mathscr{M}\mathscr{S}_{\Sigma}^{\kappa,\,\upsilon,\,\gamma}(\tilde{\phi})$, from Definition 1 we have

$$\kappa\left(1+\frac{z\mathfrak{s}''(z)}{\mathfrak{s}'(z)}\right)^{1-\upsilon}\left(\frac{z\mathfrak{s}'(z)}{\mathfrak{s}(z)}\right)^{\upsilon}+(1-\kappa)\left(\frac{z\mathfrak{s}'(z)}{\mathfrak{s}(z)}\right)^{\gamma}=\widetilde{\phi(\varphi(z))} \tag{12}$$

and

$$\kappa\left(1+\frac{w\mathfrak{j}''(w)}{\mathfrak{j}'(w)}\right)^{1-\upsilon}\left(\frac{w\mathfrak{j}'(w)}{\mathfrak{j}(w)}\right)^{\upsilon}+(1-\kappa)\left(\frac{w\mathfrak{j}'(w)}{\mathfrak{j}(w)}\right)^{\gamma}=\widetilde{\phi(\chi(w))}, \tag{13}$$

where $z,\ w \in \mathbb{D}$ and $g = \mathfrak{s}^{-1}$. Since $p \in \mathscr{P}$ and $p \prec \tilde{\phi}$. Then $\exists$ an analytic function φ such that $|\varphi(z)| < 1$ in $\mathbb{D}$ and $p(z) = \tilde{\phi}(\varphi(z))$. Therefore, define the function

$$\Phi(z)=\frac{1+\varphi(z)}{1-\varphi(z)}=1+\varphi_1 z+\varphi_2 z^2+\cdots$$

is in the class $\mathscr{P}$. It follows that

$$\varphi(z)=\frac{\Phi(z)-1}{\Phi(z)+1}=\frac{\varphi_1}{2}z+\left(\varphi_2-\frac{\varphi_1^2}{2}\right)\frac{z^2}{2}+\left(\varphi_3-\varphi_1\varphi_2+\frac{\varphi_1^3}{4}\right)\frac{z^3}{2}+\cdots$$

and

$$\begin{aligned}
\tilde{\phi}(\varphi(z)) &= 1+\tilde{\phi}\left(\frac{\varphi_1}{2}z+\left(\varphi_2-\frac{\varphi_1^2}{2}\right)\frac{z^2}{2}+\left(\varphi_3-\varphi_1\varphi_2+\frac{\varphi_1^3}{4}\right)\frac{z^3}{2}+\cdots\right)\\
&\quad+\tilde{\phi}_2\left(\frac{\varphi_1}{2}z+\left(\varphi_2-\frac{\varphi_1^2}{2}\right)\frac{z^2}{2}+\left(\varphi_3-\varphi_1\varphi_2+\frac{\varphi_1^3}{4}\right)\frac{z^3}{2}+\cdots\right)^2\\
&\quad+\tilde{\phi}_3\left(\frac{\varphi_1}{2}z+\left(\varphi_2-\frac{\varphi_1^2}{2}\right)\frac{z^2}{2}+\left(\varphi_3-\varphi_1\varphi_2+\frac{\varphi_1^3}{4}\right)\frac{z^3}{2}+\cdots\right)^3\\
&\quad+\cdots\\
&= 1+\frac{\tilde{\phi}_1\varphi_1}{2}z+\left(\frac{1}{2}\left(\varphi_2-\frac{\varphi_1^2}{2}\right)\tilde{\phi}_1+\frac{\varphi_1^2}{4}\tilde{\phi}_2\right)z^2\\
&\quad+\left(\frac{1}{2}\left(\varphi_3-\varphi_1\varphi_2+\frac{\varphi_1^3}{4}\right)\tilde{\phi}_1+\frac{1}{2}\varphi_1\left(\varphi_2-\frac{\varphi_1^2}{2}\right)\tilde{\phi}_2+\frac{\varphi_1^3}{8}\tilde{\phi}_3\right)z^3\\
&\quad+\cdots.
\end{aligned} \tag{14}$$

Similarly, $\exists$ an analytic function χ such that $|\chi(w)| < 1$ in $\mathbb{D}$ and $p(w) = \tilde{\phi}(\chi(w))$. Therefore, the function

$$\Psi(w)=\frac{1+\chi(w)}{1-\chi(w)}=1+\chi_1 w+\chi_2 w^2+\cdots$$

is in the class $\mathscr{P}$. It follows that

$$\chi(w)=\frac{\Psi(w)-1}{\Psi(w)+1}=\frac{\chi_1}{2}w+\left(\chi_2-\frac{\chi_1^2}{2}\right)\frac{w^2}{2}+\left(\chi_3-\chi_1\chi_2+\frac{\chi_1^3}{4}\right)\frac{w^3}{2}+\cdots$$

and

$$\begin{aligned}\tilde{\phi}(\chi(w)) &= 1+\tilde{\phi}\left(\frac{\chi_1}{2}w+\left(\chi_2-\frac{\chi_1^2}{2}\right)\frac{w^2}{2}+\left(\chi_3-\chi_1\chi_2+\frac{\chi_1^3}{4}\right)\frac{w^3}{2}+\cdots\right)\\ &\quad+\tilde{\phi}_2\left(\frac{\chi_1}{2}w+\left(\chi_2-\frac{\chi_1^2}{2}\right)\frac{w^2}{2}+\left(\chi_3-\chi_1\chi_2+\frac{\chi_1^3}{4}\right)\frac{w^3}{2}+\cdots\right)^2\\ &\quad+\tilde{\phi}_3\left(\frac{\chi_1}{2}w+\left(\chi_2-\frac{\chi_1^2}{2}\right)\frac{w^2}{2}+\left(\chi_3-\chi_1\chi_2+\frac{\chi_1^3}{4}\right)\frac{w^3}{2}+\cdots\right)^3\\ &\quad+\cdots\\ &= 1+\frac{\tilde{\phi}_1\chi_1}{2}w+\left(\frac{1}{2}\left(\chi_2-\frac{\chi_1^2}{2}\right)\tilde{\phi}_1+\frac{\chi_1^2}{4}\tilde{\phi}_2\right)w^2\\ &\quad+\left(\frac{1}{2}\left(\chi_3-\chi_1\chi_2+\frac{\chi_1^3}{4}\right)\tilde{\phi}_1+\frac{1}{2}\chi_1\left(\chi_2-\frac{\chi_1^2}{2}\right)\tilde{\phi}_2+\frac{\chi_1^3}{8}\tilde{\phi}_3\right)w^3\\ &\quad+\cdots. \end{aligned} \tag{15}$$

By virtue of (12), (13), (14) and (15), we have

$$[(1-\kappa)\gamma+\kappa(2-\upsilon)]a_2=\frac{\varphi_1\varsigma}{2}, \tag{16}$$

$$\begin{aligned}&2[(1-\kappa)\gamma+\kappa(3-2\upsilon)]a_3-[(1-\kappa)\gamma(3-\gamma)-\kappa(\upsilon^2+5\upsilon-8)]\frac{a_2^2}{2}\\ &= \frac{1}{2}\left(\varphi_2-\frac{\varphi_1^2}{2}\right)\varsigma+\frac{3\varphi_1^2}{4}\varsigma^2,\end{aligned} \tag{17}$$

$$-[(1-\kappa)\gamma+\kappa(2-\upsilon)]a_2=\frac{\chi_1\varsigma}{2}, \tag{18}$$

and

$$\begin{aligned}&[(1-\kappa)\gamma(5+\gamma)+\kappa(\upsilon^2-11\upsilon+16)]\frac{a_2^2}{2}-2[(1-\kappa)\gamma+\kappa(3-2\upsilon)]a_3\\ &= \frac{1}{2}\left(\chi_2-\frac{\chi_1^2}{2}\right)\varsigma+\frac{3\chi_1^2}{4}\varsigma^2.\end{aligned} \tag{19}$$

From (16) and (18), we obtain

$$\varphi_1=-\chi_1,$$

and

$$\begin{aligned}2[(1-\kappa)\gamma+\kappa(2-\upsilon)]^2a_2^2 &= \frac{(\varphi_1^2+\chi_1^2)\varsigma^2}{4}\\ a_2^2 &= \frac{(\varphi_1^2+\chi_1^2)\varsigma^2}{8[(1-\kappa)\gamma+\kappa(2-\upsilon)]^2}.\end{aligned} \tag{20}$$

By adding (17) and (19), we have

$$\begin{aligned}&\left((1-\kappa)\gamma(1+\gamma)+\kappa(\upsilon^2-3\upsilon+4)\right)a_2^2\\ &= \frac{1}{2}(\varphi_2+\chi_2)\varsigma-\frac{1}{4}\left(\varphi_1^2+\chi_1^2\right)\varsigma+\frac{3}{4}\left(\varphi_1^2+\chi_1^2\right)\varsigma^2. \quad (21)\end{aligned} \tag{22}$$

By substituting (20) in (21), we reduce that

$$a_2^2 \quad = \quad \frac{(\varphi_2 + \chi_2)\,\varsigma^2}{2\mathscr{R}}. \tag{23}$$

Now, applying Lemma 1, we obtain

$$|a_2| \quad \leq \quad \frac{\sqrt{2}\,|\varsigma|}{\sqrt{\mathscr{R}}}. \tag{24}$$

By subtracting (19) from (17), we obtain

$$a_3 = \frac{(\varphi_2 - \chi_2)\,\varsigma}{8\mathscr{Q}} + a_2^2. \tag{25}$$

Hence by Lemma 1, we have

$$|a_3| \quad \leq \quad \frac{(|\varphi_2| + |\chi_2|)\,|\varsigma|}{8\mathscr{Q}} + |a_2|^2 \leq \frac{|\varsigma|}{2\mathscr{Q}} + |a_2|^2\,. \tag{26}$$

Then in view of (24), we obtain

$$|a_3| \leq \frac{|\varsigma|\,\mathscr{P}}{2\mathscr{Q}\mathscr{R}}.$$

From (25), we have for $\nu \in \mathbb{R}$

$$a_3 - \nu a_2^2 = \frac{(\varphi_2 - \chi_2)\,\varsigma}{8\mathscr{Q}} + (1-\nu)\,a_2^2. \tag{27}$$

By substituting (23) in (27), we have

$$\begin{aligned} a_3 - \nu a_2^2 \quad &= \quad \frac{(\varphi_2 - \chi_2)\,\varsigma}{8\mathscr{Q}} + (1-\nu)\left(\frac{(\varphi_2 + \chi_2)\,\varsigma^2}{2\mathscr{R}}\right) \\ &= \quad \left(\Phi(\nu) + \frac{\varsigma}{8\mathscr{Q}}\right)\varphi_2 + \left(\Phi(\nu) - \frac{\varsigma}{8\mathscr{Q}}\right)\chi_2, \end{aligned} \tag{28}$$

where

$$\Phi(\nu) = \frac{(1-\nu)\,\varsigma^2}{2\mathscr{R}}.$$

Thus by taking modulus of (28), we conclude that

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|\varsigma|}{2\mathscr{Q}} & ; 0 \leq |\Phi(\nu)| \leq \dfrac{|\varsigma|}{8\mathscr{Q}} \\ 4\,|\Phi(\nu)| & ; |\Phi(\nu)| \geq \dfrac{|\varsigma|}{8\mathscr{Q}} \end{cases}$$

which leads to the desired inequlity (8). □

3 Corollaries and consequences

Corollary 1. *Let* $\mathfrak{s} \in \mathscr{A}$ *of the form* (1) *be in the class* $\mathscr{M}\mathscr{S}_{\Sigma}^{\kappa}(\tilde{\phi})$*. Then*

$$|a_2| \leq \frac{|\varsigma|}{\sqrt{\varsigma(1+\kappa)+(1-3\varsigma)(1+\kappa)^2}},$$
$$|a_3| \leq \frac{|\varsigma|\left[(1-3\varsigma)(1+\kappa)^2-\varsigma(1+3\kappa)\right]}{2(1+2\kappa)(\varsigma(1+\kappa)+(1-3\varsigma)(1+\kappa)^2)}$$

and for $\nu \in \mathbb{R}$,

$$\left|a_3-\nu a_2^2\right| \leq \begin{cases} \dfrac{|\varsigma|}{2(1+2\kappa)} & ; 0 \leq |\nu-1| \leq C(\varsigma,\kappa) \\ \dfrac{|1-\nu|\,|\varsigma|^2}{\varsigma(1+\kappa)+(1-3\varsigma)(1+\kappa)^2} & ; |\nu-1| \geq C(\varsigma,\kappa), \end{cases}$$

where $C(\varsigma,\kappa) = \dfrac{\varsigma(1+\kappa)+(1-3\varsigma)(1+\kappa)^2}{2\,|\varsigma|\,(1+2\kappa)}$.

Corollary 2. *Let* $\mathfrak{s} \in \mathscr{A}$ *of the form* (1) *be in the class* $\mathscr{L}\mathscr{S}_{\Sigma}^{\upsilon}(\tilde{\phi})$*. Then*

$$|a_2| \leq \frac{\sqrt{2}\,|\varsigma|}{\sqrt{\varsigma(\upsilon^2-3\upsilon+4)+2(1-3\varsigma)(2-\upsilon)^2}}, \tag{29}$$
$$|a_3| \leq \frac{|\varsigma|\left[\varsigma(\upsilon^2+5\upsilon-8)+2(1-3\varsigma)(2-\upsilon)^2\right]}{2(3-2\upsilon)\mathscr{R}_2} \tag{30}$$

and for $\nu \in \mathbb{R}$,

$$\left|a_3-\nu a_2^2\right| \leq \begin{cases} \dfrac{|\varsigma|}{2(3-2\upsilon)} & ; 0 \leq |\nu-1| \leq \dfrac{|\varsigma|}{4\,|\varsigma|\,(3-2\upsilon)} \\ \dfrac{2\,|1-\nu|\,|\varsigma|^2}{|\varsigma|} & ; |\nu-1| \geq \dfrac{|\varsigma|}{4\,|\varsigma|\,(3-2\upsilon)}. \end{cases} \tag{31}$$

Corollary 3. *[11] Let* $\mathfrak{s} \in \mathscr{A}$ *of the form* (1) *be in the class* $\mathscr{S}\mathscr{L}_{\Sigma}(\tilde{\phi})$*. Then*

$$|a_2| \leq \frac{|\varsigma|}{\sqrt{1-2\varsigma}}, \quad |a_3| \leq \frac{|\varsigma|\,(1-4\varsigma)}{2-4\varsigma}$$

and for $\nu \in \mathbb{R}$,

$$\left|a_3-\nu a_2^2\right| \leq \begin{cases} \dfrac{|\varsigma|}{2} & ; 0 \leq |\nu-1| \leq \dfrac{1-2\varsigma}{2\,|\varsigma|} \\ \dfrac{|\nu-1|\,\varsigma^2}{1-2\varsigma} & ; |\nu-1| \geq \dfrac{1-2\varsigma}{2\,|\varsigma|}. \end{cases}$$

Corollary 4. *[11] Let* $\mathfrak{s} \in \mathscr{A}$ *of the form* (1) *be in the class* $\mathscr{K}\mathscr{S}_{\Sigma}(\tilde{\phi})$*. Then*

$$|a_2| \leq \frac{|\varsigma|}{\sqrt{4-10\varsigma}}, \quad |a_3| \leq \frac{|\varsigma|\,(1-4\varsigma)}{6-15\varsigma}.$$

and for $\nu \in \mathbb{R}$,

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|\varsigma|}{6} & ; 0 \leq |\nu - 1| \leq \dfrac{2 - 5\varsigma}{3\,|\varsigma|} \\ \dfrac{|\nu - 1|\,\varsigma^2}{4 - 10\varsigma} & ; |\nu - 1| \geq \dfrac{2 - 5\varsigma}{3\,|\varsigma|}. \end{cases}$$

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