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SOME CURVATURE PROPERTIES ON GENERALIZED QUASI EINSTEIN MANIFOLDS

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Abstract

In this article, we first investigate Ricci-pseudosymmetric generalized quasi-Einstein manifolds. Next we study pseudo projectively flat generalized quasi Einstein manifolds and pseudo projective Ricci-symmetric generalized quasi Einstein manifolds.

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Key words: Quasi Einstein manifold, Generalized quasi-Einstein manifold, Pseudo projective curvature tensor, Ricci-pseudosymmetric manifold.

1 Introduction

The notion of a quasi Einstein manifold was introduced by M. C. Chaki in [2]. A non flat n-dimensional Riemannian manifold (M^n, g) is said to be a quasi Einstein manifold if its Ricci tensor S satisfies

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \quad \forall X, Y \in TM$$

for some non-zero scalars a and b, where η is a non zero 1-forms such that

$$g(X,\xi) = \eta(X), \quad g(\xi,\xi) = \eta(\xi) = 1$$

for the associated vector field ξ. The 1-form η is called the associated 1-form and the unit vector field ξ is called the generator of the manifold. For more details about quasi Einstein manifolds you can see also [1, 4, 5, 6, 8, 9, 12, 13]. As a generalization of a quasi-Einstein manifold, in [3] U. C. De and G. C. Ghosh

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introduced the notion of a generalized quasi-Einstein manifolds. A non-flat Riemannian manifold M is called a generalized quasi-Einstein manifold if its Ricci tensor S of type (0,2) is non-zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \tag{1}$$

where a, b, c are certain non-zero scalars and A, B are two non-zero 1-form such that

$$g(X,U) = A(X), \quad g(X,V) = B(X), \tag{2}$$

$$g(U,U) = g(V,V) = 1, \quad g(U,V) = 0 \tag{3}$$

i.e. U, V are orthogonal vector fields on M^n. From (1)-(3), we get

$$S(X,U) = (a+b)A(X), \quad S(X,V) = (a+c)B(X) \tag{4}$$

$$r = na + b + c, \tag{5}$$

where r denotes the scalar curvature of the manifold. If $c = 0$, then the manifold reduces to a quasi Einstein manifold. In [3], it was shown that a 2-umbilical hyper surface[10], is a generalized quasi Einstein manifold.

2 Ricci-pseudosymmetric generalized quasi-Einstein manifolds

An n-dimensional Riemannian manifold (M^n, g) is called Ricci-pseudosymmetric [7] if the tensors $R.S$ and $Q(g,S)$ are linearly dependent, where

$$(R(X,Y).S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W), \tag{6}$$

$$Q(g,S)(Z,W;X,Y) = -S((X \wedge Y)Z,W) - S(Z,(X \wedge Y)W) \tag{7}$$

and

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y \tag{8}$$

for vector fields X, Y, Z, W on M^n, R denotes the curvature tensor of M^n [11]. The condition of Ricci-pseudosymmetric equivalent to

$$(R(X,Y).S)(Z,W) = L_s Q(g,S)(Z,W;X,Y) \tag{9}$$

holding on the set

$$U_s = \{x \in M : S \neq \frac{r}{n} g \ \ at \ \ x\}$$

where L_s is some function on U_s [11]. If $R.S = 0$ then M^n is called Ricci-semisymmetric. Every Ricci-semisymmetric manifold is Ricci-pseudosymmetric but the converse is not true [7, 11].

Pseudo Projective curvature tensor $\bar{P}$ [15], on the Riemannian manifold (M^n, g) is as follows

$$\begin{aligned} \bar{P}(X,Y)Z &= \alpha R(X,Y)Z + \beta[S(Y,Z)X - S(X,Z)Y] \\ &\quad -\frac{r}{n}[\frac{\alpha}{n+1} + \beta]\{g(X,Z)X - g(X,Z)Y\}, \end{aligned} \quad (10)$$

where α and β are nonzero scalars. For $\alpha = 1$ and $\beta = -\frac{1}{n-1}$ the pseudo projective curvature tensor becomes a projective curvature tensor.

Definition 1. *An n-dimensional Riemannian manifold (M, g) is called a pseudo projective Ricci-symmetric If*

$$\bar{P}(X,Y).S(Z,W) = 0$$

for all $X, Y, Z, W \in TM$.

Definition 2. *A Riemannian mainfold (M^n, g) is called in the form of pseudo projectively Flat if*

$$\bar{P}(X,Y)Z = 0$$

for all $X, Y, Z \in TM$.

Definition 3. *An n-dimensional Riemannian manifold (M, g) is called a pseudo projective semi-symmetric if*

$$R(X,Y).P(Z,W)N = 0$$

for all $X, Y, Z, W, N \in TM$.

In this section, we prove the following theorem:

Theorem 1. *Let (M^n, g) be an n-dimensional mixed generalized quasi-Einstein manifold. If M^n is Ricci-pseudo symmetric then we have, $c - b = 0$ or*

$$R(X,Y,U,V) = L_s\{A(Y)B(X) - B(Y)A(X)\} \quad (11)$$

Proof. Let (M^n, g) be a Ricci-pseudosymmetric. Then from (6)-(9) we obtain

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = L_s\{g(Y,Z)S(X,W) \quad (12)$$

$$-g(X,Z)S(Y,W) + g(Y,W)S(X,Z) - g(X,W)S(Y,Z)\}.$$

Since (M^n, g) is a generalized quasi Einstein manifold, using the well-known properties of the curvature tensor R from (12) we get

$$b[A(R(X,Y)Z)A(W) + A(R(X,Y)W)A(Z)] \quad (13)$$

$$+c[B(R(X,Y)Z)B(W) + B(R(X,Y)W)B(Z)]$$

$$= L_s\{b[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)$$

$$+g(Y,W)A(X)A(Z) - g(X,W)A(Y)A(Z)]$$
$$+c[g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)$$
$$+g(Y,W)B(X)B(Z) - g(X,W)B(Y)B(Z)]\}.$$

Taking $Z = W = U$ in (13) we get
$(c-b)\{R(X,Y,U,V) = (c-b)L_s\{B(X)A(Y) - A(X)B(Y)\}\}$.
Then we have, $c - b = 0$ or

$$R(X,Y,Z,W) - L_s\{B(X)A(Y) - A(X)B(Y)\} = 0.$$

This completes the proof of the theorem. □

Corollary 1. *Let (M^n, g) be a pseudo symmetric Ricci generalized pseudo Einstein manifold. Then his Ricci tensor is as follows*

$$S(X,Y) = ag(X,Y) + b\{A(X)A(Y) + B(X)B(Y)\}.$$

Theorem 2. *Let (M^n, g) be an n-dimensional mixed generalized quasi-Einstein manifold. If (M^n, g) is pseudo projective flat, then $\alpha = (1-n)\beta$.*

Proof. If (M^n, g) is an n-dimensional mixed generalized quasi-Einstein manifold, then by (10), we have

$$\alpha R(X,Y) = -\beta\{S(Y,Z)X - S(X,Z)Y\} + \frac{r}{n}[\frac{\alpha}{n-1} + \beta\}\{g(Y,Z)X - g(X,Z)Y\} \tag{14}$$

If in (14) we put $X = W = e_i$, then by summation on i we have

$$[\alpha + (n-1)\beta]\{S(Y,Z) - \frac{r}{n}g(Y,Z)\} = 0.$$

Since (M^n, g) is a generalized pseudo Einstein manifold, $S(Y,Z) - \frac{r}{n}g(Y,Z) \neq 0$.
Hence $\alpha = (1-n)\beta$. □

Again, let (M^n, g) be a generalized quasi-Einstein manifold which is pseudo projective flat. By previous Theorem, $\alpha = (1-n)\beta$. Using relation (14), we have

$$R(X,Y)Z = \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}. \tag{15}$$

With replacing (1) in (15) we have

$$\begin{aligned} R(X,Y)Z &= \frac{a}{n-1}\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \frac{b}{n-1}\{A(Y)A(Z)X - A(X)A(Z)Y\} \\ &+ \frac{c}{n-1}\{B(Y)B(Z)X - B(X)B(Z)Y\}. \end{aligned}$$

Hence we have following theorem

Theorem 3. *Let (M^n, g) be a generalized quasi-Einstein manifold. If (M^n, g) is pseudo projective flat, then*

$$\begin{aligned} R(X,Y)Z &= \frac{a}{n-1}\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \frac{b}{n-1}\{A(Y)A(Z)X - A(X)A(Z)Y\} \\ &+ \frac{c}{n-1}\{B(Y)B(Z)X - B(X)B(Z)Y\}. \end{aligned}$$

for all $X, Y, Z \in TM$.

Let $U^{\perp}$ be (n-1)-dimensional orthonormal distribution of U in a pseudo projective flat generalized quasi-Einstein manifold. Then $g(X,U) = 0$ if and only if $X \in U^{\perp}$. Hence by the previous theorem we have following corollaries:

Corollary 2. *Let (M^n, g) be a generalized quasi-Einstein manifold. If (M^n, g) is pseudo projective flat, then*

$$R(X,Y)Z = \frac{a}{n-1}\{g(Y,Z) - g(X,Z)\}$$

$$+\frac{b}{n-1}\{A(Y)A(Z)X - A(X)A(Z)Y\}$$

for all $X, Y, Z \in U^{\perp}$.

Also, If $V^{\perp}$ is (n-1)-dimensional orthonormal distribution of V in a pseudo projective flat generalized quasi-Einstein manifold. Then

Corollary 3. *Let (M^n, g) be a generalized quasi-Einstein manifold. If (M^n, g) is pseudo projective flat, then*

$$R(X,Y)Z = \frac{a}{n-1}\{g(Y,Z) - g(X,Z)\}$$

$$+\frac{c}{n-1}\{B(Y)B(Z)X - B(X)B(Z)Y\}$$

for all $X, Y, Z \in V^{\perp}$.

Finally, we prove the following theorem

Theorem 4. *Let (M^n, g) be a generalized quasi-Einstein manifold. If (M^n, g) is pseudo projective symmetric Ricci, then $b = c$ or curvature tensor R has following property*

$$R(X,Y,U,V) = \frac{\alpha}{\beta}\{A(Y)B(X) - A(X)B(Y)\}.$$

Proof. Let (M^n, g) be pseudo projective symmetric Ricci. Since

$$\bar{P}(X,Y).S(Z,W) = -S(\bar{P}(X,Y)Z,W) - S(\bar{P}(X,Y)W,X),$$

then

$$S(\bar{P}(X,Y)Z,W) + S(\bar{P}(X,Y)W,X) = 0. \tag{16}$$

Since (M^n, g) is a generalized quasi-Einstein manifold, using relation (1) to (16) and with $W = Z$ and $Y = U$ in the above relation we have

$$(a+c)B(\bar{P}(X,Y)Z) + (a+b)A(\bar{P}(X,Y)Z) = 0.$$

Now, with applying relations (5) and (10) in the above relation we have

$$(c-b)[\alpha R(X,Y,U,V) - \beta\{A(Y)B(X) - A(X)B(Y)\}] = 0.$$

This implies that $b = c$ or $\alpha R(X,Y,U,V) = \beta\{A(Y)B(X) - A(X)B(Y)\}$. □

Corollary 4. *Let (M^n, g) be a generalized quasi-Einstein manifold. If (M^n, g) is pseudo projective symmetric Ricci, then $b = c$ or the curvature tensor R has the following property*

$$R(X,Y,U,V) = -\frac{1}{n-1}\{A(Y)B(X) - A(X)B(Y)\}.$$

curvature tensor R satisfies

$$R(X,Y).R = 0, \qquad X,\ Y \in TM,$$

where $R(X,Y)$ acts on R as a derivation [14]

Theorem 5. *Let (M^n, g) be an n-dimensional semi-symmetric generalized quasi-Einstein manifold. From condition $R.\bar{P} = 0$ holds on (M, g) we get $b = c$.*

Proof. Since $R.\bar{P} = 0$, then we have

$$\begin{aligned} 0 = R(X,Y)\bar{P}(Z,W)N - \bar{P}(R(X,Y)Z,W)N \\ -\bar{P}(Z,R(X,Y)W)N - \bar{P}(Z,W)R(X,Y)N \end{aligned} \tag{17}$$

Since (M, g) is semi-symmetric, from (17) we obtain

$$\begin{aligned} 0 &= \beta\{S(R(X,Y)Z,N)W - S((R(X,Y)W,N)Z \\ &+ S(R(X,Y)N,Z)W - S(R(X,Y)N,W)Z\} \\ &- \frac{r}{n}\left[\frac{\alpha}{n+1} + \beta\right]\{g(R(X,Y)Z,N)W - g((R(X,Y)W,N)Z \\ &+ g(R(X,Y)N,Z)W - g(R(X,Y)N,W)Z\} \\ &= \beta\{S(R(X,Y)Z,N)W - S((R(X,Y)W,N)Z \\ &+ S(R(X,Y)N,Z)W - S(R(X,Y)N,W)Z\} \end{aligned} \tag{18}$$

But $\beta \neq 0$. Then from (18) we get

$$0 = \{S(R(X,Y)Z,N)W - S((R(X,Y)W,N)Z \quad (19)$$

$$+S(R(X,Y)N,Z)W - S(R(X,Y)N,W)Z\}.$$

Since (M,g) is a generalized quasi-Einstein manifold. From (19) we have

$$\begin{aligned} 0 &= -bA(N)\{R(X,Y,W,U)A(Z) + R(X,Y,Z,U)A(W)\} \\ &\quad +cR(X,Y,N,V)\{B(Z)A(W) - B(W)A(Z)\} \\ &\quad -cB(Z)\{R(X,Y,W,V)A(Z) + R(X,Y,Z,V)A(W)\}. \end{aligned} \quad (20)$$

Putting $N = Z = U$ and $W = V$ in (20), then we get

$$(b-c)\{R(X,Y,U,V)\} = 0.$$

Since $R \neq 0$, we obtain $b = c$. □

Corollary 5. *Let (M^n, g) be an n-dimensional semi-symmetric generalized quasi-Einstein manifold. From the condition R.P = 0 holds on (M,g) we get $b = c$.*

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