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RICCI SOLITON ON THE TANGENT BUNDLE WITH SEMI-SYMMETRIC METRIC CONNECTION

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Abstract

In this paper, we studied the tangent bundle endowed with semi-symmetric metric connection obtained by vertical and complete lifts of a semi-symmetric metric P-connection on the base manifold. Firstly, we give a relationships between (TM, g^c) and (M, g) to be an Einstein manifolds. Secondly, we investigate necessary and sufficient conditions for (TM, g^c) with complete and vertical lift of torqued potential fields to be Ricci soliton.

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1 Introduction

An Einstein manifold is a Riemannian or pseudo-Riemannian n-manifold $(M, g)_{n\geq 2}$ in which the Ricci tensor is a scalar multiple of the Riemannian metric, i.e., $Ric(X, Y) = \lambda g(X, Y)$, where Ric denotes the Ricci tensor of (M, g) and λ is a non-zero scalar. In 2000, M. C. Chaki and R.K. Maity in [1], introduced a new type of a non-flat Riemannian manifold called a quasi-Einstein manifold if

$$Ric = ag + b\alpha \otimes \alpha \tag{1}$$

In [3], BY Chen introduced a new definition called an almost quasi-Einstein manifold if

$$Ric = ag + b(\alpha \otimes \beta + \beta \otimes \alpha), \tag{2}$$

where a, b are functions such that $b \neq 0$ and non-vanishing 1-forms α, β.

Any non-zero vector field $\mathbf{v}$ on a Riemannian manifold (or a pseudo-Riemannian manifold) satisfying this important property $\widetilde{\nabla}_X \mathbf{v} = X$ is called a concurrent vector field [15], where $\widetilde{\nabla}$ is the Levi-Civita connection and for all vector field X on M.

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A natural extension of a concurrent vector field is a concircular vector field. A concircular vector field is defined to be a vector field $\mathbf{v}$ satisfying the property

$$\widetilde{\nabla}_X \mathbf{v} = \varphi X, \tag{3}$$

for some function φ in M and any vector field X on M.

In [16], Yano extended concurrent and concircular vector fields to torse-forming vector fields τ satisfying

$$\nabla_X \tau = \varphi X + \gamma(X)\tau, \tag{4}$$

for any vectors $X \in \mathfrak{X}(M)$, where φ is a function called the conformal scalar, γ is a 1-form called the generating form, ∇ is Levi-Civita connection on M. The vector field τ is called recurrent (resp. parallel vector) if $\varphi = 0$ (resp. $\varphi = \gamma = 0$).

As a consequence of tors forming vector field, recently Chen [3] introduced a new vector field, called torqued vector field if $\gamma(\tau) = 0$, here φ and γ are known as the torqued function and the torqued form respectively.

In 1982, Hamilton [8] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows

$$\frac{\partial}{\partial t} g = -2Ric(g(t)).$$

A Ricci soliton (M, g, τ, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that

$$Ric + \frac{1}{2}\mathcal{L}_\tau g = \lambda g, \tag{5}$$

where $\mathcal{L}_\tau$ is the Lie derivative in the direction of $X \in \mathfrak{X}(M)$ and λ is a constant. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive respectively. A trivial Ricci soliton is one for which the potential field τ is zero or Killing, in which case the metric is Einsteinian.

In [3], Bang-Yen Chen prove that if a Ricci soliton (M, g, τ, λ) has torqued potential field τ, then (M, g) is an almost quasi-Einstein manifold, and (M, g) is an Einstein manifold if and only if the torqued potential field τ is a concircular vector field.

A linear connection $\overline{\nabla}$ in M is said to be Semi-Symmetric connection if its torsion $\overline{T}$ is of the form

$$\overline{T}(X, Y) = \pi(Y)X - \pi(X)Y \tag{6}$$

for $X, Y \in \mathfrak{X}(M)$, where π is a 1-form associated with the vector field P and satisfies $\pi(X) = g(X, P)$. A linear connection $\overline{\nabla}$ is said to be metric on M if $\overline{\nabla} g = 0$, otherwise it is non-metric. A systematic study of a semi-symmetric metric connection (SSMC) $\overline{\nabla}$ on a Riemannian manifold was initiated by Yano [17] in 1970. He showed that the Riemannian curvature tensor with respect to a semi-symmetric metric connection vanishes if and only if the manifold is conformably flat.

The paper is organized as follows. Section 2 is concerned with some preliminaries, we recall the notion of semi-symmetric metric P-connection on a Riemannian manifold, Tangent bundle and connection. In Section 3, we deal with the vertical and complete lifts from the base manifold M to its tangent bundle TM. We consider the tangent bundle endowed with semi-symmetric metric connection obtained by vertical and complete lifts of a semi-symmetric metric P-connection on the base manifold, we show that the complete lift of semi-symmetric metric P-connection on a Riemannian manifold is semi-symmetric metric connection in tangent bundle. We provide here the expression of the Ricci tensor field on (TM, g^c) endowed with semi-symmetric metric connection. Then we prove that (TM, g^c) is an Einstein manifold if and only if (M, g) is an Einstein manifold with some conditions. In Section 4, we investigate Ricci soliton structures with lift torqued potential fields on tangent bundles of Riemannian manifolds. We prove that every Ricci soliton structure on the tangent bundle gives rise to a Einstein structure (resp. almost-quasi Einstein structure) on the base manifold with some conditions.

2 Preliminaries

2.1 Semi-symmetric metric P-connections

Let M be an n-dimensional Riemannian manifold with a Riemannian metric g. If ∇ denotes the Levi-Civita connection corresponding to the metric g on M^n, then a linear connection $\overline{\nabla}$ on M^n is defined as

$$\overline{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X,Y)P \tag{7}$$

for $X, Y \in \mathfrak{X}(M)$, where P is a vector field and π is the 1-form satisfies $\pi(X) = g(X, P)$, the connection $\overline{\nabla}$ satisfies Eq.(6) and $\overline{\nabla} g = 0$ on (M, g) and hence, it is a semi-symmetric metric connection on (M, g) [17]. Mishra et al. [12] considered $P = \xi$ and $\overline{\nabla}\xi = 0$ on an almost contact metric manifold and proved many interesting geometrical results. The notion of a semi-symmetric metric ξ-connection on a Riemannian manifold is generalized in [2].

Definition 1. *A linear connection $\overline{\nabla}$ defined on a Riemannian manifold (M^n, g) is called a semi-symmetric metric P-connection if it satisfies Eqs.(6 and 7), $\overline{\nabla} g = 0$ and $\overline{\nabla} P = 0$.*

A direct result from Eq.(7) and Definition 1, we get $\overline{\nabla}_X P = 0$ if and only if $\nabla_X P = \pi(X)P - \pi(P)X$. The Riemannian curvature tensor R with respect to the Levi-Civita connection ∇ is connected by $\overline{R}$ as

$$\overline{R}(X,Y)Z = R(X,Y)Z + \pi(P)\{g(Y,Z)X - g(X,Z)Y\}. \tag{8}$$

Contracting Eq.(8) along the vector field X, we conclude that

$$\overline{Ric}(Y,Z) = Ric(Y,Z) + (n-1)\pi(P)g(Y,Z) \tag{9}$$

and

$$\overline{r} = r + n(n-1)\pi(P), \tag{10}$$

where $\overline{r}$, r and $\overline{Ric}$, Ric are the scalar curvatures and Ricci tensors corresponding to $\overline{\nabla}$ and ∇, respectively.

2.2 Tangent bundle

Let (M, g) be a Riemannian manifold and TM be its tangent bundle, $(x_i)_{i=\overline{1,n}}$ its local coordinates, and $(x_i, y_i)_{i=\overline{1,n}}$ its induced local coordinates on TM and the projection map $\pi : TM \longrightarrow M$ such that $\pi(\widetilde{p}) = p$.

2.3 Vertical lifts

If f is a function in M, we denote by f^v the function in $T(M)$ obtained by forming the composition of $\pi : TM \longrightarrow M$, so that $f^v = f \circ \pi$. Thus, if a point $\widetilde{p} \in \pi^{-1}(U)$ has induced coordinates (x_h, y_h), then $f^v(\widetilde{p}) = f^v(x, y) = f \circ \pi(\widetilde{p}) = f(p) = f(x)$.

Thus the value of $f^v(\widetilde{p})$ is constant along each fibre T_pM and equal to the value $f(p)$. We call f^v the vertical lift of the function f.

2.4 Complete lifts

If f is a function in M, we write f^c for the function in $T(M)$ defined by $f^c = \iota(df)$ and call f^c the complete lift of the function f. The complete lift f^c of a function f has the local expression $f^c = y^i(\frac{\partial f}{\partial x_i})$ with respect to the induced coordinates in $T(M)$.

We define a vector field X^c in TM by $X^c(f^c) = (Xf)^c$, f being an arbitrary function in M and call X^c the complete lift of X in TM. The local expression of the complete lift and vertical lift of a vector field $X = X^i\partial_i$ on M are defined as $X^c = X^i\partial_i + y^a(\partial_a X^i)\partial_{\overline{i}}$, $X^v = X^i\partial_{\overline{i}}$, where $\partial_i = \frac{\partial}{\partial x_i}$ and $\partial_{\overline{i}} = \frac{\partial}{\partial y^i}$. Moreover, the vertical and complete lifts of tensor fields obey the general properties [18]

$$\begin{aligned} (fg)^c &= f^c g^v + f^v g^c, \\ X^c f^c &= (Xf)^c,\ X^c f^v = (Xf)^v. \end{aligned} \tag{11a}$$

Let ω be a 1-form on M. Then the complete lift ω^c of ω is defined by $\omega^c(X^c) = (\omega(X))^c$, X being an arbitrary vector field in M. Moreover, these lifts have the following properties

$$\begin{aligned} \omega^c(X^v) &= (\omega(X))^v,\ \omega^v(X^c) = (\omega(X))^v, \\ [X^c, Y^c] &= [X, Y]^c,\ [X^v, Y^c] = [X, Y]^v. \end{aligned} \tag{12}$$

Any Riemannian metric g on a manifold M defines the complete lift g^c on TM at any point $(x, u) \in TM$ by

$$\begin{aligned} g^c(X^v, Y^c) &= g^c(X^c, Y^v) = (g(X, Y))^v, \\ g^c(X^c, Y^c) &= (g(X, Y))^c,\ \ g^c(X^v, Y^v) = 0. \end{aligned} \tag{13}$$

Now, we assume that M is a manifold with an affine connection ∇. Then there exists a unique affine connection ∇^c in TM which satisfies [18]

$$\begin{aligned} \nabla^c_{X^c} Y^c &= (\nabla_X Y)^c, \ \nabla^c_{X^v} Y^v = 0, \\ \nabla^c_{X^v} Y^c &= \nabla^c_{X^c} Y^v = (\nabla_X Y)^v, \end{aligned} \tag{14}$$

Proposition 1. *If T and R are respectively the torsion and the curvature tensors of ∇ in (M,g), then T^c and R^c are respectively the torsion and the curvature tensors of ∇^c in (TM, g^c), and we have [18]*

$$\begin{aligned} T^c(X^c, Y^c) &= (T(X,Y))^c; \ \ R^c(X^c, Y^c)Z^c = (R(X,Y)Z)^c, \\ R^c(X^c, Y^c)Z^v &= R^c(X^c, Y^v)Z^c = (R(X,Y)Z)^v. \end{aligned} \tag{15}$$

3 Tangent bundle endowed with semi-symmetric metric connection

Theorem 1. *Let $\overline{\nabla}$ be a semi-symmetric metric connection with respect to Riemann connection ∇ in (M,g). Then, $\overline{\nabla}^c$ is semi-symmetric complete metric connection with respect to the Riemann connection ∇^c in (TM, g^c) defined as*

$$\overline{\nabla}^c_{X^c} Y^c = \left(\overline{\nabla}_X Y\right)^c \tag{16}$$

Proof. Taking the complete lift of both sides of the Eq.(7) and using the Eq.(11a) we get

$$\begin{aligned} \left(\overline{\nabla}_X Y\right)^c &= (\nabla_X Y)^c + (\pi(Y)X)^c - (g(X,Y)P)^c \\ \overline{\nabla}^c_{X^c} Y^c &= \nabla^c_{X^c} Y^c + (\pi(Y))^c X^v + (\pi(Y))^v X^c - (g(X,Y))^c P^v - (g(X,Y))^v P^c. \end{aligned}$$

We have

$$\begin{aligned} \overline{\nabla}^c_{X^c} Y^c - \overline{\nabla}^c_{Y^c} X^c - [X^c, Y^c] &= \pi^c(Y^c)X^v + \pi^v(Y^c)X^c \\ &\quad -\pi^c(Y^c)X^v - \pi^v(Y^c)X^c. \end{aligned}$$

from Eq.(6) and Proposition 1, we obtain

$$\begin{aligned} \overline{T}^c(X^c, Y^c) &= \pi^c(Y^c)X^v + \pi^v(Y^c)X^c \\ &\quad -\pi^c(Y^c)X^v - \pi^v(Y^c)X^c. \end{aligned} \tag{17}$$

By computing

$$\begin{aligned} \overline{\nabla}^c_{X^c} g^c(Y^c, Z^c) &= g^c(\overline{\nabla}^c_{X^c} Y^c, Z^c) + g^c(Y^c, \overline{\nabla}^c_{X^c} Z^c) \\ &= g^c(\nabla^c_{X^c} Y^c + \pi^c(Y^c)X^v + \pi^v(Y^c)X^c - g^c(X^c, Y^c)P^v \\ &\quad - g^c(X^v, Y^c)P^c, Z^c) + g^c(Y^c, \nabla^c_{X^c} Z^c + \pi^c(Z^c)X^v + \pi^v(Z^c)X^c \\ &\quad - g^c(X^c, Z^c)P^v \ - g^c(X^v, Z^c)P^c) \\ &= (\nabla^c_{X^c} g^c)(Y^c, Z^c) \end{aligned}$$

we get

$$(\overline{\nabla}^c_{X^c} g^c)(Y^c, Z^c) = 0 \tag{18}$$

The Eqs.(17 and 18) imply the desired result. □

3.1 Curvature tensor of semy-symmetric metric connection

The Riemannian curvature tensor R with respect to the Levi-Civita connection ∇ is connected by $\overline{R}^c$ as

Proposition 2. *Let (M,g) be a Riemannian manifold and (TM,g^c) its tangent bundle equipped with the complete lift metric.Then the relation between the curvature tensor field R of the metric g on M and the curvature tensor field $\overline{R}^c$ of the metric g^c is given at any point $(x,u)\in TM$ by*

$$\begin{aligned}\overline{R}^c(X^c,Y^c)Z^c &= (\overline{R}(X,Y)Z)^c \\ \overline{R}^c(X^c,Y^c)Z^v &= \overline{R}^c(X^c,Y^v)Z^c = (\overline{R}(X,Y)Z)^v, \\ \overline{R}^c(X^c,Y^v)Z^v &= \overline{R}^c(X^v,Y^v)Z^c = \overline{R}^c(X^v,Y^v)Z^v = 0.\end{aligned} \tag{19}$$

for all $X,Y,Z\in\mathfrak{X}(M)$.

Proof. From Eqs.(11a and 16), we have

$$\begin{aligned}\overline{\nabla}^c_{X^c}\overline{\nabla}_{Y^c}Z^c &= \overline{\nabla}^c_{X^c}(\overline{\nabla}_Y Z)^c = \overline{\nabla}^c_{X^c}[\nabla_Y Z+\pi(Z)Y-g(Z,Y)P]^c \\ &= \overline{\nabla}^c_{X^c}(\nabla_Y Z+\pi(Z)Y)^c - \overline{\nabla}^c_{X^c}(g(Z,Y)P)^c \\ &= (\overline{\nabla}_X(\nabla_Y Z))^c + \pi(Z)^c\overline{\nabla}^c_{X^c}Y^v + X^c(\pi(Z)^c)Y^v + \pi(Z)^v\overline{\nabla}^c_{X^c}Y^c \\ &+ X^c(\pi(Z)^v)Y^c - X^c(g(Z,Y)^v)P^c - X^c(g(Z,Y)^c)P^v \\ &- g(Z,Y)^c(\overline{\nabla}_X P)^v - g(Z,Y)^v(\overline{\nabla}_X P)^c \\ &= (\overline{\nabla}_X(\nabla_Y Z))^c + [\pi(Z)\overline{\nabla}_X Y]^c + [X(\pi(Z))Y]^c - [X(g(Z,Y)P]^c \\ &- g(Z,Y)^c(\overline{\nabla}_X P)^v - g(Z,Y)^v(\overline{\nabla}_X P)^c, \\ &= (\overline{\nabla}_X[\nabla_Y Z+\pi(Z)Y-g(Z,Y)P])^c \\ &= (\overline{\nabla}_X\overline{\nabla}_Y Z)^c,\end{aligned}$$

similarly we have $\overline{\nabla}^c_{Y^c}\overline{\nabla}_{X^c}Z^c = (\overline{\nabla}_Y\overline{\nabla}_X Z)^c$.

From Eqs.(12 and 16), we get : $\overline{\nabla}^c_{[X^c,Y^c]}Z^c = \overline{\nabla}^c_{[X,Y]^c}Z^c = (\overline{\nabla}_{[X,Y]}Z)^c$. Which implies

$$\overline{R}^c(X^c,Y^c)Z^c = (\overline{R}(X,Y)Z)^c.$$

□

3.2 Ricci curvature on TM with semi-symmetric metric connection

Let $Ric(X,Y)=\sum_{i=1}^{m} g(R(X,e_i)e_i,Y)$ be the Ricci tensor field of (M,g) and similarly, let Ric^c be the Ricci tensor field of (TM,g^c), where $\{e_i\}_{i=1,\dots,m}$ is an orthonormal frame around an arbitrary point $x\in M$. Since $\{E_1=\frac{\sqrt{2}}{2}(e_1^c+e_1^v),\dots,E^m=\frac{\sqrt{2}}{2}(e_m^c+e_m^v),E^{m+1}=\frac{\sqrt{2}}{2}(e_1^c-e_1^v),\dots,E^{2m}=\frac{\sqrt{2}}{2}(e_m^c-e_m^v)\}$ is an orthonormal frame around $(x,u) \in TM$, one has

$$Ric^c(U,V) = \sum_{i=1}^{m} g(R(U, \frac{\sqrt{2}}{2}(e_i^c + e_i^v))\frac{\sqrt{2}}{2}(e_i^c + e_i^v), V) + \sum_{i=1}^{m} g(R(U, \frac{\sqrt{2}}{2}(e_i^c - e_i^v))\frac{\sqrt{2}}{2}(e_i^c - e_i^v), V) \tag{20}$$

for all $U, V \in \mathfrak{X}(TM)$. Hence, from Eqs.(15 and 20), it follows that at any point $(x,u) \in TM$, the Ricci curvature Ric^c of g^c is related by the Ricci curvature Ric of g on M by

$$\begin{aligned} Ric^c(X^c,Y^c) &= \sum_{i=1}^{m} g^c(R^c(X^c, \frac{\sqrt{2}}{2}(e_i^c + e_i^v))\frac{\sqrt{2}}{2}(e_i^c + e_i^v), Y^c) \\ &\quad + \sum_{i=1}^{m} g^c(R^c(X^c, \frac{\sqrt{2}}{2}(e_i^c - e_i^v))\frac{\sqrt{2}}{2}(e_i^c - e_i^v), Y^c) \\ &= \frac{1}{2}\sum_{i=1}^{m} g^c(R^c(X^c, (e_i^c + e_i^v))e_i^c, Y^c) \\ &\quad + \frac{1}{2}\sum_{i=1}^{m} g^c(R^c(X^c, (e_i^c + e_i^v))e_i^v, Y^c) \\ &\quad + \frac{1}{2}\sum_{i=1}^{m} g^c(R^c(X^c, (e_i^c - e_i^v))e_i^c, Y^c) \\ &\quad - \frac{1}{2}\sum_{i=1}^{m} g^c(R^c(X^c, (e_i^c - e_i^v))e_i^v, Y^c) \\ &= \frac{1}{2}\sum_{i=1}^{m} g^c(R^c(X^c, e_i^c)e_i^c, Y^c) + \frac{1}{2}\sum_{i=1}^{m} g^c(R^c(X^c, e_i^v)e_i^c, Y^c) \\ &\quad + \frac{1}{2}\sum_{i=1}^{m} g^c(R^c(X^c, e_i^c)e_i^v, Y^c) + \frac{1}{2}\sum_{i=1}^{m} g^c(R^c(X^c, e_i^v)e_i^v, Y^c) \\ &\quad + \frac{1}{2}\sum_{i=1}^{m} g^c(R^c(X^c, e_i^c)e_i^c, Y^c) - \frac{1}{2}\sum_{i=1}^{m} g^c(R^c(X^c, e_i^v)e_i^c, Y^c) \\ &\quad - \frac{1}{2}\sum_{i=1}^{m} g^c(R^c(X^c, e_i^c)e_i^v, Y^c) + \frac{1}{2}\sum_{i=1}^{m} g^c(R^c(X^c, e_i^v)e_i^v, Y^c) \\ &= \sum_{i=1}^{m} g^c(R^c(X^c, e_i^c)e_i^c, X^c) = (Ric(X,Y))^c, \end{aligned}$$

Hence, we get

$$Ric^c(X^c,Y^c) = (Ric(X,Y))^c, \;\; Ric^c(X^c,Y^v) = (Ric(X,Y))^v. \tag{21}$$

Proposition 3. *Let $(M, g, \overline{\nabla})$ be an m-dimensional manifold $(m > 2)$ endowed with semi-symmetric metric connection and let $(TM, g^c, \overline{\nabla}^c)$ be the complete lift metric endowed with semi-symmetric metric connection $\overline{\nabla}^c$. Then the Ricci tensor field $\overline{Ric}^c$ on $(TM, g^c, \overline{\nabla}^c)$ satisfies:*

$$\begin{aligned}\overline{Ric}^c(X^c, Y^c) &= (\overline{Ric}(X, Y))^c = (Ric(X, Y))^c + (m-1)(\pi(P)g(X, Y))^c, \quad (22)\\ \overline{Ric}^c(X^c, Y^v) &= (\overline{Ric}(X, Y))^v = (Ric(X, Y))^v + (m-1)(\pi(P)g(X, Y))^v\end{aligned}$$

Proof. According to Eq.(20), we have

$$\begin{aligned}\overline{Ric}^c(X^c, Y^c) &= \sum_{i=1}^{m} g^c(\overline{R}^c(X^c, \frac{\sqrt{2}}{2}(e_i^c + e_i^v))\frac{\sqrt{2}}{2}(e_i^c + e_i^v), Y^c)\\ &\quad + \sum_{i=1}^{m} g^c(\overline{R}^c(X^c, \frac{\sqrt{2}}{2}(e_i^c - e_i^v))\frac{\sqrt{2}}{2}(e_i^c - e_i^v), Y^c)\\ &= \sum_{i=1}^{m} g^c(\overline{R}^c(X^c, e_i^c)e_i^c, Y^c)\end{aligned}$$

In view of Proposition 2 and Eq.(8), we have

$$\begin{aligned}\overline{Ric}^c(X^c, Y^c) &= \sum_{i=1}^{m} g^c(R^c(X^c, e_i^c)e_i^c, Y^c) + g^c(X^v, Y^c)\sum_{i=1}^{m}(\pi(P)g(e_i, e_i))^c\\ &+ g^c(X^c, Y^c)\sum_{i=1}^{m}(\pi(P)g(e_i, e_i))^v - \sum_{i=1}^{m}(\pi(P)g(X, e_i))^c g^c(e_i^v, Y^c)\\ &- \sum_{i=1}^{m}(\pi(P)g(X, e_i))^v g^c(e_i^c, Y^c)\end{aligned}$$

From Eqs.(11a and 13) we get

$$\begin{aligned}\overline{Ric}^c(X^c, Y^c) &= Ric^c(X^c, Y^c) + g^c(X^v, Y^c)(\pi(P))^c\sum_{i=1}^{m}(g(e_i, e_i))^v\\ &\quad + g^c(X^v, Y^c)(\pi(P))^v\sum_{i=1}^{m}(g(e_i, e_i))^c\\ &\quad + g^c(X^c, Y^c)(\pi(P))^v\sum_{i=1}^{m}(g(e_i, e_i))^v\\ &\quad - (\pi(P))^c\sum_{i=1}^{m}(g(X, e_i))^v g^c(e_i^v, Y^c)\\ &\quad - (\pi(P))^v\sum_{i=1}^{m}(g(X, e_i))^v g^c(e_i^c, Y^c)\\ &\quad - (\pi(P))^v\sum_{i=1}^{m}(g(X, e_i))^c g^c(e_i^v, Y^c)\end{aligned}$$

$$= \quad Ric^c(X^c, Y^c) + mg^c(X^v, Y^c)(\pi(P))^c + mg^c(X^c, Y^c)(\pi(P))^v$$
$$-(\pi(P))^c(g(X,Y))^v - (\pi(P))^v g^c(X^c, Y^c)$$

Hence

$$\overline{Ric}^c(X^c, Y^c) = Ric^c(X^c, Y^c) + (m-1)(g(X,Y)\pi(P))^c$$

Applying the same argument, we get the second equation. □

Therefore we proved the following result.

Theorem 2. *Let (M, g, ∇) be a Riemannian manifold $(n \geq 3)$ and g^c be the complete lift metric on TM endowed with semi-symmetric metric connection $\overline{\nabla}^c$. Then $(TM, g^c, \overline{\nabla}^c)$ is Einstein manifold if and only if (M, g, ∇) is an Einstein manifold and $\pi(P)$ is a non-zero constant.*

Proof. If we assume that $(g^c, \overline{\nabla}^c)$ is Einstein on TM then

$$\overline{Ric}^c(X^*, Y^*) = \overline{\lambda} g^c(X^*, Y^*), \ \forall X, Y \in \mathfrak{X}(M).$$

Using Proposition 3, Eqs.(9 and 13) we get

$$\{.\overline{Ric}^c(X^c, Y^c) = (\overline{Ric}(X,Y))^c \overline{\lambda} g^c(X^c, Y^c) = \overline{\lambda}(g(X,Y))^c$$

and

$$Ric(X,Y) + (m-1)\pi(P)g(X,Y) = \overline{\lambda} g(X,Y)$$

which give

$$Ric(X,Y) = [\overline{\lambda} - (m-1)\pi(P)]g(X,Y), \tag{23}$$

then (M, g, ∇) is an Einstein manifold on M. Since $m \geq 3$ and M is Einsteinian, M has constant scalar curvature. So $\overline{\lambda} - (m-1)\pi(P)$ is constant. Thus the function with $\pi(P)$ is also non-zero constant.

Conversely, if (M, g, ∇) is Einstein manifold i.e. $Ric(X,Y) = \overline{\lambda} g(X,Y)$. If we suppose that $\pi(P)$ is a non-zero constant. Taking the complete and vertical lift of both sides of the Equation, we found

$$\begin{aligned}
(Ric(X,Y))^c &= \overline{\lambda}(g(X,Y))^c \\
(\overline{Ric}(X,Y))^c &= \overline{\lambda} g^c(X^c, Y^c) + (m-1)(\pi(P)g(X,Y))^c \\
&= [\overline{\lambda} + (m-1)\pi(P)]g^c(X^c, Y^c) \\
\overline{Ric}^c(X^c, Y^c) &= [\overline{\lambda} + (m-1)\pi(P)]g^c(X^c, Y^c).
\end{aligned}$$

Similarly,

$$\begin{aligned}
(Ric(X,Y))^v &= \overline{\lambda}(g(X,Y))^v \\
(\overline{Ric}(X,Y))^v &= \overline{\lambda} g^c(X^c, Y^v) + (m-1)(\pi(P)g(X,Y))^v, \\
&= [\overline{\lambda} + (m-1)\pi(P)]g^c(X^c, Y^v) \\
\overline{Ric}^c(X^c, Y^v) &= [\overline{\lambda} + (m-1)\pi(P)]g^c(X^c, Y^v).
\end{aligned}$$

We deduce that

$$\overline{Ric}^c = [\overline{\lambda} + (m-1)\pi(P)]g^c, \tag{24}$$

Therefore $(g^c, \overline{\nabla}^c)$ is an Einstein structure on TM. □

4 Ricci solitons on tangent bundle

In this section we have studied Ricci solitons, whose potential vector field is torqued, on Riemannian manifolds with respect to Riemannian connection and semi-symmetric metric P-connection.

4.1 Case when the potential is a vertical lift vector field

We start with the description of the Lie derivative of g^c with respect to the vertical lift of torqued vector field.

Let (M, g, ∇, τ) be a Riemannian manifold and let $(TM, g^c, \nabla^c, \tau^v)$ be a pseudo-Riemannian metric on TM. If τ is torqued vector field on M, using equation Eq.(4 and 14), we get

$$\begin{aligned} \mathcal{L}_{\tau^v} g^c(X^c, Y^c) &= g^c(\nabla^c_{X^c}\tau^v, Y^c) + g^c(\nabla^c_{Y^c}\tau^v, X^c) \\ &= g^c((\nabla_X\tau)^v, Y^c) + g^c((\nabla_Y\tau)^v, X^c) \\ &= [g((\nabla_X\tau), Y) + g((\nabla_Y\tau), X)]^v \\ &= [2\varphi g(X,Y) + \gamma(X)g(\tau, Y) + \gamma(Y)g(\tau, X)]^v, \\ \mathcal{L}_{\tau^v} g^c(X^c, Y^v) &= g^c(\nabla^c_{X^c}\tau^v, Y^v) + g^c(\nabla^c_{Y^v}\tau^v, X^c) = 0. \end{aligned} \tag{25}$$

First, we recall the following result proved in [3].

Corollary 1. *A torqued vector field τ on a Riemannian manifold M is a Killing vector field if and only if τ is a recurrent vector field satisfies*

$$\nabla_X\tau = \alpha(X)\tau \text{ and } \alpha(\tau) = 0,$$

where α is a 1-form.

In views of Corollary 1, we have

Theorem 3. *Let (M, g, ∇, τ) be Riemannian manifold with torqued vector field τ. Then $(TM, g^c, \nabla^c, \tau^v, \lambda)$ is Ricci soliton on TM if and only if (M, g) is Einstein manifold with τ is a Killing vector field.*

Proof. Assume that $(TM, g^c, \nabla^c, \tau^v, \lambda)$ is a Ricci soliton whose potential field is a recurrent potential field τ^v, i.e.

$$Ric^c(X^c, Y^c) + \frac{1}{2}(\mathcal{L}_{\tau^v} g^c)(X^c, Y^c) = \lambda g^c(X^c, Y^c) \tag{26}$$

combining Eq.(25 and 26), we find

$$(Ric(X,Y))^c = (\lambda g(X,Y))^c - \frac{1}{2}[2\varphi g(X,Y) + \gamma(X)g(\tau, Y) + \gamma(Y)g(\tau, X)]^v$$

with $\gamma(\tau) = 0$, then Eq.(27) yields

$$Ric(X,Y) = \lambda g(X,Y)\varphi g(X,Y) + \frac{1}{2}[\gamma(X)g(\tau, Y) + \gamma(Y)g(\tau, X)] = 0, \tag{27}$$

for any vector field X, Y. Taking contraction of the second equation over X and Y in Eq.(27), we get

$$Ric(X,Y) = \lambda g(X,Y) m\varphi + \gamma(\tau) = 0,$$

Combining this with $\gamma(\tau) = 0$ gives $\varphi = 0$. Hence the potential field τ is a recurrent vector field. Therefore, from Corollary 1, τ is a Killing vector field and (M, g) is an Einstein manifold.
Conversaly, if (M, g) is Einstein manifold with τ is torqued vector field. If we suppose that τ is a Killing vector field, from Corollary 1, τ is recurrent vector field that satisfies $\gamma(\tau) = 0$. Then we have

$$Ric(X,Y) = \lambda g(X,Y)\gamma(X)g(\tau,Y) + \gamma(Y)g(\tau,X) = 0$$

Hence, from Eq.(25) we get

$$\begin{aligned} Ric^c(X^c,Y^c) + \frac{1}{2}(\mathcal{L}_{\tau^v} g^c)(X^c,Y^c) &= \lambda g^c(X^c,Y^c) \\ Ric^c(X^c,Y^v) + \frac{1}{2}(\mathcal{L}_{\tau^v} g^c)(X^c,Y^c) &= \lambda g^c(X^c,Y^v). \end{aligned}$$

Consequently, $(TM, g^c, \nabla^c, \tau^v, \lambda)$ is a Ricci soliton and it is a trivial one. □

In the same way, let (M, g, ∇, τ) be a Riemannian manifold and let $(TM, g^c, \overline{\nabla}^c, \tau^v)$ be a pseudo-Riemannian metric on TM. If τ is a torqued vector field on M, using Eqs. (4, 7, 16 and 25), we get

$$\begin{aligned} \overline{\mathcal{L}}_{\tau^v} g^c(X^c,Y^c) &= g^c(\overline{\nabla}^c_{X^c}\tau^v, Y^c) + g^c(\overline{\nabla}^c_{Y^c}\tau^v, X^c) \qquad (28) \\ &= g^c((\overline{\nabla}_X\tau)^v, Y^c) + g^c((\overline{\nabla}_Y\tau)^v, X^c) \\ &= g(\overline{\nabla}_X\tau, Y) + g(\overline{\nabla}_Y\tau, X) \\ &= g(\nabla_X\tau + \pi(\tau)X - g(X,\tau)P, Y) + g(\nabla_Y\tau + \pi(\tau)Y - \pi(Y)P, X) \\ &= (\mathcal{L}_\tau g)(X,Y) + 2\pi(\tau)g(X,Y) - [g(X,\tau)\pi(Y) + g(Y,\tau)\pi(X)] \\ &= 2(\varphi + \pi(\tau))g(X,Y) + (\gamma(X) - \pi(X))g(\tau,Y) \\ &\quad + (\gamma(Y) - \pi(Y))g(\tau,X), \\ \overline{\mathcal{L}}_{\tau^v} g^c(X^c,Y^v) &= g^c(\overline{\nabla}^c_{X^c}\tau^v, Y^v) + g^c(\overline{\nabla}^c_{Y^v}\tau^v, X^c) = 0. \end{aligned}$$

Theorem 4. *Let (M, g, ∇, τ) be Riemannian manifolds with torqued vector field τ. Then the triple $(TM, g^c, \overline{\nabla}^c, \tau^v, \overline{\lambda})$ is Ricci soliton on TM if and only if (M, g, τ) is an Einstein manifold and the following three conditions hold:*

1. $\frac{1}{m}\pi(\tau)g - \frac{1}{2}[(\pi - \gamma)\otimes\eta + \eta\otimes(\pi - \gamma)] = 0,$
2. *The function φ in 4 is $(\frac{1-m}{m})\pi(\tau)$,*
3. *The function $\pi(P)$ is a non zero constant.*

Proof. Assume that $(TM, g^c, \overline{\nabla}^c, \tau^v, \overline{\lambda})$ is a Ricci soliton whose potential field is a recurrent potential field τ^v. By combining Eqs.(22, 25 and 28), we find

$$\begin{aligned} \overline{Ric}^c(X^c, Y^c) &= \overline{\lambda} g^c(X^c, Y^c) - \frac{1}{2}(\overline{\mathcal{L}}_{\tau^v} g^c)(X^c, Y^c) \\ &= [(\overline{\lambda} - (m-1)\pi(P))g(X,Y)]^c \\ &\quad -\frac{1}{2}[2(\varphi + \pi(\tau))g(X,Y) + (\gamma - \pi)(X)g(\tau, Y) \\ &\quad +(\gamma - \pi)(Y)g(\tau, X)] \end{aligned} \tag{29}$$

In view of Eq.(29) yields

$$\begin{aligned} &Ric(X,Y) = [\overline{\lambda} - (m-1)\pi(P)]g(X,Y), \\ &(\varphi + \pi(\tau))g(X,Y) = \frac{1}{2}[(\pi - \gamma)(Y)g(X,\tau) + (\pi - \gamma)(X)g(Y,\tau)] \end{aligned}$$

for any vertical vector field X, Y. Taking contraction of the second equation over X and Y, we get

$$Ric(X,Y) = [\overline{\lambda} - (m-1)\pi(P)]g(X,Y)m(\varphi + \pi(\tau)) = \pi(\tau) - \gamma(\tau),$$

Since $\gamma(\tau) = 0$, then $\varphi = (\frac{1-m}{m})\pi(\tau)$. If we denote the dual 1-form of τ by η, then yields

$$Ric = [\overline{\lambda} - (m-1)\pi(P)]g, \frac{1}{m}\pi(\tau)g - \frac{1}{2}[(\pi - \gamma)\otimes\eta + \eta\otimes(\pi - \gamma)] = 0. \tag{30}$$

Therefore (M, g) is Einstein manifold with recurrent vector field such that the following relation

$$\frac{1}{m}\pi(\tau)g - \frac{1}{2}[(\pi - \gamma)\otimes\eta + \eta\otimes(\pi - \gamma)] = 0. \tag{31}$$

holds for any arbitrary vector fields.
Conversely, assume that (M, g) is Einstein manifold and Eq.(31) holds, then from Eqs. (9, 21, 22 and 28), we get

$$\begin{aligned} \overline{Ric}^c(X^c, Y^c) + \frac{1}{2}(\overline{\mathcal{L}}_{\tau^v} g^c)(X^c, Y^c) &= (\overline{Ric}(X,Y))^c + (\varphi g(X,Y))^v + (\pi(\tau)g(X,Y))^v \\ &\quad + \frac{1}{2}[(\gamma - \pi)(X)g(\tau, Y) + (\gamma - \pi)(Y)g(\tau, X)]^v \\ &= [(\lambda + (m-1)\pi(P))g(X,Y)]^c \\ &\quad + [\varphi + \frac{m-1}{m}\pi(\tau)]^v(g(X,Y))^v \\ &= (\lambda + (m-1)\pi(P))^v(g(X,Y))^c \\ &\quad + (\lambda + (m-1)\pi(P))^c(g(X,Y))^v \\ &\quad + [\varphi + \frac{m-1}{m}\pi(\tau)]^v(g(X,Y))^v \end{aligned} \tag{32}$$

$$\overline{Ric}^c(X^c,Y^v)+\frac{1}{2}(\mathcal{L}_{\tau^v}g^c)(X^c,Y^v) = (\overline{Ric}(X,Y))^c+(\varphi g(X,Y))^v+(\pi(\tau)g(X,Y))^v$$
$$+\frac{1}{2}[(\gamma-\pi)(X)g(\tau,Y)+(\gamma-\pi)(Y)g(\tau,X)]^v$$
$$=[(\lambda+(m-1)\pi(P))g(X,Y)]^c$$
$$+[\varphi+\frac{m-1}{m}\pi(\tau)]^v(g(X,Y))^v$$
$$=(\lambda+(m-1)\pi(P))^v(g(X,Y))^c$$
$$+(\lambda+(m-1)\pi(P))^c(g(X,Y))^v$$
$$+[\varphi+\frac{m-1}{m}\pi(\tau)]^v(g(X,Y))^v$$

Hence, $(TM,g^c,\overline{\nabla}^c,\tau^v,\overline{\lambda})$ is a Ricci soliton with $\overline{\lambda}=\lambda+(m-1)\pi(P)$ if $\varphi+\frac{m-1}{m}\pi(\tau)=0$ and $\pi(P)$ is a non zero constant, which complete the proof. □

4.2 Case when the potential is a complete lift vector field

Now we consider complete vector fields. We start with the following result, obtained starting from

$$\begin{aligned}
\overline{\mathcal{L}}_{\tau^c}g^c(X^c,Y^c) &= g^c(\overline{\nabla}^c_{X^c}\tau^c,Y^c)+g^c(\overline{\nabla}^c_{Y^c}\tau^c,X^c) \\
&= g^c((\overline{\nabla}_X\tau)^c,Y^c)+g^c((\overline{\nabla}_Y\tau)^c,X^c) \\
&= (g(\overline{\nabla}_X\tau,Y))^c+(g(\overline{\nabla}_Y\tau,X))^c \\
&= [2(\varphi+\pi(\tau))g(X,Y)]^c \\
&\quad +[(\gamma-\pi)(X)g(\tau,Y)+(\gamma-\pi)(Y)g(X,\tau)]^c \\
\overline{\mathcal{L}}_{\tau^c}g^c(X^c,Y^v) &= g^c(\overline{\nabla}^c_{X^c}\tau^c,Y^v)+g^c(\overline{\nabla}^c_{Y^v}\tau^c,X^c) \\
&= g^c((\overline{\nabla}_X\tau)^c,Y^v)+g^c((\overline{\nabla}_Y\tau)^v,X^c) \\
&= (g(\overline{\nabla}_X\tau,Y))^v+(g(\overline{\nabla}_Y\tau,X))^v \\
&= [2(\varphi+\pi(\tau))g(X,Y)]^v \\
&\quad +[(\gamma-\pi)(X))g(\tau,Y)+(\gamma-\pi)(Y)g(X,\tau)]^v
\end{aligned} \tag{33}$$

Theorem 5. *Let (M,g,∇,τ) be a Riemmannian manifold with torqued vector field τ. Then, if $(TM,g^c,\overline{\nabla}^c,\tau^c,\overline{\lambda})$ is Ricci soliton then (M,g,∇,τ) is an almost quasi-Einstein manifold.*

Proof. Assume that $(TM,g^c,\overline{\nabla}^c,\tau^c,\overline{\lambda})$ is Ricci soliton. By combining Eqs.(5 and 33), we find

$$\begin{aligned}
\overline{Ric}^c(X^c,Y^c) &= \overline{\lambda}g^c(X^c,Y^c)-\frac{1}{2}(\mathcal{L}_{\tau^c}g^c)(X^c,Y^c) \\
(\overline{Ric}(X,Y))^c &= (\overline{\lambda}g(X,Y))^c-\frac{1}{2}[2(\varphi+\pi(\tau))g(X,Y)]^c \\
&\quad -\frac{1}{2}[(\gamma-\pi)(X))g(\tau,Y)+(\gamma-\pi)g(X,\tau)]^c \\
&= [(\overline{\lambda}-\varphi-\pi(\tau))g(X,Y)]^c \\
&\quad +\frac{1}{2}[(\pi-\gamma)(X)g(\tau,Y)+(\pi-\gamma)(Y)g(X,\tau)]^c
\end{aligned} \tag{34}$$

If we denote the dual 1-form of τ by η, then, from Eqs.(9 and 34) yields

$$\begin{aligned} Ric(X,Y) &= [\overline{\lambda} - (n-1)\pi(P) - \varphi - \pi(\tau)]g(X,Y) \\ &+\frac{1}{2}[(\pi-\gamma)(Y)\eta(X) + (\pi-\gamma)(X)\eta(Y)] \end{aligned}$$

for any vertical vector field X, Y ,i.e.

$$Ric = [\overline{\lambda} - (n-1)\pi(P) - \varphi - \pi(\tau)]g + \frac{1}{2}[(\pi-\gamma)\otimes\eta + \eta\otimes(\pi-\gamma)]$$

Therefore (M, g) is an almost quasi-Einstein manifold. □

Now, we can state the following result which gives conditions for $(TM, g^c, \overline{\nabla}^c, \tau^v, \overline{\lambda})$ to be Ricci soliton

Theorem 6. *Let (M, g, ∇, τ) be a Riemmannian manifold with torqued vector field τ. Then, if (M, g, ∇, τ) is Ricci soliton then $(TM, g^c, \overline{\nabla}^c, \tau^c, \overline{\lambda})$ is Ricci soliton and the following two conditions hold*

1. $\pi(X)g(\tau, Y) + \pi(Y)g(\tau, X) = 0,$
2. *The function $\lambda + (n-1)\pi(P) + \pi(\tau)$ is a non zero constant.*

Proof. We suppose that (M, g, λ, τ) is a Ricci soliton whose potential field is a torqued vector field. Then Eq.(4) holds, which implies

$$\mathcal{L}_\tau g(X,Y) = 2\varphi g(X,Y) + \gamma(X)g(\tau,Y) + \gamma(Y)g(\tau,X). \tag{35}$$

for any vector fields X, Y tangent to M. By combining Eqs.(35 and 5), we find

$$Ric(X,Y) = (\lambda-\varphi)g(X,Y) - \frac{1}{2}\gamma(X)g(\tau,Y) - \frac{1}{2}\gamma(Y)g(\tau,X) \tag{36}$$

Hence, by Proposition (3.1) in [3], (M, g) is an almost quasi-Einstein manifold. From Eqs.(7 and 35), the Lie derivative along the torqued vector field τ with respect to semi-symmetric metric connection $\overline{\nabla}$ is given by

$$\begin{aligned} \overline{\mathcal{L}}_\tau g(X,Y) &= g(\overline{\nabla}_X\tau, Y) + g(\overline{\nabla}_Y\tau, X) \qquad (37)\\ &= (\mathcal{L}_\tau g)(X,Y) + 2\pi(\tau)g(X,Y) - [g(X,\tau)\pi(Y) + g(Y,\tau)\pi(X)] \\ &= 2(\varphi+\pi(\tau))g(X,Y) + (\gamma-\pi)(X)g(\tau,Y) + (\gamma-\pi)(Y)g(X,\tau). \end{aligned}$$

By combining Eqs.(9, 19 and 37), we find

$$\begin{aligned} \overline{Ric}^c(X^c,Y^c) + \frac{1}{2}(\mathcal{L}_{\tau^c}g^c)(X^c,Y^c) &= (\overline{Ric}(Y,Z))^c + \frac{1}{2}[(\overline{\mathcal{L}}_\tau g)(X,Y)]^c \\ &= [(\lambda-\varphi)g(X,Y)]^c - \frac{1}{2}[\gamma(X)g(\tau,Y) + \gamma(Y)g(\tau,X)]^c \\ &+[(n-1)\pi(P)g(X,Y)]^c + [(\varphi+\pi(\tau))g(X,Y)]^c \\ &+\frac{1}{2}[(\gamma-\pi)(X)g(\tau,Y) + (\gamma-\pi)(Y)g(X,\tau)]^c \\ &= [(\lambda+(n-1)\pi(P)+\pi(\tau))g(X,Y)]^c \\ &-\frac{1}{2}[\pi(X)g(\tau,Y) + \pi(Y)g(\tau,X)]^c, \end{aligned}$$

Therefore, $\overline{Ric}^c(X^c, Y^c) + \frac{1}{2}(\mathcal{L}_{\tau^c} g^c)(X^c, Y^c) = \overline{\lambda} g^c(X^c, Y^c)$ if $\lambda + (n-1)\pi(P) + \pi(\tau)$ is a non zero constant function and the relation

$$\pi(X)g(\tau, Y) + \pi(Y)g(\tau, X) = 0$$

holds for arbitrary vector fields X, Y. □

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