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ON SOME THEOREMS OF THE JACOBI-LIPSCHITZ CLASS FOR THE JACOBI TRANSFORM

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Abstract

Using a generalized Jacobi translation, we obtain a generalization of the theorem 84 of Titchmarsh for the Jacobi transform satisfying the Jacobi-Lipschitz and Dini Lipschitz conditions in the space $L^p(\mathbb{R}^+, \Delta(t)dt)$, where $1 < p \leq 2$.

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1 Introduction and preliminaries

Titchmarsh established in ([15], Theorem 84) that if $f(x)$ satisfies the Lipschitz condition $Lip(\alpha, p)$ in the L^p, $1 < p \leq 2$, on the real line $\mathbb{R}$, that is

$$\int_{-\infty}^{+\infty} |f(x+h) - f(x-h)|^p dx = O(h^{\alpha p}),\ 0 < \alpha \leq 1,\ h \to 0,$$

then its Fourier transform $\widehat{f}$ belongs to L^β for

$$\frac{p}{p + \alpha p - 1} < \beta \leq \frac{p}{p-1}.$$

On the other hand, Younis proved this theorem to higher differences and to functions on $\mathbb{R}$ (see [16])) satisfying the Dini-Lipschitz condition. More precisely, we have

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Theorem 1. *[16] Let $f \in \mathrm{L}^p(\mathbb{R})$ with $1 < p \leq 2$ such that*

$$\left(\int_{-\infty}^{+\infty} |f(x+h) - f(x)|^p dx\right)^{1/p} = O\left(\frac{h^\alpha}{(\log \frac{1}{h})^\gamma}\right), \ h \to 0, \ 0 < \alpha \leq 1, \quad \gamma > 0$$

Then, its Fourier transform $\widehat{f}$ belongs to L^β for

$$\frac{p}{p + \alpha p - 1} < \beta \leq \frac{p}{p-1}.$$

The main aim of this paper is to establish an analogue of these theorems for the Jacobi transform setting by means of the generalized Jacobi translation. There are many analogues of these theorems: for the Bessel transform on $\mathbb{R}^+$, for the Dunkl transform on $\mathbb{R}^d$, for the Laguerre Hypergroup and etc (for exemple, see [7, 9, 12])

Now, we recall some notations and results about harmonic analysis on Jacobi transform and we refer for more details to the articles [1, 8, 11, 13, 14]

Throughout the paper α, β and ρ are arbitrary real numbers with $\alpha > \beta \geq -\frac{1}{2}$ and $\rho = \alpha + \beta + 1$

We consider the Jacobi differential operator

$$\mathrm{D} = \mathrm{D}_{\alpha,\beta} = \frac{d^2}{dt^2} + ((2\alpha+1)\coth t + (2\beta+1)\tanh t)\frac{d}{dt}.$$

It is known that for any $\lambda \in \mathbb{C}$ there exists a unique even C^∞-solution $u(t)$ of the differential equation

$$\begin{cases} \mathrm{D}u + (\lambda^2 + \rho^2)u = 0 \\ u(0) = 1, \frac{d}{dx}u(0) = 0. \end{cases}$$

This function $u(t)$ is called Jacobi function and it is denoted $\phi_\lambda(t) = \phi_\lambda^{(\alpha,\beta)}(t)$. The function $\phi_\lambda(t)$ can be expressed in terms of the hypergeometric function

$$\phi_\lambda(t) ={}_2 F_1(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda), \alpha + 1, -\sinh^2 t).$$

where ${}_2F_1(a, b, c, z)$ is the hypergeometric function.

For $\alpha \geq -\frac{1}{2}$, we introduce the normalized spherical Bessel function j_α defined by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\alpha+1)}(\frac{z}{2})^{2k}, z \in \mathbb{C}, \quad (1)$$

where $\Gamma(x)$ is the gamma-function.

Moreover, from (1) we see that

$$\lim_{x\to 0} \frac{j_\alpha(x) - 1}{x^2} \neq 0.$$

by consequence, there exist $c > 0$ and $\eta > 0$ satisfying

$$|x| \leq \eta \Longrightarrow |j_\alpha(x) - 1| \geq c|x|^2 \tag{2}$$

Lemma 1. *Let $\alpha > -1/2$, $\alpha > \beta \geq -1/2$, and let $t_0 > 0$. Then for $|\eta| \leq \rho$, there exists a positive constant C_1 such that*

$$|1 - \phi_{\mu+i\eta}(t)| \geq C_1|1 - j_\alpha(\mu t)|,$$

for all $0 \leq t \leq t_0$.

Proof. (see Lemma 9 in [4]). □

We adhere to the conventions and normalization used [10], the c-function

$$c(\lambda) = \frac{2^\rho \Gamma(i\lambda)\Gamma(\frac{1}{2}(1 + i\lambda))}{\Gamma(\frac{1}{2}(i\lambda + \rho))\Gamma(\frac{1}{2}(i\lambda + \rho) - \beta)}.$$

The function $|c(\lambda)|^{-2}$ is an even continuous function on $\mathbb{R}$ and satisfies the following estimates. There exist positive constants $k_1,\ k_2,\ k$ such that

1. If $\rho \geq 0$ and $\alpha > -\frac{1}{2}$, then

$$k_1|\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^{2\alpha+1},\ \lambda \in \mathbb{R},\ |\lambda| > k \tag{3}$$

2. If $\rho > 0$ and $\alpha > -\frac{1}{2}$, then

$$k_1|\lambda|^2 \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^2,\ \lambda \in \mathbb{R},\ |\lambda| \leq k. \tag{4}$$

For more details, see [2, 3] and references therein.

We denote $\mathcal{D}(\mathbb{R}^+)$ is the space of space of even C^∞-function on $\mathbb{R}^+$ with compact support.

Definition 1. *[14] For every $f \in \mathcal{D}(\mathbb{R}^+)$ the Jacobi transform of f is defined by:*

$$\mathcal{F}(f)(\lambda) = \widehat{f}(\lambda) = \int_0^\infty f(t)\phi_\lambda(t)\Delta(t)dt,\ \lambda \in \mathbb{R}^+,$$

where $\Delta(t) = (2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1}$.

The mapping $\mathcal{F} : f \to \widehat{f}$ extended by continuity from the Banach space $\mathrm{L}^p(\mathbb{R}^+, \Delta(t)dt)$ onto the Banach space $\mathrm{L}^q(\mathbb{R}^+, d\mu(\lambda))$, where $d\mu(\lambda) = \frac{1}{\sqrt{2\pi}}|c(\lambda)|^{-2}d\lambda$ and $p^{-1} + q^{-1} = 1$. The extended mapping is also denoted by $\mathcal{F} : f \to \widehat{f}$ and it is called Jacobi transform.

We have the inversion formula (cf. [11])

$$f(t) = \int_0^\infty \widehat{f}(\lambda)\phi_\lambda(t)d\mu(\lambda),$$

We put $\|f\|_{\rho,p} = \|f\|_{\mathrm{L}^p(\mathbb{R}^+,\Delta(t)dt)}$ and $\|\widehat{f}\|_{\mu,q} = \|\widehat{f}\|_{\mathrm{L}^q(\mathbb{R}^+,d\mu(\lambda))}$.

From [5], we have the Hausdorff-Young inequality

$$\|\widehat{f}\|_{\mu,q} \leq C_2\|f\|_{\rho,p},$$

where $C_2 > 0$ is a positive constant.

The generalized Jacobi translation was defined by Flensted-Jensen and Koornwinder [10] (analogue of translation operator for Fourier transform) given by

$$\mathrm{T}_y f(x) = \int_0^\infty f(z)K(x,y,z)\Delta(z)dz,$$

with kernel

$$\begin{aligned} K(x,y,z) &= \frac{2^{-2\rho}\Gamma(\alpha+1)(\cosh x cosh y \cosh z)^{-\alpha-\beta-1}}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})(\sinh x \sinh y \sinh z)^{2\alpha}}(1-B^2)^{\alpha-\frac{1}{2}} \\ &\times \ {}_2F_1((\alpha+\beta, \alpha-\beta, \alpha+\frac{1}{2}, \frac{1}{2}(1-B)), \end{aligned}$$

for $|x-y| < z < x+y$ and $K(x,y,z) = 0$ elsewhere and

$$B = \frac{\cosh^2 x + \cosh^2 y + \cosh^2 z - 1}{2\cosh x \cosh y \cosh z}.$$

In [4], we have

$$\widehat{(\mathrm{T}_h f)}(\lambda) = \phi_\lambda(h)\widehat{f}(\lambda), \tag{5}$$

2 Main results

In this section, we give the main results of the paper but first we need to define the Lipschitz-Jacobi class.

Definition 2. *Let $0 < \delta \leq 1$. A function $f(x) \in \mathrm{L}^p(\mathbb{R}^+, \Delta(t)dt)$ is said to be in the Lipschtz-Jacobi class, denoted by Lip$(\delta,\ p)$, if it satisfies*

$$\|\mathrm{T}_h f(x) - f(x)\|_{\rho,p} = O(h^{\delta+2}),\ \text{as } h \to 0. \tag{6}$$

for all x in $\mathbb{R}^+$.

Theorem 2. *Let f belongs to the Lipschitz-Jacobi class Lip$(\delta,\ p)$, $0 < \delta \leq 1$ and $1 < p \leq 2$. Then $\widehat{f} \in \mathrm{L}^\gamma(\mathbb{R}^+, d\mu(\lambda))$ for all γ satisfying*

$$\frac{3p}{3p+\delta p-3} < \gamma \leq \frac{p}{p-1}.$$

Proof. Let $f \in \mathrm{L}^p(\mathbb{R}^+, \Delta(t)dt)$ satisfying relation (6)

i.e.,

$$\|\mathrm{T}_h f(x) - f(x)\|_{\rho,p} = O(h^{\delta+2}),\ as\ h \to 0.$$

Thus, from formula (5) and the Hausdorff-Young inequality, we obtain

$$\int_0^{+\infty} |1-\phi_\lambda(h)|^q |\widehat{f}(\lambda)|^q d\mu(\lambda) \leq C_2^q \|\mathrm{T}_h f(x) - f(x)\|_{\rho,p}^q$$

Then

$$\int_0^{+\infty} |1-\phi_\lambda(h)|^q |\widehat{f}(\lambda)|^q d\mu(\lambda) \leq K h^{q(\delta+2)},$$

where $K > 0$ is a positive constant.

Lemma 1 implies that

$$C_1 \int_0^{+\infty} |1-j_\alpha(\lambda h)|^q |\widehat{f}(\lambda)|^q d\mu(\lambda) \leq K h^{q(\delta+2)}$$

i.e.,

$$\int_0^{+\infty} |1-j_\alpha(\lambda h)|^q |\widehat{f}(\lambda)|^q d\mu(\lambda) \leq C_3 h^{q(\delta+2)},$$

where $C_3 = \frac{K}{C_1} > 0$ is a positive constant.

It follows from (2) that

$$c^q \int_0^{\frac{\eta}{h}} |\lambda h|^{2q} |\widehat{f}(\lambda)|^q d\mu(\lambda) \leq C_3 h^{q(\delta+2)}$$

Therefore

$$\int_0^{\frac{\eta}{h}} |\lambda^2 \widehat{f}(\lambda)|^q d\mu(\lambda) \leq C_4 h^{q\delta}, \tag{7}$$

where $C_4 = C_3 c^{-q}$.

Now, let

$$\psi(t) = \int_1^t |\lambda^2 \widehat{f}(\lambda)|^\gamma d\mu(\lambda)$$

Then, for $\gamma \leq q$, where $\frac{1}{p} + \frac{1}{q} = 1$, by Hölder inequality, we have

$$\begin{aligned}
\psi(t) &\leq \left(\int_1^t |\lambda^2 \widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{\gamma}{q}} \left(\int_1^t d\mu(\lambda)\right)^{1-\frac{\gamma}{q}} \\
&\leq \left(\int_1^t |\lambda^2 \widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{\gamma}{q}} \left(\int_1^t |c(\lambda)|^{-2} d\lambda\right)^{1-\frac{\gamma}{q}}
\end{aligned}$$

whence, in view of (4) and (7), we obtain

$$\psi(t) \leq O(t^{-\delta\gamma})t^{3-3\frac{\gamma}{q}}$$

That is

$$\int_1^t |\lambda^2 \widehat{f}(\lambda)|^\gamma d\mu(\lambda) = O(t^{-\delta\gamma+3-3\frac{\gamma}{q}}) = O(t^{-\delta\gamma+3-3\gamma+\frac{3\gamma}{p}})$$

Since $\lambda \geq 1$, then

$$\begin{aligned} \int_1^t |\widehat{f}(\lambda)|^\gamma d\mu(\lambda) &\leq \int_1^t |\lambda^2 \widehat{f}(\lambda)|^\gamma d\mu(\lambda) \\ &= O(t^{-\delta\gamma+3-3\gamma+\frac{3\gamma}{p}}) \end{aligned}$$

and this is bounded as $t \to \infty$ if

$$-\delta\gamma + 3 - 3\gamma + \frac{3\gamma}{p} < 0$$

which gives

$$\frac{3p}{3p+p\delta-3} < \gamma \leq \frac{p}{p-1}$$

and this ends the proof. □

By analog with the proof of Theorem 2, we can establish the following result

Theorem 3. *If $f \in \mathrm{L}^p(\mathbb{R}^+, \Delta(t)dt)$ with $1 < p \leq 2$ such that*

$$\|\mathrm{T}_h f(x) - f(x)\|_{\rho,p} = O\left(\frac{h^{\delta+2}}{(\log\frac{1}{h})^\sigma}\right), \ \textit{as } h \to 0, \ 0 < \delta \leq 1, \ \sigma > 0.$$

Then $\widehat{f} \in \mathrm{L}^\gamma(\mathbb{R}^+, d\mu(\lambda))$ for all γ satisfying

$$\frac{3p}{3p+p\delta-3} < \gamma \leq \frac{p}{p-1}.$$

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