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RICCI SOLITONS ON SASAKIAN MANIFOLDS UNDER A NEW DEFORMATION

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Abstract

The object of the present paper is to introduce a new transformation of almost contact metric manifolds. Firstly, starting from a Sasakian manifold we construct another Sasakian manifold and we prove some geometric properties. Secondly, we study Ricci solitons in Sasakian manifolds under this deformation. Concrete examples are given.

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1 Introduction

There exist several types of deformations of almost contact metric structures. For example D-homothetic deformations [16], conformal deformations [17, 8], deformation of Marrero [13], $\mathcal{D}$-homothetic warping [7], $\mathcal{D}$-homothetic bi-warping [1], $\mathcal{D}$-isometric warping [3] etc. Recently, in [2], we have investigated a new deformation of almost contact metric manifolds where we have deformed the structural tensor φ and metric tensor g at the same time unlike the previous deformations.

The study of deformations of a Sasakian structure $(M, \varphi, \xi, \eta, g)$ is feasible when one keeps some of the tensors or structures fixed and varies others.

Thus far we have considered a fixed Sasakian manifold $(M, \varphi, \xi, \eta, g)$. It will be important later to understand how one can deform such a structure to another Sasakian structure on the same manifold M.

It is known that the deformation of Tanno [16] preserves the Sasakian structure. For a Sasakian structure (φ, ξ, η, g) and positive constant a, the structure

$$\overline{\varphi} = \varphi, \qquad \overline{\eta} = a\eta, \qquad \overline{\xi} = \frac{1}{a}\xi, \qquad \overline{g} = ag + a(a-1)\eta \otimes \eta,$$

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is again a Sasakian structure.
Recently, in [4] we introduced a new deformation of almost contact metric structures where we deform the metric g and the structural tensor φ simultaneously. In this work, we use our deformation particularly on Sasakian manifolds and we show some interesting results. This paper is organized in the following way:
Section 2, contains basic results about Sasakian manifolds and generalized Sasakian space forms. In Section 3, we give the new deformation which preserves the Sasakian structures with concrete examples. Section 3 is devoted to studying some geometric properties on the deformed Sasakian manifold. In the last section, we study deformed Sasakian metrics as Ricci solitons and we give an example.

2 Review of needed notions

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exist on M a $(1,1)$ tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\eta(\xi) = 1,\ \varphi^2(X) = -X + \eta(X)\xi \quad and \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad (1)$$

for any vector fields X,Y on M. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$.

Such a manifold is said to be a contact metric manifold if $d\eta = \phi$, where $\phi(X,Y) = g(X, \varphi Y)$ is called the fundamental 2-form of M. If, in addition, ξ is a Killing vector field, then M is said to be a K-contact manifold. It is well-known that a contact metric manifold is a K-contact manifold if and only if $\nabla_X \xi = -\varphi X$, for any vector field X on M.

On the other hand, the almost contact metric structure of M is said to be normal if $[\varphi,\varphi](X,Y) = -2d\eta(X,Y)\xi$, for any X, Y, where $[\varphi,\varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

A normal contact metric manifold is called a Sasakian manifold. It can be proved that a Sasakian manifold is K-contact, and that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \quad (2)$$

for any X,Y. Moreover, from the formula (2) easily obtains

$$\nabla_X \xi = -\varphi X, \qquad (\nabla_X \eta)Y = g(X, \varphi Y) = \phi(X,Y). \quad (3)$$

For a Sasakian manifold the following equations hold:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \quad (4)$$

$$S(X,\xi) = 2n\eta(X), \tag{5}$$

where R is the curvature tensor and S denotes the Ricci curvature defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{6}$$

$$S(X,Y) = \sum_{i=1}^{2n+1} g\big(R(X,e_i)e_i, Y\big), \tag{7}$$

where $\{e_i\}$ is a local orthonormal frame fields.

A metric g is Einstein if $S = \lambda g$ for some constant λ. It turns out that a Sasakian manifold can be Einstein only for $\lambda = 2n$ (i.e. $S = 2ng$), so that g has positive Ricci curvature.

A Sasakian manifold M of dimension $2n+1$ with a Sasakian structure (φ,ξ,η,g) is said to be η–Einstein if the Ricci curvature tensor of the metric g satisfies the equation $S = ag + b\eta\otimes\eta$ for some constants $a, b \in \mathbb{R}$. These metrics were introduced and studied by Okumura [14] and then named by Sasaki [6]. Okumura assumed that both a and b are functions, and then proved, similar to the case of Einstein metrics, that they must be constant when $n > 1$. Obviously $b = 0$ reduces to the more familiar Sasakian-Einstein condition. In general, $a + b = 2n$.

The sectional curvature of the plane section spanned by the unit tangent vector field X orthogonal to ξ and φX is called a φ-sectional curvature. If any Sasakian manifold M has a constant φ-sectional curvature c, then M is called a Sasakian space form and denoted by $M^{2n+1}(c)$. The Riemannian curvature tensor of Sasakian space form is given by the following formula:

$$R(X,Y) = X\wedge Y + \frac{c-1}{4}\big(\varphi^2X\wedge\varphi^2Y + \varphi X\wedge\varphi Y + 2g(X,\varphi Y)\varphi\big), \tag{8}$$

where

$$(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{9}$$

For more background on these manifolds, we recommend the references [6, 5, 18, 9].

3 Deformation of Sasakian structures

Let (φ,ξ,η,g) be an almost contact metric structure on M^{2n+1}.

For all X and Y vector fields on M, we mean a change of structure tensors of the form

$$\begin{cases} \tilde\varphi X = \varphi X + \theta(\varphi X)\xi, \\ \tilde\xi = \xi, \\ \tilde\eta = \eta - \theta, \\ \tilde g(X,Y) = fg(X,Y) - f\eta(X)\eta(Y) + \tilde\eta(X)\tilde\eta(Y), \end{cases} \tag{10}$$

where θ is a closed 1-form orthogonal to η on M.

Proposition 1. *$(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is an almost contact metric structure.*

Proof. The proof follows by a routine calculation, just using (1). □

Note that the simplest case for this deformation is for $\theta = \mathrm{d}f$ where $f \in \mathcal{C}^{\infty}(M)$ and $\xi(f) = 0$.

Proposition 2. *Let ∇ and $\tilde{\nabla}$ denote the Riemannian connections of g and $\tilde{g}$ respectively. If (φ, ξ, η, g) is a Sasakian structure then for all X, Y vector fields on M, we have the relation:*

$$\tilde{\nabla}_X Y = \nabla_X Y + \theta(X)\tilde{\varphi}Y + \theta(Y)\tilde{\varphi}X - \big(\nabla_X \theta\big)(Y)\xi. \tag{11}$$

Proof. Firstly, the metric $\tilde{g}$ may also written as

$$\tilde{g}(X, Y) = g(X, Y) - \eta(X)\eta(Y) + \tilde{\eta}(X)\tilde{\eta}(Y), \tag{12}$$

for all X and Y vector fields on M. Using Koszul's formula for the metric g,

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_X Y, Z) &= X\tilde{g}(Y, Z) + Y\tilde{g}(Z, X) - Z\tilde{g}(X, Y) \\ &- \tilde{g}(X, [Y, Z]) + \tilde{g}(Y, [Z, X]) + \tilde{g}(Z, [X, Y], \end{aligned}$$

one can obtain

$$\begin{aligned} \tilde{g}(\tilde{\nabla}_X Y, Z) &= \tilde{g}(\nabla_X Y, Z) \\ &- \frac{1}{2}\big((\nabla_X \eta)Y + (\nabla_Y \eta)X\big)\eta(Z) - \mathrm{d}\eta(X, Z)\eta(Y) - \mathrm{d}\eta(Y, Z)\eta(X) \\ &+ \frac{1}{2}\big((\nabla_X \tilde{\eta})Y + (\nabla_Y \tilde{\eta})X\big)\tilde{\eta}(Z) + \mathrm{d}\tilde{\eta}(X, Z)\tilde{\eta}(Y) + \mathrm{d}\tilde{\eta}(Y, Z)\tilde{\eta}(X). \end{aligned}$$

Knowing that $\mathrm{d}\theta = 0$ and $\tilde{\eta} = \eta - \theta$, we get

$$\tilde{g}(\tilde{\nabla}_X Y, Z) = \tilde{g}(\nabla_X Y, Z) - (\nabla_X \theta)(Y)\tilde{\eta}(Z) - \mathrm{d}\eta(X, Z)\theta(Y) - \mathrm{d}\eta(Y, Z)\theta(X).$$

Since (φ, ξ, η, g) is a Sasakian structure then we have

$$\begin{aligned} \mathrm{d}\eta(X, Z) &= \phi(X, Z) \\ &= -g(\varphi X, Z) \\ &= -\tilde{g}(\tilde{\varphi}X, Z), \end{aligned}$$

and therefore

$$\tilde{\nabla}_X Y = \nabla_X Y + \theta(X)\tilde{\varphi}Y + \theta(Y)\tilde{\varphi}X - \big(\nabla_X \theta\big)(Y)\xi.$$

□

Theorem 1. *If (φ, ξ, η, g) is a Sasakian structure then $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Sasakian too.*

Proof. Suppose that (φ, ξ, η, g) is a Sasakian structure. Knowing that

$$\big(\tilde{\nabla}_X \tilde{\varphi}\big)Y = \tilde{\nabla}_X \tilde{\varphi} Y - \tilde{\varphi} \tilde{\nabla}_X Y,$$

and using $\tilde{\varphi} X = \varphi X + \theta(\varphi X)\xi$, we get

$$\big(\tilde{\nabla}_X \tilde{\varphi}\big)Y = \tilde{\nabla}_X \varphi Y + X\big(\theta(\varphi Y)\big)\xi + \theta(\varphi Y)\tilde{\nabla}_X \xi - \varphi \tilde{\nabla}_X Y - \theta\big(\varphi \tilde{\nabla}_X Y\big)\xi,$$

using propostion 2, we obtain

$$\begin{aligned} \big(\tilde{\nabla}_X \tilde{\varphi}\big)Y &= (\nabla_X \varphi)Y + \theta\big((\nabla_X \varphi)Y\big)\xi + \theta(\varphi Y)\nabla_X \xi \\ &+ \theta(\varphi Y)\varphi X + \theta(Y)X - \eta(X)\theta(Y)\xi + \theta(X)\theta(Y)\xi. \end{aligned}$$

Since (φ, ξ, η, g) is a Sasakian structure, we can use formulas (2) and (3) and we get

$$\big(\tilde{\nabla}_X \tilde{\varphi}\big)Y = \tilde{g}(X, Y)\tilde{\xi} - \tilde{\eta}(Y)X,$$

which shows that $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Sasakian structure. □

Example 1. *For this example, we rely on the example of Blair [6]. We know that* $\mathbb{R}^{2n+1}$ *with coordinates* (x^i, y^i, z), $i = 1..n$, *admits the Sasakian structure*

$$g = \frac{1}{4}\begin{pmatrix} \delta_{ij} + y^i y^j & 0 & -y^i \\ 0 & \delta_{ij} & 0 \\ -y^j & 0 & 1 \end{pmatrix}, \qquad \varphi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix},$$

$$\xi = 2\left(\frac{\partial}{\partial z}\right), \qquad \eta = \frac{1}{2}(dz - y^i dx^i).$$

There is an infinite number of possibilities to choose θ *under the two conditions* $\mathrm{d}\theta = 0$ *and* $\theta(\xi) = 0$. *Let us take* $\theta = \frac{1}{2}dy^i$ *and using the formulas (10) we obtain*

$$\tilde{g} = \frac{1}{4}\begin{pmatrix} \delta_{ij} + y^i y^j & y^i & -y^i \\ y^j & 1 + \delta_{ij} & -1 \\ -y^j & -1 & 1 \end{pmatrix}, \qquad \tilde{\varphi} = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ -\delta_{ij} & y^j & 0 \end{pmatrix},$$

$$\tilde{\xi} = 2\left(\frac{\partial}{\partial z}\right), \qquad \tilde{\eta} = \frac{1}{2}(dz - y^i dx^i - dy^i).$$

So, one can check that $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ *is a Sasakian structure.*

4 Curvature formulas and main results

Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold. By a direct computation using the proposition (11), we get the following:

Proposition 3. *For all X, Y vector field on M, we have:*
Let $\tilde{R}$ (resp. R) denote the curvature tensors for $\tilde{g}$ (resp. g). Then, we have

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + \theta\big(R(X,Y)Z\big)\xi + \theta(Y)g(X,Z)\xi - \theta(X)g(Y,Z)\xi \\ &+ \eta(Z)\big(\theta(X)Y - \theta(Y)X\big) + \theta(Z)\big(\tilde{\eta}(X)Y - \tilde{\eta}(Y)X\big). \qquad (13)\end{aligned}$$

Proof. Suppose that, at $x_0 \in M$, $\nabla_{U_i} U_j = 0$, $\forall i, j \in \{1..., 2n+1\}$. We compute:

$$\begin{aligned}\tilde{\nabla}_X \tilde{\nabla}_Y Z &= \tilde{\nabla}_X\Big(\nabla_X Y + \theta(X)\tilde{\varphi}Y + \theta(Y)\tilde{\varphi}X - \big(\nabla_X\theta\big)(Y)\xi\Big) \\ &= \nabla_X\nabla_Y Z + X\big(\theta(Y)\big)\tilde{\varphi}Z + \theta(Y)\tilde{\nabla}_X\tilde{\varphi}Z + X\big(\theta(Z)\big)\tilde{\varphi}Y + \theta(Z)\tilde{\nabla}_X\tilde{\varphi}Y \\ &- g\big(\nabla_X\nabla_Y\psi, Z\big)\xi - g\big(\nabla_Y\psi, Z\big)\tilde{\nabla}_X\xi,\end{aligned}$$

where θ is the g-dual of ψ i.e. $\theta(X) = g(\psi, X)$. By using formulas (10), (11), (2) and (3), we get

$$\begin{aligned}\tilde{\nabla}_X\tilde{\nabla}_Y Z &= \nabla_X\nabla_Y Z + (\nabla_X\theta)(Y)\tilde{\varphi}Z + (\nabla_X\theta)(Z)\tilde{\varphi}Y \\ &+ \theta(Y)\big(g(X,Z)\xi - \eta(Z)X + \theta(X)\varphi^2 Z - \theta(X)\theta(Z)\xi - \theta(X)\eta(Z)\xi\big) \\ &+ \theta(Z)\big(g(X,Y)\xi - \eta(Y)X + \theta(X)\varphi^2 Y - \theta(X)\theta(Y)\xi - \theta(X)\eta(Y)\xi\big) \\ &- g(\nabla_X\nabla_Y\psi, Z)\xi + (\nabla_Y\theta)(Z)\tilde{\varphi}X.\end{aligned}$$

From the definition of curvature tensor $\tilde{R}$ of M (see (6)),

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z,$$

with at x_0, $[X,Y] = 0$ and $\mathrm{d}\theta = 0$, we get

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + \theta\big(R(X,Y)Z\big)\xi + \theta(Y)g(X,Z)\xi - \theta(X)g(Y,Z)\xi \\ &+ \eta(Z)\big(\theta(X)Y - \theta(Y)X\big) + \theta(Z)\big(\tilde{\eta}(X)Y - \tilde{\eta}(Y)X\big).\end{aligned}$$

□

Now, suppose that the sectional curvature of the Sasakian manifold $(M, \varphi, \xi, \eta, g)$ is a constant c, that is

$$R(X,Y)Z = c(X \wedge Y)Z.$$

Then we have

$$\begin{aligned}\tilde{R}(X,Y)Z &= c(X\tilde{\wedge}Y)Z + (c-1)\theta\big((X\wedge Y)Z\big)\xi \\ &+ (1-c)\Big(\eta(Z)\big(\theta(X)Y - \theta(Y)X\big) + \theta(Z)\big(\tilde{\eta}(X)Y - \tilde{\eta}(Y)X\big)\Big),\end{aligned}$$

which give the following:

Proposition 4. *If $(M, \varphi, \xi, \eta, g)$ is a Sasakian manifold of sectional curvature $c = 1$ then, $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Sasakian manifold of sectional curvature $c = 1$ too.*

Example 2. *Let S^{2n+1} be a $(2n+1)$-dimensional unit sphere equipped with the canonical Sasakian structure (φ, ξ, η, g) of constant sectional curvature 1. One can check that S^{2n+1} equipped with the structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is also a Sasakian manifold of constant sectional curvature equal 1.*

Proposition 5. *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian space form with constant φ-sectional curvature c and apply our deformation. Then, $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Sasakian space form with constant φ-sectional curvature c too.*

Proof. Let $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ be obtained by the above deformation of a Sasakian manifold $(M, \varphi, \xi, \eta, g)$. Using formulas (9) and (12), one can easily prove that:

(1): $(X \tilde{\wedge} Y)Z = (X \wedge Y)Z + \eta(Z)\big(\eta(X)Y - \eta(Y)X\big) - \tilde{\eta}(Z)\big(\tilde{\eta}(X)Y - \tilde{\eta}(Y)X\big),$

(2): $(\tilde{\varphi}X \tilde{\wedge} \tilde{\varphi}Y)Z = (\varphi X \wedge \varphi Y)Z + \theta\big((\varphi X \wedge \varphi Y)Z\big)\xi,$

(3): $(\tilde{\varphi}^2 X \tilde{\wedge} \tilde{\varphi}^2 Y)Z = (\varphi^2 X \wedge \varphi^2 Y)Z + \theta\big((\varphi^2 X \wedge \varphi^2 Y)Z\big)\xi,$

(4): $\tilde{g}(X, \tilde{\varphi}Y)\tilde{\varphi}Z = g(X, \varphi Y)\varphi Z + \theta(\varphi Z) g(X, \varphi Y)\xi.$

Now, suppose that $(M, \varphi, \xi, \eta, g)$ is a Sasakian space form with constant φ-sectional curvature c that is:

$$R(X,Y)Z = (X \wedge Y)Z + \frac{c-1}{4}\big((\varphi^2 X \wedge \varphi^2 Y)Z + (\varphi X \wedge \varphi Y)Z + 2g(X, \varphi Y)\varphi Z\big), \tag{14}$$

note that

$$\theta\big(\tilde{\varphi}X \tilde{\wedge} \tilde{\varphi}Y)Z\big) = \theta\big((\varphi X \wedge \varphi Y)Z\big) \quad and \quad \theta\big(\tilde{\varphi}^2 X \tilde{\wedge} \tilde{\varphi}^2 Y)Z\big) = \theta\big((\varphi^2 X \wedge \varphi^2 Y)Z\big).$$

So, using the four equations above in (14), we obtain:

$$\begin{aligned} R(X,Y) &= (X \tilde{\wedge} Y)Z + \frac{c-1}{4}\big((\tilde{\varphi}^2 X \tilde{\wedge} \tilde{\varphi}^2 Y)Z + (\tilde{\varphi}X \tilde{\wedge} \tilde{\varphi}Y)Z + 2\tilde{g}(X, \tilde{\varphi}Y)\tilde{\varphi}Z\big) \\ &- \frac{c-1}{4}\theta\big((\tilde{\varphi}^2 X \tilde{\wedge} \tilde{\varphi}^2 Y)Z + (\tilde{\varphi}X \tilde{\wedge} \tilde{\varphi}Y)Z + 2\tilde{g}(X, \tilde{\varphi}Y)\tilde{\varphi}Z\big)\xi \\ &- \eta(Z)\big(\eta(X)Y - \eta(Y)X\big) + \tilde{\eta}(Z)\big(\tilde{\eta}(X)Y - \tilde{\eta}(Y)X\big), \end{aligned} \tag{15}$$

substituting the equation (15) in formulas (13), we get

$$\tilde{R}(X,Y)Z = (X \tilde{\wedge} Y)Z + \frac{c-1}{4}\big((\tilde{\varphi}^2 X \tilde{\wedge} \tilde{\varphi}^2 Y)Z + (\tilde{\varphi}X \tilde{\wedge} \tilde{\varphi}Y)Z + 2\tilde{g}(X, \tilde{\varphi}Y)\tilde{\varphi}Z\big),$$

which completes the proof. □

Lemma 1. *We choose a g-orthonormal basis $\{e_i\}_{0 \le i \le 2n}$ of the tangent space T_xM at each point $x \in M$ where $e_0 = \xi$, then*

$$\{\tilde{e}_i = e_i + \theta(e_i)\xi\}_{0 \le i \le 2n} \tag{16}$$

is a $\tilde{g}$-orthonormal basis of T_xM.

Proof. Firstly, we have

$$\tilde{g}(\tilde{e}_0, \tilde{e}_0) = \tilde{g}(\xi, \xi) = 1,$$

and for any $i, j \in \{1, ..., 2n\}$, $\tilde{g}(e_i, \xi) = -\theta(e_i)$, then

$$\begin{aligned} \tilde{g}(\tilde{e}_i, \tilde{e}_j) &= \tilde{g}\big(e_i + \theta(e_i)\xi, e_j + \theta(e_j)\xi\big) \\ &= \tilde{g}(e_i, e_j) + \theta(e_j)\tilde{g}(e_i, \xi) + \theta(e_i)\tilde{g}(\xi, e_j) + \theta(e_i)\theta(e_j)\tilde{g}(\xi, \xi) \\ &= \delta_{ij}. \end{aligned}$$

We compute

$$\begin{aligned} \tilde{g}(\tilde{e}_i, \xi) &= \tilde{g}(e_i + \theta(e_i)\xi, \xi) \\ &= \tilde{g}(e_i, \xi) + \theta(e_i)\tilde{g}(\xi, \xi) \\ &= -\theta(e_i) + \theta(e_i) = 0. \end{aligned}$$

□

Proposition 6. *Let $\tilde{S}$ (resp. S) denote the Ricci curvatures for $\tilde{g}$ (resp. g). Then, we have*

$$\tilde{S}(X, Y) = S(X, Y) - 2n\eta(X)\eta(Y) + 2n\tilde{\eta}(X)\tilde{\eta}(Y). \tag{17}$$

Proof. Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold and $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ the Sasakian manifold obtained by the deformation (10). Using (16) then (13), we get

$$\begin{aligned} \tilde{S}(X, Y) &= \sum_{i=0}^{2n} \tilde{g}\big(\tilde{R}(X, \tilde{e}_i)\tilde{e}_i, Y\big) \\ &= \sum_{i=0}^{2n} \tilde{g}\Big(\tilde{R}(X, e_i)e_i + \tilde{R}(X, \psi)\xi + \tilde{R}(X, \xi)\psi + \theta(\psi)\tilde{R}(X, \xi)\xi, Y\Big) \\ &= \sum_{i=0}^{2n} \tilde{g}\big(R(X, e_i)e_i, Y\big) + S(X, \psi)\tilde{\eta}(Y) + (1 - 2n)\theta(X)\tilde{\eta}(Y) \\ &+ \tilde{g}\big(R(X, \xi)\psi, Y\big). \end{aligned} \tag{18}$$

The first term of (18), is given by

$$\sum_{i=0}^{2n} \tilde{g}\big(R(X, e_i)e_i, Y\big) = S(X, Y) - S(X, \psi)\tilde{\eta}(Y) - 2n\eta(X)\theta(Y). \tag{19}$$

For the last term in (18), using (4) with the formula

$$R(X, Y, Z, W) = R(Z, W, X, Y) = -R(Z, W, Y, X),$$

for all X, Y, Z and W vector fields on M, we obtain

$$\begin{aligned} \tilde{g}\big(\tilde{R}(X, \xi)\psi, Y\big) &= R(X, \xi, \psi, Y) - \theta(Y)R(X, \xi, \psi, \xi) \\ &= -g\big(R(\psi, Y)\xi, X\big) + \theta(Y)g\big(R(X, \xi)\xi, \psi\big) \\ &= -\theta(X)\tilde{\eta}(Y). \end{aligned} \tag{20}$$

Substituting the formulas (19) and (20) in (18), we obtain:

$$\tilde{S}(X,Y) = S(X,Y) - 2n\eta(X)\eta(Y) + 2n\tilde{\eta}(X)\tilde{\eta}(Y).$$

□

Corollary 1. *Let $\tilde{r}$ (resp. r) denote the scalar curvatures for $\tilde{g}$ (resp. g). Then, we have*

$$\tilde{r} = r. \tag{21}$$

Proof. Contracting (17) with respect to X and Y, we get

$$\begin{aligned} \tilde{r} &= \sum_{i=0}^{2n} \tilde{S}(\tilde{e}_i, \tilde{e}_i) \\ &= r + 2n\theta(\psi) + 2\tilde{S}(\psi,\xi) + \theta(\psi)\tilde{S}(\xi,\xi), \end{aligned}$$

using the formula $\tilde{S}(X,\xi) = 2n\tilde{\eta}(X)$, we obtain

$$\tilde{r} = r.$$

□

Corollary 2. *$(M^{2n+1}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Sasaki-Einstein manifold if and only if $(M, \varphi, \xi, \eta, g)$ is a Sasaki-Einstein manifold too.*

Proof. Just use the formula

$$\tilde{S}(X,Y) - 2n\tilde{g}(X,Y) = S(X,Y) - 2ng(X,Y),$$

obtained by formulas (12) and (17). □

Corollary 3. *$(M^{2n+1}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is an η-Einstein manifold if and only if $(M, \varphi, \xi, \eta, g)$ is an η-Einstein manifold too.*

Proof. Suppose that $(M^{2n+1}, \varphi, \xi, \eta, g)$ is an η-Einstein manifold, i.e. for all X and Y vector fields on M, we have

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y).$$

So, using formula (17) with $a + b = 2n$, we get

$$\begin{aligned} \tilde{S}(X,Y) &= ag(X,Y) + (b-2n)\eta(X)\eta(Y) + 2n\tilde{\eta}(X)\tilde{\eta}(Y) \\ &= ag(X,Y) + a\eta(X)\eta(Y) + 2n\tilde{\eta}(X)\tilde{\eta}(Y) \\ &= a\tilde{g}(X,Y) + (2n-a)\tilde{\eta}(X)\tilde{\eta}(Y) \\ &= a\tilde{g}(X,Y) + b\tilde{\eta}(X)\tilde{\eta}(Y). \end{aligned}$$

With the same reasoning we show the opposite. □

5 Ricci solitons in deformed Sasakian manifold

One of the important topics in the study of almost contact metric manifolds is the study of Ricci flow and Ricci solitons. Ricci solitons introduced by Hamilton [10] are natural generalizations of an Einstein metric. A complete Riemannian metric g on a smooth manifold M is a Ricci soliton if there is a vector field V and a constant λ such that

$$(\mathcal{L}_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{22}$$

where $\mathcal{L}_V g$ denotes the Lie derivative of g along a vector field V, λ a constant, and arbitrary vector fields X, Y on M. Sharma [15] initiated the study of Ricci solitons in K- contact geometry. Also in [11] He and Zhu established these results for the Sasakian case. Recently, A. Ghosha, R. Sharma [12] generalized these results and also proved the existence of shrinking Ricci soliton on a Sasakian manifold, which is not Einstein. Among the equations that they proved, we mention the equation that we need here (see Theorem 1, [12])

$$\mathcal{L}_V \xi = 4(n+1)\xi \qquad and \qquad \lambda = 2n+4. \tag{23}$$

Theorem 2. *Let (g, V, λ) be a Ricci soliton on the Sasakian manifold $(M, \varphi, \xi, \eta, g)$. If*

$$X\big(\theta(V)\big) = -4(n+1)\theta(X), \tag{24}$$

for all vector field X on M then $(\tilde{g}, V, \lambda)$ is a Ricci soliton on the deformed Sasakian manifold $(M^{2n+1}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$.

Proof. Using (22) and (17), we can write

$$\begin{aligned}(\tilde{\mathcal{L}}_V \tilde{g})(X,Y) + 2\tilde{S}(X,Y) + 2\lambda\tilde{g}(X,Y) &= \tilde{g}\big(\tilde{\nabla}_X V, Y\big) + \tilde{g}\big(\tilde{\nabla}_Y V, X\big) + 2S(X,Y) \\ &+ 2\lambda g(X,Y) - 2(2n+\lambda)\big(\eta(X)\eta(Y) \\ &- \tilde{\eta(X)}\tilde{\eta}(Y)\big). \end{aligned} \tag{25}$$

Making use of (10)-(12) in (25), we obtain

$$\begin{aligned}(\tilde{\mathcal{L}}_V \tilde{g})(X,Y) + 2\tilde{S}(X,Y) + 2\lambda\tilde{g}(X,Y) &= (\mathcal{L}_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) \\ &+ \theta(X)\Big(g(\varphi V, Y) - \eta\big(\nabla_Y V\big)\Big) \\ &+ \theta(Y)\Big(g(\varphi V, X) - \eta\big(\nabla_X V\big)\Big) \\ &- X\big(\theta(V)\big)\tilde{\eta}(Y) - Y\big(\theta(V)\big)\tilde{\eta}(X) \\ &- 2(2n+\lambda)\big(\eta(X)\eta(Y) - \tilde{\eta}(X)\tilde{\eta}(Y)\big). \end{aligned} \tag{26}$$

Since g is a Ricci soliton, i.e.

$$(\mathcal{L}_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{27}$$

then we have

$$(\mathcal{L}_V g)(X,\xi) + 2(2n+\lambda)\eta(X) = 0,$$

which implies

$$\eta\big(\nabla_X V\big) + g(\nabla_\xi V, X) + 2(2n+\lambda)\eta(X).$$

From equation (23), we get

$$\nabla_\xi V = -\varphi V - 4(n+1)\xi,$$

which gives

$$\eta\big(\nabla_X V\big) = g(\varphi V, X) - 4(n+1)\eta(X). \tag{28}$$

Substituting (27) and (28) in (26) , we obtain

$$\begin{aligned}(\tilde{\mathcal{L}}_V\tilde{g})(X,Y) + 2\tilde{S}(X,Y) + 2\lambda\tilde{g}(X,Y) &= -\tilde{\eta}(X)\Big(Y\big(\theta(V)\big) + 4(n+1)\theta(Y)\Big)\\ &- \tilde{\eta}(Y)\Big(X\big(\theta(V)\big) + 4(n+1)\theta(X)\Big).\end{aligned} \tag{29}$$

Which completes the proof. □

Example 3. *We'll employ the example above for dimension three ($n = 1$). The Sasakian manifold*

$$g = \frac{1}{4}\begin{pmatrix} 1+y^2 & 0 & -y \\ 0 & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}, \qquad \varphi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y & 0 \end{pmatrix},$$

$$\xi = 2\left(\frac{\partial}{\partial z}\right), \qquad \eta = \frac{1}{2}(dz - ydx)$$

admits a Ricci soliton (g, V, λ) such that $V = -4\big(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}\big)$ and $\lambda = 6$ (see [12]).

For the 1-form θ we have

$$\begin{cases} \theta = \rho dx + \tau dy + \gamma dz \\ \mathrm{d}\theta = 0 \\ \theta(\xi) = 0 \end{cases} \Rightarrow \begin{cases} \theta = \rho dx + \tau dy \\ \rho_1 = \tau_2 \\ \rho_3 = \tau_3 = 0 \end{cases}$$

where ρ, τ and γ are three functions on M and $\rho_i = \frac{\partial \rho}{\partial x_i}$ and $\tau_i = \frac{\partial \tau}{\partial x_i}$. Using condition (24), we get the following system of ODEs

$$\begin{cases} x\rho_1 + y\tau_1 = \rho \\ x\rho_2 + y\tau_2 = \tau \\ \rho_1 = \tau_2 \end{cases}$$

which gives

$$\theta = (ax + by)dx + (ay + bx)dy,$$

where a and b are real numbers. Now, using the formulas (10) we obtain the Sasakian manifold

$$\tilde{g} = \frac{1}{4}\begin{pmatrix} 1+y^2 & y & -y \\ y & 2 & -1 \\ -y & -1 & 1 \end{pmatrix}, \qquad \tilde{\varphi} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & y & 0 \end{pmatrix},$$

$$\tilde{\xi} = 2\left(\frac{\partial}{\partial z}\right), \qquad \tilde{\eta} = -\frac{1}{2}\Big(\big(2ax + (2b+1)y\big)dx + 2(ay + bx)dy - dz\Big),$$

which admits the Ricci soliton $(\tilde{g}, V, 6)$.

Open question:
The above theorem keeps the same potential vector field V in both initial and deformed structures. The question that arises is the following one: Can we have another potential vector field in the deformed structure?

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