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APPROXIMATION BY SZÁSZ-MIRAKJAN-BASKAKOV OPERATORS BASED ON SHAPE PARAMETER λ

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Abstract

In this paper, we aim to obtain several approximation properties of Szász-Mirakjan-Baskakov operators with shape parameter $\lambda \in [-1,1]$. We reach some preliminary results such as moments and central moments. Next, we estimate the order of convergence with respect to the usual modulus of continuity, for the functions belong to Lipschitz-type class and Peetre's K-functional, respectively. Also, we prove a result concerning the weighted approximation for these operators. Finally, we give the comparison of the convergence of these newly defined operators to certain functions with some graphics.

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1 Introduction

Szász [31] and Mirakjan [19] proposed the following linear positive operators, which are related to the Poisson distribution, as

$$S_m(\mu;y) = \sum_{j=0}^{\infty} s_{m,j}(y)\mu\left(\frac{j}{m}\right), \tag{1}$$

where $y \geq 0$, $m \in \mathbb{N}$, $\mu \in C[0,\infty)$ and Szász-Mirakjan basis functions $s_{m,j}(y)$ are defined as below:

$$s_{m,j}(y) = e^{-my}\frac{(my)^j}{j!}. \tag{2}$$

In 1983, Prasad et al. [26] considered Baskakov type integral modifications of (1) operators as follows:

$$K_m(\mu;y) = (m-1)\sum_{j=0}^{\infty} s_{m,j}(y)\int_0^{\infty} q_{m,j}(t)\mu(t)\,dt,\ \ y \in [0,\infty), \tag{3}$$

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where $s_{m,j}(y)$ given in (2) and $q_{m,j}(t)=\binom{m+j-1}{j}\frac{t^j}{(1+t)^{m+j}}$.

Later, some various approximation properties of (1) and (3) operators have been presented by several authors. We refer the readers to some recent papers on these directions [2, 16, 15, 1, 14, 13, 5, 32, 12, 33].

A short while ago, using the shape parameter $\lambda \in [-1,1]$, which plays an important role in computer graphics and computer-aided geometric design, become a new research field in the theory of approximation. In 2019, Qi et al. [27] introduced a new generalization of $\lambda-$Szász-Mirakjan polynomials with shape parameter $\lambda \in [-1,1]$ as below:

$$S_{m,\lambda}(\mu;y)=\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\mu\left(\frac{j}{m}\right), \tag{4}$$

where Szász-Mirakjan bases functions $\widetilde{s}_{m,j}(\lambda;y)$ with shape parameter $\lambda \in [-1,1]$:

$$\begin{aligned}\widetilde{s}_{m,0}(\lambda;y)&=s_{m,0}(y)-\frac{\lambda}{m+1}s_{m+1,1}(y);\\ \widetilde{s}_{m,i}(\lambda;y)&=s_{m,i}(y)+\lambda\left(\frac{m-2i+1}{m^2-1}s_{m+1,i}(y)-\frac{m-2i-1}{m^2-1}s_{m+1,i+1}(y)\right)\\ &\qquad (i=1,2,...,\infty,\ y\in[0,\infty)).\end{aligned} \tag{5}$$

They obtained several theorems such as Korovkin type approximation, local approximation, Lipschitz type convergence, Voronovskaja and Grüss-Voronovskaja type for the operators (4). We can mention some recent works based on shape parameter $\lambda \in [-1,1]$, see: [8, 9, 6, 7, 29, 30, 20, 21, 3, 28, 18, 22, 23, 24, 25].

Motivated by all of the above mentioned papers, we define $\lambda-$Szász-Mirakjan-Baskakov polynomials as follows:

$$R_{m,\lambda}(\mu;y)=(m-1)\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_0^{\infty}q_{m,j}(t)\mu(t)\,dt, \quad y\in[0,\infty), \tag{6}$$

where $q_{m,j}(t)$ given in (3), $\widetilde{s}_{m,j}(\lambda;y)$ $(j=0,1,..\infty)$ in (5) and $\lambda \in [-1,1]$.

The structure of this work is organized as follows: In Sect. 2, we calculate some moments and central moments. In Sect. 3, we establish the order of convergence with respect to the usual modulus of continuity, for the functions belonging to Lipschitz class and Peetre's K-functional, respectively. In Sect. 4, we give a result concerning the weighted approximation. In the final Section, we show the comparison of the convergence of operators (6) to the certain functions for the different values of m and λ. We also compare the convergence of operators (3) and (6) to the certain function to see the role of λ parameter.

2 Preliminaries

Lemma 1. *[27]. For the $\lambda-$Szász-Mirakjan operators $S_{m,\lambda}(\mu;y)$, the following results are satisfied:*

$$S_{m,\lambda}(1;y) = 1,$$
$$S_{m,\lambda}(t;y) = y + \left[\frac{1-e^{-(m+1)y}-2y}{m(m-1)}\right]\lambda,$$
$$S_{m,\lambda}(t^2;y) = y^2 + \frac{y}{m} + \left[\frac{2y+e^{-(m+1)y}-1-4(m+1)y^2}{m^2(m-1)}\right]\lambda,$$
$$S_{m,\lambda}(t^3;y) = y^3 + \frac{3y^2}{m} + \frac{y}{m^2} + \left[\frac{1-e^{-(m+1)y}-2y+3(m-3)(m+1)y^2-6(m+1)y^3}{m^3(m-1)}\right]\lambda,$$
$$S_{m,\lambda}(t^4;y) = y^4 + \frac{6y^3}{m} + \frac{7y^2}{m^2} + \frac{y}{m^3} + \left[\frac{e^{-(m+1)y}-1+2my+2(3m-11)(m+1)y^2}{m^4(m-1)} + \frac{4(m-8)(m+1)^2y^3-8(m+1)^3y^4}{m^4(m-1)}\right]\lambda.$$

Lemma 2. *Let the operators $R_{m,\lambda}$ be defined by (6). Then, we obtain*

$$R_{m,\lambda}(1;y) = 1, \tag{7}$$
$$R_{m,\lambda}(t;y) = \frac{my+1}{m-2} + \left[\frac{1-e^{-(m+1)y}-2y}{(m-1)(m-2)}\right]\lambda, \tag{8}$$
$$R_{m,\lambda}(t^2;y) = \frac{m^2y^2+4my+2}{(m-2)(m-3)} + \left[\frac{1-e^{-(m+1)y}-2y-2(m+1)y^2}{(m-1)(m-2)(m-3)}\right]2\lambda, \tag{9}$$
$$R_{m,\lambda}(t^3;y) = \frac{m^3y^3+9m^2y^2+18my+6}{(m-2)(m-3)(m-4)} + \left[\frac{2-2e^{-(m+1)y}-4y+(m-11)(m+1)y^2-2(m+1)y^3}{(m-1)(m-2)(m-3)(m-4)}\right]3\lambda, \tag{10}$$
$$R_{m,\lambda}(t^4;y) = \frac{m^4y^4+16m^3y^3+72m^2y^2+96my+24}{(m-2)(m-3)(m-4)(m-5)} + \left[\frac{12-12e^{-(m+1)y}+2y(m-25)+18(m-7)(m+1)y^2}{(m-1)(m-2)(m-3)(m-4)(m-5)} - \frac{2(m^2-7m-23)(m+1)y^3+4(m+1)^3y^4}{(m-1)(m-2)(m-3)(m-4)(m-5)}\right]2\lambda. \tag{11}$$

Proof. By the definition of (6) and $\widetilde{s}_{m,i}(\lambda;y)$ (5), it is easy to see $\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y) = 1$, hence we get (7).

Now, with the help of Lemma 1 and definition of beta function, we will compute expressions (8) and (9).

$$\begin{aligned}
R_{m,\lambda}(t;y) &= (m-1)\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_{0}^{\infty}\binom{m+j-1}{j}\frac{t^{j+1}}{(1+t)^{m+j}}dt \\
&= (m-1)\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\binom{m+j-1}{j}\frac{\Gamma(j+2)\Gamma(m-2)}{\Gamma(m+j)} \\
&= (m-1)\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{(m+j-1)!}{j!(m-1)!}\frac{(j+1)!(m-3)!}{(m+j-1)!} \\
&= \sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\left(\frac{j+1}{m-2}\right) \\
&= \frac{m}{m-2}S_{m,\lambda}(t;y)+\frac{1}{m-2}S_{m,\lambda}(1;y) \\
&= \frac{my+1}{m-2}+\left[\frac{1-e^{-(m+1)y}-2y}{(m-1)(m-2)}\right]\lambda.
\end{aligned}$$

$$\begin{aligned}
R_{m,\lambda}(t^2;y) &= (m-1)\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_{0}^{\infty}\binom{m+j-1}{j}\frac{t^{j+2}}{(1+t)^{m+j}}dt \\
&= (m-1)\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\binom{m+j-1}{j}\frac{\Gamma(j+3)\Gamma(m-3)}{\Gamma(m+j)} \\
&= (m-1)\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{(m+j-1)!}{j!(m-1)!}\frac{(j+2)!(m-4)!}{(m+j-1)!} \\
&= \sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{j^2+3j+2}{(m-2)(m-3)} \\
&= \frac{m^2}{(m-2)(m-3)}S_{m,\lambda}(t^2;y)+\frac{3m}{(m-2)(m-3)}S_{m,\lambda}(t;y) \\
&+ \frac{2}{(m-2)(m-3)}S_{m,\lambda}(1;y) \\
&= \frac{m^2y^2+4my+2}{(m-2)(m-3)}+\left[\frac{1-e^{-(m+1)y}-2y-2(m+1)y^2}{(m-1)(m-2)(m-3)}\right]2\lambda.
\end{aligned}$$

Similarly, from Lemma 1, we can get expressions (10) and (11) by simple computation, thus we have omitted details. □

Corollary 1. *Let $y \in [0,\infty)$, $m > 5$ and $\lambda \in [-1,1]$. As a consequence of Lemma 2, we obtain the following relations:*

$$(i)\ R_{m,\lambda}(t-y;y) = \frac{2y+1}{m-2} + \left[\frac{1-e^{-(m+1)y}-2y}{(m-1)(m-2)}\right]\lambda := \alpha_{m,\lambda}(y)$$

$$\begin{aligned}(ii)\ R_{m,\lambda}((t-y)^2;y) &= \frac{(m+6)y^2+2(m+3)y+2}{(m-2)(m-3)} \\ &+ \left[\frac{1+[(m-3)y-1]\,e^{-(m+1)y}+y-8y^2}{(m-1)(m-2)(m-3)}\right]2\lambda \\ &\leq \frac{(m+6)y^2+2(m+3)y+2}{(m-2)(m-3)} \\ &+ \frac{2\left(1+[(m-3)y-1]\,e^{-(m+1)y}+y+8y^2\right)}{(m-1)(m-2)(m-3)} := \beta_m(y)\end{aligned}$$

$$\begin{aligned}(iii)\ R_{m,\lambda}((t-y)^4;y) &= \frac{\left(3m^2+286m+120\right)y^4+4\left(3m^2+73m+60\right)y^3}{(m-2)(m-3)(m-4)(m-5)} \\ &+ \frac{12\left(m^2+21m+20\right)y^2+24(3m+5)y+24}{(m-2)(m-3)(m-4)(m-5)} \\ &+ \left(\frac{12-12e^{-(m+1)y}+2y(m-25)+18(m-7)(m+1)y^2}{(m-1)(m-2)(m-3)(m-4)(m-5)}\right. \\ &\frac{-2(m^2-7m-23)(m+1)y^3-4(m+1)^3y^4}{(m-1)(m-2)(m-3)(m-4)(m-5)} \\ &+\frac{6y\left(2e^{-(m+1)y}-2+4y-(m-11)(m+1)y^2+2(m+1)y^3\right)}{(m-1)(m-2)(m-3)(m-4)} \\ &+\frac{12y^2\left(1-e^{-(m+1)y}-2y^3+2(m+1)y^2\right)}{(m-1)(m-2)(m-3)} \\ &\left.-\frac{4y^3(1-e^{-(m+1)y}-2y)}{(m-1)(m-2)}\right)2\lambda.\end{aligned}$$

3 Direct theorems of $R_{m,\lambda}$ operators

In this section, we establish the order of convergence in connection with the usual modulus of continuity, for the function belonging to Lipschitz type continuous and Peetre's K-functional. Let the space $C_B[0,\infty)$ denotes all continuous and bounded functions on $[0,\infty)$ and be equipped with the sup-norm for a function μ as follows:

$$\|\mu\|_{[0,\infty)} = \sup_{y\in[0,\infty)} |\mu(y)|.$$

Further, the Peetre's K-functional is defined by

$$K_2(\mu,\eta) = \inf_{\nu \in C^2[0,\infty)} \left\{ \|\mu - \nu\| + \eta \left\|\nu''\right\| \right\},$$

where $\eta > 0$ and $C_B^2[0,\infty) = \{\nu \in C_B[0,\infty) : \nu', \nu'' \in C_B[0,\infty)\}$.

Taking into account [10], there exists an absolute constant $C > 0$ such that

$$K_2(\mu;\eta) \leq C\omega_2(\mu;\sqrt{\eta}), \qquad \eta > 0, \tag{12}$$

where

$$\omega_2(\mu;\eta) = \sup_{0<z\leq\eta} \sup_{y\in[0,\infty)} \left|\mu(y+2z) - 2\mu(y+z) + \mu(y)\right|,$$

is the second order modulus of smoothness of the function $\mu \in C_B[0,\infty)$. Also, by

$$\omega(\mu;\eta) := \sup_{0<\alpha\leq\eta} \sup_{y\in[0,\infty)} \left|\mu(y+z) - \mu(y)\right|,$$

we denote the usual modulus of continuity of $\mu \in C_B[0,\infty)$. Since $\eta > 0$, $\omega(\mu;\eta)$ has some useful properties see details: [4].

Moreover, we give an element of Lipschitz continuous function with $Lip_L(\zeta)$, where $L > 0$ and $0 < \zeta \leq 1$. If the expression below:

$$|\mu(t) - \mu(y)| \leq L\left|t-y\right|^{\zeta}, \qquad (t, y \in \mathbb{R}), \tag{13}$$

holds, then one can say that the function μ belongs to $Lip_L(\zeta)$.

Theorem 1. *Let $\mu \in C_B[0,\infty)$, $y \in [0,\infty)$ and $\lambda \in [-1,1]$. Then, the following inequality verifies*

$$|R_{m,\lambda}(\mu;y) - \mu(y)| \leq 2\omega(\mu;\sqrt{\beta_m(y)}),$$

where $\beta_m(y)$ given in Corollary 1.

Proof. Taking into account the well-known property of modulus of continuity $|\mu(t) - \mu(y)| \leq \left(1 + \frac{|t-y|}{\delta}\right)\omega(\mu;\delta)$ and operating $R_{m,\lambda}(.;y)$, it gives

$$|R_{m,\lambda}(\mu;y) - \mu(y)| \leq \left(1 + \frac{1}{\delta}R_{m,\lambda}(\left|t-y\right|;y)\right)\omega(\mu;\delta).$$

Utilizing the Cauchy-Bunyakovsky-Schwarz inequality and by Corollary 1, we obtain

$$\begin{aligned}|R_{m,\lambda}(\mu;y) - \mu(y)| &\leq \left(1 + \frac{1}{\delta}\sqrt{R_{m,\lambda}((t-y)^2;y)}\right)\omega(\mu;\delta)\\ &\leq \left(1 + \frac{1}{\delta}\sqrt{\beta_m(y)}\right)\omega(\mu;\delta).\end{aligned}$$

Taking $\delta = \sqrt{\beta_m(y)}$, hence we get the proof of this theorem. □

Theorem 2. *Let $\mu \in Lip_L(\zeta)$, $y \in [0,\infty)$ and $\lambda \in [-1,1]$. Then, we obtain*

$$|R_{m,\lambda}(\mu;y)-\mu(y)| \leq L(\beta_m(y))^{\frac{\zeta}{2}}.$$

Proof. From (13) and using the Hölder's inequality, it follows

$$|R_{m,\lambda}(\mu;y)-\mu(y)| \leq R_{m,\lambda}(|\mu(t)-\mu(y)|\,;y) \leq LR_{m,\lambda}(|t-y|^{\zeta}\,;y).$$

Therefore,

$$\begin{aligned}
|R_{m,\lambda}(\mu;y)-\mu(y)| &\leq L(m-1)\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_0^{\infty}\binom{m+j-1}{j}\frac{t^j}{(1+t)^{m+j}}\,|t-y|^{\zeta}\,dt \\
&\leq L(m-1)\sum_{j=0}^{\infty}\left(\widetilde{s}_{m,j}(\lambda;y)\right)^{\frac{2-\zeta}{2}} \\
&\times\left(\widetilde{s}_{m,j}(\lambda;y)\right)^{\frac{\zeta}{2}}\int_0^{\infty}\binom{m+j-1}{j}\frac{t^j}{(1+t)^{m+j}}\,|t-y|^{\zeta}\,dt \\
&\leq L\left((m-1)\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_0^{\infty}\binom{m+j-1}{j}\frac{t^j}{(1+t)^{m+j}}dt\right)^{\frac{2-\zeta}{2}} \\
&\times\left((m-1)\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_0^{\infty}\binom{m+j-1}{j}\frac{t^j}{(1+t)^{m+j}}\,|t-y|^2\,dt\right)^{\frac{\zeta}{2}} \\
&= L\left\{R_{m,\lambda}((t-y)^2;y)\right\}^{\frac{\zeta}{2}} \\
&\leq L(\beta_m(y))^{\frac{\zeta}{2}}.
\end{aligned}$$

Thus, we get the proof of this theorem. □

Theorem 3. *For all $\mu \in C_B[0,\infty)$, $y \in [0,\infty)$ and $\lambda \in [-1,1]$, the following inequality holds:*

$$|R_{m,\lambda}(\mu;y)-\mu(y)| \leq C\omega_2(\mu;\frac{1}{2}\sqrt{\beta_m(y)+(\alpha_{m,\lambda}(y))^2}+\omega(\mu;|\alpha_{m,\lambda}(y)|),$$

where $C>0$ is a constant, $\alpha_{m,\lambda}(y)$ and $\beta_m(y)$ are same as in Corollary 1.

Proof. We denote $\gamma_{m,\lambda}(y) := \frac{my+1}{m-2}+\left[\frac{1-2y-e^{-(m+1)y}}{(m-1)(m-2)}\right]\lambda$, it is obvious that $\gamma_{m,\lambda}(y) \in [0,\infty)$ for sufficently large m. We define the following auxiliary operators:

$$\widehat{R}_{m,\lambda}(\mu;y) = R_{m,\lambda}(\mu;y)-\mu(\gamma_{m,\lambda}(y))+\mu(y). \tag{14}$$

In view of (7) and (8), we find

$$\widehat{R}_{m,\lambda}(t-y;y) = 0.$$

Using Taylor's formula, one has

$$\xi(t)=\xi(y)+(t-y)\xi'(y)+\int_y^t (t-u)\xi''(u)du, \quad (\xi\in C_B^2[0,\infty)). \tag{15}$$

After operating $\widehat{R}_{m,\lambda}(.;y)$ to (15), yields

$$\begin{aligned}\widehat{R}_{m,\lambda}(\xi;y)-\xi(y) &= \widehat{R}_{m,\lambda}((t-y)\xi'(y);y)+\widehat{R}_{m,\lambda}(\int_y^t (t-u)\xi''(u)du;y)\\ &= \xi'(y)\widehat{R}_{m,\lambda}(t-y;y)+R_{m,\lambda}(\int_y^t (t-u)\xi''(u)du;y)\\ &- \int_y^{\gamma_{m,\lambda}(y)} (\gamma_{m,\lambda}(y)-u)\xi''(u)du\\ &= R_{m,\lambda}(\int_y^t (t-u)\xi''(u)du;y)-\int_y^{\gamma_{m,\lambda}(y)} (\gamma_{m,\lambda}(y)-u)\xi''(u)du.\end{aligned}$$

Taking Lemma 2 and (14) into the account, we get

$$\begin{aligned}\left|\widehat{R}_{m,\lambda}(\xi;y)-\xi(y)\right| &\le \left|R_{m,\lambda}(\int_y^t (t-u)\xi''(u)du;y)\right|+\left|\int_y^{\gamma_{m,\lambda}(y)} (\gamma_{m,\lambda}(y)-u)\xi''(u)du\right|\\ &\le R_{m,\lambda}\left(\int_y^t (t-u)\left|\xi''(u)\right|du;y\right)\\ &+\left(\int_y^{\gamma_{m,\lambda}(y)} \left|\gamma_{m,\lambda}(y)-u\right|\left|\xi''(u)\right|du\right)\\ &\le \left\|\xi''\right\|\left\{R_{m,\lambda}((t-y)^2;y)+(\gamma_{m,\lambda}(y)-y)^2\right\}\\ &\le \left\{\beta_m(y)+(\alpha_{m,\lambda}(y))^2\right\}\left\|\xi''\right\|.\end{aligned}$$

From (7), (8) and (14), it deduce the following

$$\left|\widehat{R}_{m,\lambda}(\mu;y)\right|\le |R_{m,\lambda}(\mu;y)|+2\|\mu\|\le \|\mu\|R_{m,\lambda}(1;y)+2\|\mu\|\le 3\|\mu\|. \tag{16}$$

Also, by (15) and (16) we get

$$\begin{aligned}|R_{m,\lambda}(\mu;y)-\mu(y)| &\le \left|\widehat{R}_{m,\lambda}(\mu-\xi;y)-(\mu-\xi)(y)\right|\\ &+\left|\widehat{R}_{m,\lambda}(\xi;y)-\xi(y)\right|+|\mu(y)-\mu(\alpha_{m,\lambda}(y))|\\ &\le 4\|\mu-\xi\|+\left\{\beta_m(y)+(\gamma_{m,\lambda}(y))^2\right\}\left\|\xi''\right\|+\omega\left(\mu;|\alpha_{m,\lambda}(y)|\right).\end{aligned}$$

On account of this, if we take the infimum on the right hand side over all $\xi \in C_B^2[0,\infty)$ and by (12), we arrive

$$\begin{aligned}|R_{m,\lambda}(\mu;y)-\mu(y)| &\leq 4K_2(\mu;\frac{\{\beta_m(y)+(\alpha_{m,\lambda}(y))^2\}}{4})+\omega(\mu;|\alpha_{m,\lambda}(y)|)\\ &\leq C\omega_2(\mu;\frac{1}{2}\sqrt{\beta_m(y)+(\alpha_{m,\lambda}(y))^2})+\omega(\mu;|\alpha_{m,\lambda}(y)|).\end{aligned}$$

Hence, we obtain the proof of this theorem. □

4 Weighted approximation

In this section, we prove a result concerning the weighted approximation for the sequence of operators $(R_{m,\lambda})_n$. Let $B_{y^2}[0,\infty)$ be the set of all functions h verifying the condition $|h(y)| \leq M_h(1+y^2)$, $y \in [0,\infty)$ with constant M_h, which depend only on h. We denote by $C_{y^2}[0,\infty)$ the set of all continuous functions belonging to $B_{y^2}[0,\infty)$ endowed with the norm $\|h\|_{y^2} = \sup\limits_{y\in[0,\infty)} \frac{|h(y)|}{1+y^2}$ and $C^*_{y^2}[0,\infty) := \{h : \ h \in C_{y^2}[0,\infty), \ \lim\limits_{y\to\infty} \frac{|h(y)|}{1+y^2} < \infty\}$.

Theorem 4. *For all $\mu \in C^*_{y^2}[0,\infty)$, we obtain*

$$\lim_{m\to\infty} \sup_{y\in[0,\infty)} \frac{|R_{m,\lambda}(\mu;y)-\mu(y)|}{1+y^2} = 0.$$

Proof. Taking into account the Korovkin type theorem given by Gadzhiev [11], we have to show that (3) operators satisfy the following condition:

$$\lim_{m\to\infty} \sup_{y\in[0,\infty)} \frac{|R_{m,\lambda}(t^s;y)-y^s|}{1+y^2} = 0, \qquad s=0,1,2. \tag{17}$$

Using (7), the first condition in (17) is clear for $s=0$.

For $s=1$, by (8), we find

$$\begin{aligned}\sup_{y\in[0,\infty)} \frac{|R_{m,\lambda}(t;y)-y|}{1+y^2} &\leq \left|\frac{m-1+\lambda}{(m-1)(m-2)}\right| \sup_{y\in[0,\infty)} \frac{1}{1+y^2}\\ &+\left|\frac{2(m-1)-3\lambda}{(m-1)(m-2)}\right| \sup_{y\in[0,\infty)} \frac{y}{1+y^2},\end{aligned}$$

which implies that

$$\lim_{m\to\infty} \sup_{y\in[0,\infty)} \frac{|R_{m,\lambda}(t;y)-y|}{1+y^2} = 0.$$

Likewise for $s = 2$, from (9), it becomes

$$\sup_{y\in[0,\infty)} \frac{\left|R_{m,\lambda}(t^2;y) - y^2\right|}{1+y^2} \le \left|\frac{2(m-1+\lambda)}{(m-1)(m-2)(m-3)}\right| \sup_{y\in[0,\infty)} \frac{1}{1+y^2}$$
$$+ \left|\frac{2(2m(m-1)-3\lambda)}{(m-1)(m-2)(m-3)}\right| \sup_{y\in[0,\infty)} \frac{y}{1+y^2}$$
$$+ \left|\frac{(5m-6)(m-1)-4(m+1)\lambda}{(m-1)(m-2)(m-3)}\right| \sup_{y\in[0,\infty)} \frac{y^2}{1+y^2}.$$

Hence, we get

$$\lim_{m\to\infty} \sup_{y\in[0,\infty)} \frac{\left|R_{m,\lambda}(t^2;y) - y^2\right|}{1+y^2} = 0.$$

This completes the proof. □

5 Graphical analysis

In this section, we show several graphics to see the convergence of operators (6) to certain functions with different values of m and λ. Also, we compare the convergence operators (6) and (3) for a certain function to see the role of λ parameter.

In Figure 1, we choose the function $\mu(y) = (y-2/3)(y-3/4)$ (black), $\lambda = 0.9$, $m = 10$ (red), $m = 20$ (green) and $m = 50$ (blue). In Figure 2, we choose the function $\mu(y) = y^2$ (black), $\lambda = 0.4$, $m = 10$ (red), $m = 20$ (green) and $m = 50$ (blue). It is clear from Figure 1 and Figure 2 that, for the different values of λ, as the values of m increases than the convergence of operators (6) to the functions $\mu(y)$ becomes better. In Figure 3, we choose the function $\mu(y) = y^2$ (black), $\lambda = 1$ and $m = 15$. We compare the convergence of operators (3) (green) and (6) (red) to the function $\mu(y)$. As a result of this comparison, if we increase the value of m, we get better approximation for the operators (6) than operators (3).

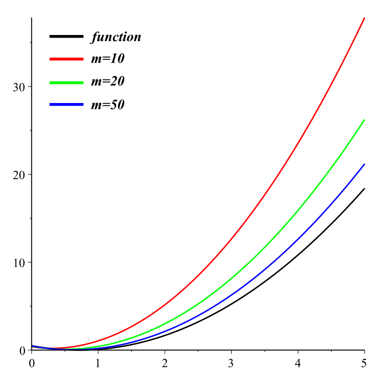

Figure 1: The convergence of operators $R_{m,\lambda}(\mu;y)$ to $\mu(y) = (y-2/3)(y-3/4)$ for $\lambda = 0.9$ and $m = 10, 20, 50$

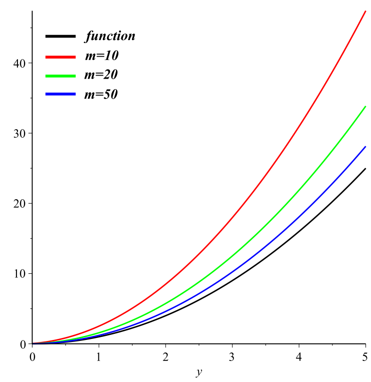

Figure 2: The convergence of operators $R_{m,\lambda}(\mu; y)$ to $\mu(y) = y^2$ for $\lambda = 0.4$ and $m = 10, 20, 50$

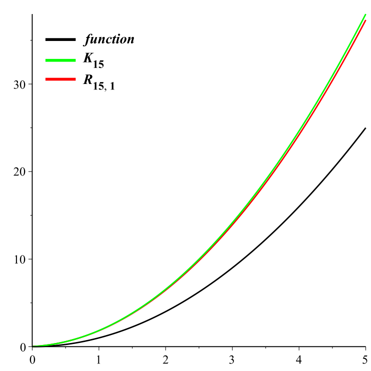

Figure 3: The convergence of operators $R_{m,\lambda}(\mu; y)$ and $K_m(\mu; y)$ to $\mu(y) = y^2$ for $\lambda = 1$ and $m = 15$

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