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REGULARITY OF THE SOLUTIONS TO QUASI-LINEAR PARABOLIC SYSTEMS WITH THE SINGULAR COEFFICIENTS

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Dedicated to Professor Marin Marin on the occasion of his 70th anniversary

Abstract

This article establishes the regularity properties of solutions to the parabolic quasilinear parabolic systems in the divergent form

$$\frac{\partial}{\partial t}\vec{u} - \frac{d}{dx_i}\vec{a}_i\left(x,\ t,\ \vec{u},\ \nabla\vec{u}\right) + \vec{b}\left(x,\ t,\ \vec{u},\ \nabla\vec{u}\right) = 0,$$

under rather general conditions on its coefficients. To prove solvability, we apply the Leray-Schauder theory and method of apriori estimations.

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1 Introduction

In the l-dimensional Euclidean space, we consider a parabolic differential system in the divergent form

$$\frac{\partial}{\partial t}\vec{u} - \frac{d}{dx_i}\vec{a}_i\left(x,\ t,\ \vec{u},\ \nabla\vec{u}\right) + \vec{b}\left(x,\ t,\ \vec{u},\ \nabla\vec{u}\right) = 0, \tag{1}$$

where $\vec{u}\left(x,\ t\right) = \left(u^1\left(x,\ t\right),\ ...,\ u^N\left(x,\ t\right)\right)$ is an unknown N-dimensional vector-function defined over $clos\left(D_T\right)$, domain $D_T = \Omega \times (0,\ T)$, $\Omega \subset R^l$, $l \geq 3$.

We assume that the matrix $\vec{a}\ :\ \Omega \times [0,\ T] \times R^N \times R^l \times R^N \to R^l \times R^N$ satisfies the parabolic conditions in the form

$$\left|\vec{a}_i\left(x,\ t,\ \vec{u},\ \vec{k}\right)k\right| \geq \nu\left(\left|\vec{u}\right|\right)\left|\vec{k}\right|^2 - \gamma_1\left(x,\ t\right), \tag{2}$$

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$$\left|\vec{a}_i\left(x,\ t,\ \vec{u},\ \vec{k}\right)\right| \leq \mu\left(|\vec{u}|\right)\left|\vec{k}\right| + \gamma_2\left(x,\ t\right), \tag{3}$$

$$\left|b\left(x,\ t,\ \vec{u},\ \vec{k}\right)\right| \leq \mu_0\left(|\vec{u}|\right)\left|\vec{k}\right|^2 + \gamma_3\left(x,\ t\right), \tag{4}$$

where ν, μ and μ_0 are continuous positive functions so that ν is a monotone decreasing and μ is a monotone increasing function; in classical theory, functions γ_i satisfy the conditions $\|\gamma_1\|_{q,r,D_T} \leq \mu_2$, $\|\gamma_3\|_{q,r,D_T} \leq \mu_2$ and $\|\gamma_2\|_{2q,2r,D_T} \leq \mu_2$ with $\frac{1}{r} + \frac{l}{2q} = 1 - \chi$, $q \in \left(\frac{l}{2(1-\chi)},\ \infty\right]$ and $r \in \left((1-\chi)^{-1},\ \infty\right]$ for $\chi \in (0,\ 1)$, where the norm of $L_{q,r}\left(D_T\right)$ is given by

$$\|\vec{u}\|_{q,r,(D_T)} = \left(\int_{[0,\ T]}\left(\int_\Omega |\vec{u}\left(x,t\right)|^q\, dx\right)^{\frac{r}{q}} dt\right)^{\frac{1}{r}}. \tag{5}$$

We are assuming that functions γ_2 and ${\gamma_1}^{\frac{1}{2}}$, ${\gamma_3}^{\frac{1}{2}}$ are form-bounded [13].

The essential tool of the theory of partial differential equations is the maximum principle, the general form of which establishes the estimations of $\max\limits_{D_T}|u|$. The existence of the solutions to the boundary problems for the parabolic quasilinear system is proven by the method of the Leray-Schauder theory with the employment of apriori estimations of its solutions [10].

We will call a generalized weak solution to the system (1) *a vector-function $\vec{u} \in L^1_{loc}\left(R^l \times (0,\ T)\right)$ such that the equality*

$$\int_{R^n} \vec{u}\left(x,\ t\right)\vec{\phi}\left(x,\ t\right) dxdt\Big|_0^T - \int_{[0,\ T]}\int_{R^n} \vec{u}\partial_t\vec{\phi}dxdt + \\ + \int_{[0,\ T]}\int_{R^n} \vec{a}_i\left(x,\ t,\ \vec{u},\ \nabla\vec{u}\right)\left(\nabla_i\vec{\phi}\right) dxdt + \\ + \int_{[0,\ T]}\int_{R^n} \vec{b}\vec{\phi}dxdt = 0 \tag{6}$$

holds for allvector-functions $\vec{\phi} \in C_0^\infty\left(R^l \times (0,\ T)\right)$.

We introduce the norm of the functional space $V^p\left(D_T\right)$ by

$$\|u\|_{V^p} = ess\max_{t\in[0,\ T]} \|u\left(\cdot,\ t\right)\|_{L^p(\Omega)} + \|\nabla u\|_{p,(D_T)} \tag{7}$$

where

$$\|\nabla u\|_{p,(D_T)} = \left(\int_{[0,\ t]}\int_\Omega |\nabla u|^p\, dxdt\right)^{\frac{1}{p}} \tag{8}$$

and

$$\|\vec{u}\|_{V^p} = ess\max_{t\in[0,\ T]} \|\vec{u}\left(\cdot,\ t\right)\|_{L^p(\Omega)} + \|\nabla\vec{u}\|_{p,(D_T)}, \tag{9}$$

$$\|\nabla\vec{u}\|_{p,(D_T)} = \left(\int_{[0,\ T]}\int_\Omega |\vec{u}_x|^p\, dxdt\right)^{\frac{1}{p}}. \tag{10}$$

The space $V_{1,0}^{p}(D_T)$ consists of all elements of $V^p(D_T)$ continuous at t respectively to $L^p(D_T)$ with the norm

$$\|\vec{u}\|_{V^p} = \max_{t\in[0,\,T]} \|\vec{u}(\cdot,\ t)\|_{L^p(\Omega)} + \|\nabla\vec{u}\|_{p,(D_T)}.$$

Definition 1. *A bounded generalized solution of the system* (1) *is a vector-function* $\vec{u} \in V_{1,0}^{2}(D_T)$ *such that the identity* (2) *is satisfied for all*

$$\vec{\phi} \in clos\left(W_{1,1}^{2}(D_T) \cap C_0^{\infty}\left(R^l \times (0,\ T)\right)\right) \text{ and } ess\max_{D_T}\left|\vec{\phi}\right| < \infty.$$

Quasilinear parabolic systems have been intensely investigated for many years by methods of PDE perturbation theory. The main results are concerned with the existence of solutions in a certain functional class, many works deal with time-dependent solutions.

Employing the Leray-Schauder method, V. Ladyzenskaja studies the solvability of one quasilinear equation of the general type

$$\partial_t u - a_{ij}(x,\ t,\ u,\ \nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j} + a(x,\ t,\ u,\ \nabla u) = 0$$

under the Dirichlet boundary condition $u|_{\partial\Omega\times[0,T]} = 0,\ u|_{t=0} = \phi(x)$ [10].

E. Heinz built an example that clarifies if the condition

$$|a(x,\ t,\ u,\ k)| \le (\varepsilon(|u|) + P(|k|,\ |u|))(1+|k|)^2$$

here $\lim\limits_{|k|\to\infty} P(|k|,\ |u|) = 0$ and ε is a small enough constant, is not satisfied then an apriori estimation of $\max\limits_{[0,\ 2\pi]} |\nabla u|$ does not necessarily hold, indeed, the system

$$\begin{aligned} \partial_t u^1 - \partial_{xx} u^1 &= u^1\left(\left(\partial_x u^1\right)^2 + \left(\partial_x u^2\right)^2\right) \\ \partial_t u^2 - \partial_{xx} u^2 &= u^2\left(\left(\partial_x u^1\right)^2 + \left(\partial_x u^2\right)^2\right), \end{aligned}$$

has a solution $u^1 = \cos(mx)$ and $u^2 = \sin(mx)$, however, there is no estimation of $\max\limits_{[0,\ 2\pi]} |\nabla u|$.

The partial differential equations in the divergent form were considered by Amann, who considered the solvability of the Neumann problem in the Sobolev spaces [3]. In recent works, H. Dong, S. Kim, and S. Lee constructed the fundamental solution of second-order parabolic equations in the non-divergence form working with the Dini mean oscillation classes of functions [25], Dini conditions were also considered by V. Ladyzenskaja [10]. For some modern literature see the list of references [1 - 48].

In the present work, we establish sufficient conditions for the existence of the solution to the generalized quasilinear parabolic systems

$$\frac{\partial\vec{u}}{\partial t} - \frac{d}{dx_i}\vec{a}_i(x,\ t,\ \vec{u},\ \nabla\vec{u}) + \vec{b}(x,\ t,\ \vec{u},\ \nabla\vec{u}) = 0,$$

under fair weak conditions by applying the modified Leray-Schauder approach.

2 Existence of the solution to the quasilinear parabolic equations

First, we summarize the newest results for the simplest case of one linear equation in the form

$$\partial_t u - \nabla_i \left(a_{ij}(x,\ t)\,\nabla_j u\right) + b_i(x,\ t)\,\nabla_i u = 0$$

under the Cauchy condition

$$u(x,\ +0) = \psi(x) \in L^2\left(R^l\right)$$

in the functional spaces L^p. Assume that the matrix $a(x,\ t)$ is uniformly elliptic and the perturbation vector is $|b| \in L^1_{loc}$, $div(b) = 0$, the estimation

$$\begin{aligned}&\textstyle\int_{[0,\ T]}\int_{R^n} |b(x,\ t)|\,|\varphi(x,\ t)|^2\,dxdt \le \\ &\le c_1 \textstyle\int_{[0,\ T]} \|\nabla\varphi(t)\|\,\|\varphi(t)\|\,dt + \int_{[0,\ T]}\int_{R^n} (c_2 + c_3(t))\,|\varphi(x,\ t)|^2\,dxdt\end{aligned}$$

holds for all $\varphi \in C_0^\infty\left(R^l \times (0,\ T)\right)$ and some positive constants $c_1,\ c_2,\ c\ ,\ c_3 \in L^1_{loc}$ and $\int_{[s,\ t]} c_3(\tau)\,d\tau \le c\sqrt{t-s}$ for all $0 \le s < t < \infty$. Then, there exists a classical solution for all $t > 0$ and $x \in R^l$ for each initially given function $\psi(x)$ that belongs to $L^2\left(R^l\right)$. Next, if $c_3 \in C((s,\ \infty))$ then we have the Gaussian estimation of the fundamental solution

$$P(x,t;y,s) \le \beta^{\frac{l}{2}} \exp\left(\frac{c^2}{\beta-1}\right)\Gamma_{\beta(t-s)}(x-y)$$

for all $\beta > 1$. To formulate more refined results we need the definition of form-bounded fields.

Definition 2. *A vector-function $f\ :\ R^l \to R^l$ is called form-bounded if $|f| \in L^2_{loc}$ and there exist constants $\varepsilon > 0$ and $c(\varepsilon)$ such that*

$$\|f\varphi\|_2^2 \le \varepsilon\,\|\nabla\varphi\|_2^2 + c(\varepsilon)\,\|\varphi\|_2^2$$

for all $\varphi \in C_0^\infty\left(R^l\right)$.

A vector-function $f : R^l \to R^l$ is called multiplicative form-bounded if $|f| \in L^1_{loc}$ and there exist constants $\varepsilon > 0$ and $c(\varepsilon)$ such that

$$\langle f\varphi,\ \varphi\rangle\,\|\|_2^2 \le \varepsilon\,\|\varphi\|_2\sqrt{\|\nabla\varphi\|_2^2 + c(\varepsilon)\,\|\varphi\|_2^2}$$

for all $\varphi \in W_1^2\left(R^l\right)$.

The Kato class K^l_ν consist of all potentials $V \in L^1_{loc}$ such that

$$\left\|(\lambda-\Delta)^{-1}|V|\right\|_\infty \le \nu$$

holds for some $\lambda = \lambda(\nu) \ge 0$.

Assume that $l \geq 3$, the vector b is form-bounded for some $0 < \varepsilon < \infty$ and $|div\,(b)| \in K_\nu^l$. Then the fundamental solution to

$$\partial_t u - \nabla_i \left(a_{ij}\left(x,\ t\right) \nabla_j u\right) + b_i\left(x,\ t\right) \nabla_i u = 0$$

satisfies the two-sided Gaussian bound.

Comparing these results to the classical, we see that conditions on coefficients are weaker, indeed, according to V. Ladyzenskaja, the vector b has to be such that $|b| \in L^q\left(\Omega\right)$.

For the quasilinear parabolic system

$$\vec{u}_t - \frac{d}{dx_i}\vec{a}_i\left(x,\ t,\ \vec{u},\ \nabla\vec{u}\right) + \vec{b}\left(x,\ t,\ \vec{u},\ \nabla\vec{u}\right) = 0,$$

the uniform parabolic condition states that there exist positive constants ν and μ such that

$$\nu\left(|\vec{u}|\right)\xi^2 \leq \left| \sum_{ij=1,\ldots,l} \frac{\partial \vec{a}_i\left(x,\ t,\ \vec{u},\ \vec{k}\right)}{\partial {k_j}^k}\xi_i\xi_j \right| \leq \mu\left(|\vec{u}|\right)\xi^2$$

holds for all vectors $\xi \in R^l$; and the necessary growth conditions are given by

$$\sum_i \left(|\vec{a}_i| + \left|\frac{\partial\vec{a}_i}{\partial u^k}\right|\right)\left(1 + \left|\vec{k}\right|\right) + \sum_{i,j}\left|\frac{\partial\vec{a}_i}{\partial x_j}\right| + \left|\vec{b}\right| \leq \mu\left(1 + \left|\vec{k}\right|\right)^2$$

and we postulate

$$\left|\vec{b}\left(x,\ t,\ \vec{u},\ \vec{k}\right)\right| \leq \left(\varepsilon\left(|\vec{u}|\right) + P\left(\left|\vec{k}\right|,\ |\vec{u}|\right)\right)\left(1 + \left|\vec{k}\right|\right)^2$$

where $\lim\limits_{|\vec{k}|\to\infty} P\left(\left|\vec{k}\right|,\ |\vec{u}|\right) = 0$ and for small enough constants ε. These growth conditions are necessary to eliminate wild systems, for example, E. Heinz's type of systems [10].

3 Leray-Schauder method for a quasilinear parabolic system

The agreement condition is

$$\frac{\partial\vec{\phi}}{\partial t} - \frac{d\vec{a}_i\left(x,\ t,\ \vec{\phi},\ \nabla\vec{\phi}\right)}{dx_i} + \vec{b}\left(x,\ t,\ \vec{\phi},\ \nabla\vec{\phi}\right) = 0;$$

and the boundary conditions are given by $\vec{u}|_{\partial D_T} = \vec{\phi}\Big|_{\partial D_T}$, $D_T = \Omega \times [0, T]$

We will use the Leray-Schauder approach. The system (1) can be written in the form

$$\partial_t u^k - \frac{\partial {a_i}^k\left(x,\ t,\ \vec{u},\ \nabla\vec{u}\right)}{\partial \nabla_j u^k}\nabla_i\nabla_j u^k + \Lambda^k\left(x,\ t,\ \vec{u},\ \nabla\vec{u}\right) = 0, \qquad (11)$$

where we denote

$$\Lambda^k(x,\ t,\ \vec{u},\ \nabla\vec{u}) = b^k(x,\ t,\ \vec{u},\ \nabla\vec{u}) - \\ - \frac{\partial {a_i}^k(x,\ t,\ \vec{u},\ \nabla\vec{u})}{\partial u^k}\nabla_i u^k - \frac{\partial {a_i}^k(x,\ t,\ \vec{u},\ \nabla\vec{u})}{\partial x_i}.$$

Then, we compose the family of linear systems dependent on the parameter τ in the form

$$ {v^k}_t - \left(\tau\frac{\partial {a_i}^k(x,\ t,\ \vec{w},\ \nabla\vec{w})}{\partial\nabla_j w^k} + (1-\tau)\,\delta_i^j\right)\nabla_i\nabla_j v^k + \\ +\tau\Lambda^k(x,\ t,\ \vec{w},\ \nabla\vec{w}) - (1-\tau)\left(\partial_t\phi^k - \Delta\phi^k\right) = 0 \quad (12)$$

with the conditions $\vec{v}|_{\partial D_T} = \vec{\phi}\Big|_{\partial D_T}$ for all $\tau\in[0,\ 1]$, where the v is an unknown vector-function and vector-function w is given.

Let Ξ_δ be the space of all continuous functions w with the continuous derivatives with the Holder norm over D_T given by

$$\|\vec{w}\|_{\Xi_\delta} = |\vec{w}|^\delta_{D_T} + |\nabla\vec{w}|^\delta_{D_T}$$

where $|\vec{w}|^\delta_{D_T} = |\vec{w}|^\delta_{x,D_T} + |\vec{w}|^\delta_{t,D_T}$ and we denote

$$|\vec{w}|^\delta_{x,D_T} = \sup_{\substack{(x,\ t),\ (\tilde{x},\ \tilde{t})\in clos(D_T)\\ |x-\tilde{x}|\le\rho}} \frac{\left|\vec{w}(x,\ t) - \vec{w}(\tilde{x},\ \tilde{t})\right|}{|x-\tilde{x}|^\delta}$$

and

$$|\vec{w}|^\delta_{x,D_T} = \sup_{\substack{(x,\ t),\ (\tilde{x},\ \tilde{t})\in clos(D_T)\\ |t-\tilde{t}|\le\rho}} \frac{\left|\vec{w}(x,\ t) - \vec{w}(\tilde{x},\ \tilde{t})\right|}{\left|t-\tilde{t}\right|^\delta}$$

for $\delta\in(0,\ 1)$.

The system (12) defines the mapping $\Xi_\delta\to\Xi_\delta$ given $w\mapsto v$ so that we denote $\Im(\vec{w},\ \tau) = \vec{v}$ the τ-parametrized mapping $\Xi_\delta\to\Xi_\delta$. The fixed point of $\Im$ at $\tau=1$ is the solution of the boundary problem for system (1).

If $u^\tau$is a fixed point of the nonlinear mapping $\Im$ then u^τ is a solution to the boundary problem

$$P^\tau(\vec{u}) \overset{def}{=} {u^k}_t - \frac{d}{dx_i}\left(\tau {a_i}^k(x,\ t,\ \vec{u},\ \nabla\vec{u}) + (1-\tau)\nabla_i u^k\right) + \\ +\tau b^k(x,\ t,\ \vec{u},\ \nabla\vec{u}) - (1-\tau)\left(\partial_t\phi^k - \Delta\phi^k\right) = 0 \quad (13)$$

under the conditions $\vec{u}|_{\partial D_T} = \vec{\phi}\Big|_{\partial D_T}$ for all $\tau\in[0,\ 1]$.

Now, assume

$$\max_{D_T} |\vec{u}^{\tau}| \leq M_1, \max_{D_T} |\nabla \vec{u}^{\tau}| \leq M_2, \|\vec{u}^{\tau}\|_{\Xi_\alpha} \leq M_3$$

hold for small enough $\alpha > 0$ for all solutions $\vec{u}^{\tau}$. Then, each solution $\vec{u}^{\tau}$ is a fixed point of the mapping $\Im$, and vice versa each fixed point $\vec{u}^{\tau}$ of $\Im$ is a solution to (13).

Let the set Θ be a convex bounded subset of the functional space Ξ_δ, consisting of all elements w of Ξ_δ such that

$$\max_{D_T} |\vec{w}| \leq M_1 + \varepsilon, \max_{D_T} |\nabla \vec{w}| \leq M_2 + \varepsilon, \|\vec{w}\|_{\Xi_\alpha} \leq M_3 + \varepsilon.$$

All fixed points of $\Im$ are belonging to the interior of Θ.

We can consider the fixed point $\vec{u}^{\tau}$ of $\Im$ as a solution to (12) under the condition $\vec{w} = \vec{u}^{\tau}$ that, therefore, $\vec{v} = \vec{u}^{\tau}$ is the solution to the problem (12) for all $\tau \in [0,\ 1]$. Hence, problems (13) and (1) have the solutions in the relative spaces in our case $H^{2+\beta,\ 1+\frac{\beta}{2}} (clos\,(D_T))$.

4 Existence of the solution, apriori estimations

To complete the proving of the solvability of the problem (1) we must prove the apriori assumptions

$$\max_{D_T} |\vec{u}^{\tau}| \leq M_1, \max_{D_T} |\nabla \vec{u}^{\tau}| \leq M_2, \|\vec{u}^{\tau}\|_{\Xi_\alpha} \leq M_3.$$

All operators P^{τ} for all $\tau \in [0,\ 1]$ shear the same properties as the operator P^1 so norm estimations for P^1 hold for P^{τ}, $\tau \in [0,\ 1]$.

From the generalized weak solution to the system (1), we obtain the integral estimation

$$\int_{R^n} \vec{u}\,(x,\ t)\,\vec{\phi}\,(x,\ t)\,dxdt\Big|_{t_1}^{t_2} - \int_{[t_1,\ t_2]} \int_{R^n} \vec{u}\partial_t \vec{\phi} dxdt +$$
$$+ \int_{[t_1,\ t_2]} \int_{R^n} \vec{a}_i\,(x,\ t,\ \vec{u},\ \nabla \vec{u}) \left(\nabla_i \vec{\phi}\right) dxdt \leq$$
$$\leq \int_{[t_1,\ t_2]} \int_{R^n} \left(\mu_0 \,|\nabla \vec{u}|^2 + \gamma_3\right) \left|\vec{\phi}\right| dxdt$$

for all vector-functions $\vec{\phi} \in C_0^{\infty}\left(R^l \times (0,\ T)\right)$ and $0 \leq t_1 \leq t_2 \leq T$.

Thus, we have the lemma.

Lemma. *Let functions $\vec{a}_i$ and $\vec{b}$ satisfy conditions* (2)-(4) *where functions γ_2 and ${\gamma_1}^{\frac{1}{2}}$, ${\gamma_3}^{\frac{1}{2}}$ are form-bounded; and u is a solution to* (1). *Then there is a positive number α such that $\vec{u} \in H^{2+\alpha,\ 1+\frac{\alpha}{2}}\,(D_T)$.*

Indeed, let $\upsilon \in W^2_{1,1}\,(D_{-h,T}) \cap clos\,(C^{\infty}\,(\Omega))$ such that $\upsilon\,(x,\ t) = 0,\quad t \leq 0,\ \ t \geq T - h$ and denote

$$\upsilon_{h^-}\,(x,\ t) = \frac{1}{h} \int_{[t-h,\ t]} \upsilon\,(x,\ \tau)\,d\tau$$

and

$$\upsilon_h\left(x,\ t\right)=\frac{1}{h}\int_{[t,\ t+h]}\upsilon\left(x,\ \tau\right)d\tau.$$

Then, we compose the identity

$$\int_{[t_1,\ t_2]}\int_{\Omega}\left(\upsilon\partial_t u_h^k+{a_i}^k\left(x,\ t,\ \vec{u},\ \nabla\vec{u}\right)\nabla_i\upsilon_{h^-}+b^k\left(x,\ t,\ \vec{u},\ \nabla\vec{u}\right)\upsilon_{h^-}\right)dxdt=0,$$

for $0\le h\le t_1\le t_2\le T-h$. Assume ξ is a positive continuous smooth function such that $\xi|_{\partial\Omega_T}=0$ and $\upsilon\left(x,\ t\right)=\xi^2\left(x,\ t\right)\max\left\{u_h^k\left(x,\ t\right)-n,\ 0\right\}=\xi^2u_h^k\left(n\right)$, we obtain

$$\int_{[t_1,\ t_2]}\int_{\Omega}\xi^2u_h^k\left(n\right)\partial_t u_h^k dxdt=$$
$$=\frac{1}{2}\int_{\Omega}\xi^2\left(u_h^k\left(n\right)\right)^2dx\bigg|_{t_1}^{t_2}-\int_{[t_1,\ t_2]}\int_{\Omega}\xi\partial_t\xi\left(u_h^k\left(n\right)\right)^2dxdt.$$

Denoting $\Theta\left(n,\rho\right)=\left\{x\in B\left(\rho\right)\ :\ u^k\left(x,t\right)>n\right\}$ and applying conditions, we have the estimation as $h\to 0$

$$\frac{1}{2}\left\|u^k\left(n\right)\xi\right\|_{2,B(\rho)}^2\bigg|_{t_1}^{t}+\nu\int_{[t_1,\ t]}\int_{B(\rho)}\left|\nabla u^k\left(n\right)\right|^2\xi^2dxdt\le$$
$$\le\int_{[t_1,\ t]}\int_{\Theta(n,\rho)}\left(\begin{array}{l}\gamma_1\xi^2+\\ \left(2\xi\left|\nabla\xi\right|\mu\left|\nabla u^k\right|+2\xi\left|\nabla\xi\right|\gamma_2+\mu_1\left|\nabla u^k\right|^2\xi^2+\gamma_2\xi^2\right)\cdot\\ \qquad\cdot\left(u^k-n\right)+\left(u^k-n\right)^2\xi\left|\partial_t\xi\right|\end{array}\right)dxdt.$$

If numbers n such that $\max\limits_{B(\rho)\times(t_1,\ t_2)}u^k\left(x,\ t\right)-n\le\frac{\nu}{4\mu_1}$, then, we obtain

$$\left\|u^k\left(n\right)\left(x,t\right)\xi\left(x,t\right)\right\|_{2,B(\rho)}^2+\nu\int_{[t_1,\ t]}\int_{B(\rho)}\left|\nabla u^k\left(n\right)\right|^2\xi^2dxdt\le$$
$$\le\left\|u^k\left(n\right)\left(x,t_1\right)\xi\left(x,t_1\right)\right\|_{2,B(\rho)}^2+$$
$$+2\left(\frac{4\mu^2}{\nu}+1\right)\int_{[t_1,\ t]}\int_{B(\rho)}\left(u^k\left(n\right)\right)^2\left(\xi\left|\partial_t\xi\right|+\left|\nabla\xi\right|^2\right)dxdt+$$
$$+\int_{[t_1,\ t]}\int_{\Theta(n,\rho)}2\xi^2\left(\gamma_1+\gamma_2^2+\frac{\nu}{4\mu_1}\gamma_3\right)dxdt.$$

Thus, if γ_2 and ${\gamma_1}^{\frac{1}{2}},\ {\gamma_3}^{\frac{1}{2}}$ are form-bounded functions then the lemma's statement is proven.

Now, we assume additional conditions on coefficients:

$$\nu\xi^2\le\left|\sum_{ij=1,\ldots,l}\frac{\partial\vec{a}_i\left(x,\ t,\ \vec{u},\ \vec{k}\right)}{\partial{k_j}^k}\xi_i\xi_j\right|\le\mu\xi^2 \qquad (14)$$

for all $\xi \in R^l$,

$$\left|\vec{a}_i\left(x,\ t,\ \vec{u},\ \vec{k}\right)\right| + \left|\frac{\partial \vec{a}_i\left(x,\ t,\ \vec{u},\ \vec{k}\right)}{\partial u^k}\right| \le \mu\left|\vec{k}\right| + \gamma_2\left(x,\ t\right), \tag{15}$$

$$\left|\frac{\partial \vec{a}_i\left(x,\ t,\ \vec{u},\ \vec{k}\right)}{\partial x_j}\right| \le \mu\left|\vec{k}\right|^2 + \gamma_3\left(x,\ t\right), \tag{16}$$

$$\left|\vec{b}\left(x,\ t,\ \vec{u},\ \vec{k}\right)\right| \le \mu\left|\vec{k}\right|^2 + \gamma_4\left(x,\ t\right). \tag{17}$$

Our goal is to find conditions on singularities of $\gamma_2,\ \gamma_3,\ \gamma_4$ under which for solutions $\vec{u}\left(x,t\right)$ to the system (1) values of $\max\limits_{\tilde{D}_T}\left|\nabla\vec{u}\right|$ and $\left\|\vec{u}\right\|_{\Xi_\alpha}$ can be estimated above by a constant for some $\alpha > 0$.

Let $\xi_1\left(x,t\right)$ be a positive smooth function so that $0 \le \xi_1\left(x,t\right) \le 1$, zero when $t = 0$ and on the sides of the cylinder. We calculate

$$\int_{[0,\ t]}\int_\Omega \frac{\lambda\nu}{2}\left|\nabla u^k\right|^2 \exp\left(\lambda\left|\vec{u}\right|^2\right){\xi_1}^2 dxdt \le$$
$$\le \int_{[0,\ t]}\int_\Omega \exp\left(\lambda\left|\vec{u}\right|^2\right)\left(\frac{2}{\lambda}\left|\xi_1\partial_t\xi_1\right| + 2\mu\left|\nabla u^k\right|^2{\xi_1}^2 + \mu\left|\nabla{\xi_1}^2\right|^2\right)dxdt+$$
$$+\int_{[0,\ t]}\int_\Omega \exp\left(\lambda\left|\vec{u}\right|^2\right)\left(\frac{\lambda l}{2\nu}\gamma_2^2 + 2\gamma_2\xi_1\left|\nabla\xi_1\right| + \gamma_4{\xi_1}^2\right)dxdt.$$

Assuming that functions γ_2 and ${\gamma_4}^{\frac{1}{2}}$ are form-bounded, we obtain the estimation $\int_{D_T}\left|\nabla u^k\right|^2{\xi_1}^2 dxdt \le c$ where the constant c depends on $l,\ \nu,\ \mu,\ \mu_0,\ M_1,$ $\max\limits_{D_T}\left(\left|\nabla\xi_1\right|,\ \left|\partial_t\xi_1\right|\right)$ and $mes\left(D_T\right)$.

To estimate the integral $\int_{B(r)}\left|\nabla u^k\right|^{2s}dx$, we consider the identity

$$\frac{1}{2\left(1+s\right)}\int_{B(2\rho)}\left|\nabla\vec{u}\left(x,\ t\right)\right|^{2+2s}{\xi_2}^2\left(x,\ t\right)dx-$$
$$-\int_{[0,\ t]}\int_{B(2\rho)}\frac{1}{s+1}\left|\nabla\vec{u}\right|^{2s+2}\xi_2\partial_t\xi_2 dxdt+$$
$$+\int_{[0,\ t]}\int_{B(2\rho)}\frac{d{a_i}^k}{dx_j}\frac{d}{dx_i}\left(\left|\vec{u}\right|^{2s}\xi_2\nabla_j u^k\right)dxdt-$$
$$-\int_{[0,\ t]}\int_{B(2\rho)}b^k\frac{da_i^k}{dx_j}\left(\left|\nabla_j\vec{u}\right|^{2s}\xi_2\nabla_j u^k\right)dxdt = 0.$$

After standard calculation, we have

$$\frac{1}{2(1+s)}\int_{B(2\rho)}|\nabla\vec{u}(x,\ t)|^{2+2s}{\xi_2}^2(x,\ t)\,dx+$$
$$+\int_{[0,\ t]}\int_{B(2\rho)}\frac{\nu}{4}|\nabla\vec{u}|^{2s}{\xi_2}^2\left|\nabla\nabla u^k\right|^2dxdt+$$
$$+s\nu\int_{[0,\ t]}\int_{B(2\rho)}\left|\nabla u^k\right|^{2s-2}{\xi_2}^2\sum_i\left(\sum_m\left(\nabla_m u^k\right)\left(\nabla_m\nabla_i u^k\right)\right)^2dxdt$$
$$\leq c(l,\nu,\mu)\left(\int_{[0,\ t]}\int_{B(2\rho)}(\gamma_2+\gamma_3+\gamma_4)(1+|\nabla\vec{u}|)\sum_{i,m}\nabla_i\left(|\nabla\vec{u}|^{2s}{\xi_1}^2\nabla_m u^k\right)dxdt\right.$$
$$+\int_{[0,\ t]}\int_{B(2\rho)}|\partial_t\xi_1|\,|\nabla\vec{u}|^{2s+2}dxdt+$$
$$\left.+\left(\check{c}\tilde{\beta}^{\beta_1}(1+s)+1\right)\int_{[0,\ t]}\int_{B(2\rho)}|\nabla\vec{u}|^{2s+2}|\nabla\xi_1|^2dxdt\right).$$

Thus, we have

$$\max_{t\in[\varepsilon,\ T]}\int_{B(\rho)}|\nabla\vec{u}(x,\ t)|^{2s+2}dx+\int_{[\varepsilon,\ T]}\int_{B(\rho)}|\nabla\vec{u}|^{2s+4}dxdt\leq c(\tilde{s}),$$

which holds for $\varepsilon>0$, $0\leq s\leq\tilde{s}$.

Let η be a smooth function equal to zero on the bottom of the cylinder $B(\rho)\times\left(0,\ \tilde{t}\right)$. By integration by parts, we have

$$\int_{B(2\rho)}\eta(x,\ t)\nabla_m u^k(x,\ t)\,dx-\int_{[0,\ t]}\int_{B(\rho)}\partial_t\eta\nabla_m u^k dxdt$$
$$+\int_{[0,\ t]}\int_{B(\rho)}\frac{\partial{a_i}^k(x,\ t,\ \vec{u},\ \nabla\vec{u})}{\partial\nabla_j u^k}\left(\nabla_j\nabla_m u^k\right)(\nabla_i\eta)\,dxdt$$
$$+\int_{[0,\ t]}\int_{B(\rho)}\frac{\partial{a_i}^k(x,\ t,\ \vec{u},\ \nabla\vec{u})}{\partial u^k}\left(\nabla_m u^k\right)(\nabla_i\eta)\,dxdt$$
$$+\int_{[0,\ t]}\int_{B(\rho)}\frac{\partial{a_i}^k(x,\ t,\ \vec{u},\ \nabla\vec{u})}{\partial x_m}(\nabla_i\eta)\,dxdt$$
$$-\int_{[0,\ t]}\int_{B(\rho)}b^k\nabla_m\eta dxdt=0.$$

We denote

$$a_{ij}(x,\ t)=\frac{\partial{a_i}^k(x,\ t,\ \vec{u}(x,\ t),\ \nabla\vec{u}(x,\ t))}{\partial\nabla_j u^k}$$

and

$$\Upsilon_i^{m,k}(x,\ t) = \frac{\partial {a_i}^k(x,\ t,\ \vec{u}(x,\ t),\ \nabla\vec{u}(x,\ t))}{\partial u^k}\nabla_m u^k(x,\ t) + \frac{\partial {a_i}^k(x,\ t,\ \vec{u}(x,\ t),\ \nabla\vec{u}(x,\ t))}{\partial x_m} - \delta_m^i b^k(x,\ t,\ \vec{u}(x,\ t),\ \nabla\vec{u}(x,\ t)).$$

Therefore, we can consider the function $u_m^k(x,\ t) = \nabla_m u^k(x,\ t)$ as a general classical solution to the linear equation

$$\frac{\partial \nabla_m u_m^k}{\partial t} - \frac{\partial}{\partial x_i}\Big(a_{ij}(x,\ t)\nabla_j u_m^k + \Upsilon_i^{m,k}(x,\ t)\Big) = 0. \tag{18}$$

Functions a_{ij} satisfy the ellipticity condition

$$\nu(M_1)\xi^2 \le |a_{ij}(x,\ t)\xi_i\xi_j| \le \mu(M_1)\xi^2$$

and functions $\Upsilon_i^{m,k}$ must satisfy the solvability conditions for the linear parabolic equation. Thus, the function u_m^k is a solution to the linear equation (18) therefore we can apply the theory of linear parabolic equations.

Now, we can formulate the following propositions.

Proposition 1. *Let function $\vec{u} \in C_{2,1}(D_T)$ be a solution to the system* (1) *such that $\max\limits_{D_T}|\vec{u}| \le M_1$, and the functions $\vec{a}_i(x,\ t,\ \vec{u},\ \nabla\vec{u})$ and $\vec{b}(x,\ t,\ \vec{u},\ \nabla\vec{u})$ satisfy the condition* (14) – (17) *then for any domain $\tilde{D} \subset D_T$ with the distance d to the boundary ∂D_T, the norm $\|\vec{u}\|_{\Xi_\alpha(\tilde{D}_T)}$ for some $\alpha > 0$ has the upper estimation depended on l, ν, μ, distance d, and form-bounded constant.*

Proposition 2. *Let function $\vec{u} \in C_{2,1}(clos(D_T))$ be a solution to the system* (1) *such that $\max\limits_{D_T}|\vec{u}| \le M_1$, $\max\limits_{D_T}|\nabla\vec{u}| \le M_2$ and let for all $|\vec{u}| \le M_1$, $\left|\vec{k}\right| \le M_2$, $(x,\ t) \in clos(D_T)$ functions $\vec{a}_i(x,\ t,\ \vec{u},\ \nabla\vec{u})$ and $\vec{b}(x,\ t,\ \vec{u},\ \nabla\vec{u})$ satisfy the Lipschitz condition at t, differential at u^k and k_i^k and such that*

$$\nu\xi^2 \le \left|\sum_{ij=1,\dots,l}\frac{\partial\vec{a}_i\left(x,\ t,\ \vec{u},\ \vec{k}\right)}{\partial {k_j}^k}\xi_i\xi_j\right| \le \mu\xi^2, \tag{19}$$

$$\left|\frac{\partial {a_i}^k(x,\ t,\ \vec{u}(x,\ t),\ \nabla\vec{u}(x,\ t))}{\partial u^s}\right| \le \gamma(x,\ t) \tag{20}$$

$$\left|\frac{\partial {a_i}^k(x,\ t,\ \vec{u}(x,\ t),\ \nabla\vec{u}(x,\ t))}{\partial x_m}\right| \le \gamma(x,\ t) \tag{21}$$

$$\left|\frac{\partial b^s}{\partial k^k}\right| \le \gamma(x,\ t),\ \left|\frac{\partial b^s}{\partial u^k}\right| \le \gamma(x,\ t), \tag{22}$$

where the function γ is form-bounded. Then, $\max\limits_{D_T}|\partial_t\vec{u}|$ can be estimated by a constant depending on l, ν, $\tilde{\mu}$ its maximum on the boundary and form-bounded constant.

Proposition 3. ***If*** $\max\limits_{t\in[0,\,T]}|\nabla\vec{u}|^{\alpha}_{x,D_T}\le c$ ***and*** $\max\limits_{D_T}|\partial_t\vec{u}|\le\check{c}$ ***then*** $|\nabla\vec{u}|^{\frac{\alpha}{1+\alpha}}_{t,D_T}\le c_4$ ***where constant*** c_4 ***depends on the boundary.***

Proposition 4. *Let functions* $\vec{a}_i\,(x,\ t,\ \vec{u},\ \nabla\vec{u})$ *and* $\vec{b}\,(x,\ t,\ \vec{u},\ \nabla\vec{u})$ *satisfy* (19) – (22) *and boundary be smooth enough then for any solution* $\vec{u}\in C_{2,1}\,(clos\,(D_T))$ *to the system* (1) *such that* $\max\limits_{D_T}|\vec{u}|\le M_1$, $\max\limits_{D_T}|\nabla\vec{u}|\le M_2$ *the norm* $\|\vec{u}\|_{\Xi_\alpha}$ *has upper estimation by constant depending on* $l,\ \nu,\ \mu,\ \tilde{\mu},\ M_1,\ M_2$, *its maximum on the boundary and form-bounded constant.*

5 The existence theorem

We formulate the theorem of the existence of the Holder solutions to the parabolic quasilinear system.

Theorem 1. *Let functions* $\vec{a}_i\,(x,\ t,\ \vec{u},\ \nabla\vec{u})$ *and* $\vec{b}\,(x,\ t,\ \vec{u},\ \nabla\vec{u})$ *satisfy our general assumptions* (2)-(4) *and* (19)-(22), *and*

$$\sum_i\left(|\vec{a}_i|+\left|\frac{\partial\vec{a}_i}{\partial u^k}\right|\right)\left(1+\left|\vec{k}\right|\right)+\sum_{i,j}\left|\frac{\partial\vec{a}_i}{\partial x_j}\right|+\left|\vec{b}\right|\le\mu\left(1+\left|\vec{k}\right|\right)^2$$

and

$$\left|\vec{b}\left(x,\ t,\ \vec{u},\ \vec{k}\right)\right|\le\left(\varepsilon\left(|\vec{u}|\right)+P\left(\left|\vec{k}\right|,\ |\vec{u}|\right)\right)\left(1+\left|\vec{k}\right|\right)^2$$

where $\lim\limits_{|\vec{k}|\to\infty}P\left(\left|\vec{k}\right|,\ |\vec{u}|\right)=0$, $\varepsilon>0$. *Let for* $(x,\ t)\subset clos\,(D_T)$, $|\vec{u}|\le M_1$, $\left|\vec{k}\right|\le M_2$, *functions* $\vec{a}_i$, $\frac{\partial\vec{a}_i}{\partial k^m_j}$, $\frac{\partial\vec{a}_i}{\partial u^m}$, $\frac{\partial\vec{a}_i}{\partial x_j}\in H^{\beta}$ *and* $\vec{b}\in H^{\frac{\beta}{2}}$. *Let* $\vec{\phi}\Big|_{\partial D_T}$ *be a continuous in* $clos\,(\partial D_T)$ *with the continuous bounded derivatives second order at* x *and first order at* t *and with the first differential at* x, *and let* $\max\limits_{\Omega}\left|\nabla\vec{\phi}\,(x,\ 0)\right|<\infty$. *Then there exists a unique solution* $\vec{u}\in H^{\alpha,\,\frac{\alpha}{2}}\,(clos\,(D_T))$.

The uniqueness of the solution $\vec{u}\in H^{\alpha,\,\frac{\alpha}{2}}\,(clos\,(D_T))$ to the system (1) can be proven by classical methods. Let us assume that there are two different solutions $\breve{\vec{u}}$ and $\tilde{\vec{u}}$ then they both must satisfy the integral identity

$$\int_{[0,\,t]}\int_\Omega\partial_t\vec{u}\vec{\phi}dxdt+\int_{[0,\,t]}\int_\Omega\vec{a}_i\,(x,\ t,\ \vec{u},\nabla\vec{u})\left(\nabla_i\vec{\phi}\right)dxdt+\int_{[0,\,t]}\int_\Omega\vec{b}\vec{\phi}dxdt=0$$

for all vector-functions $\vec{\phi}\in C_0^\infty\,(\Omega\times[0,\ T])$. We subtract one identity from another, we obtain the linear system

$$\int_{[0,\,t]}\int_\Omega\partial_tw^k\varphi^kdxdt+$$
$$+\int_{[0,\,t]}\int_\Omega\left(\tilde{a}_{ij}{}^k\nabla_jw^k+\tilde{b}^kw^k\right)\left(\nabla_i\varphi^k\right)dxdt+$$
$$+\int_{[0,\,t]}\int_\Omega\left(\left(\tilde{c}^k{}_i\right)\nabla_iw^k+\tilde{c}^kw^k\right)\varphi^kdxdt=0$$

where we denote

$$a_i{}^k\left(x,\ t,\ \vec{\breve{u}},\ \nabla\vec{\breve{u}}\right) - a_i{}^k\left(x,\ t,\ \vec{\tilde{u}},\ \nabla\vec{\tilde{u}}\right) =$$

$$\nabla_j w^k \int_{[0,\,1]} \frac{\partial a_i{}^k\left(x,\ t,\ \tau\vec{\breve{u}} + (1-\tau)\,\vec{\tilde{u}},\ \tau\nabla\vec{\breve{u}} + (1-\tau)\,\nabla\vec{\tilde{u}}\right)}{\partial\nabla_j u^k} d\tau +$$

$$+w^k \int_{[0,\,1]} \frac{\partial a_i{}^k\left(x,\ t,\ \tau\vec{\breve{u}} + (1-\tau)\,\vec{\tilde{u}},\ \tau\nabla\vec{\breve{u}} + (1-\tau)\,\nabla\vec{\tilde{u}}\right)}{\partial u^k} d\tau =$$

$$\stackrel{def}{=} \tilde{a}_{ij}{}^k \nabla_j w^k + \tilde{b}^k w^k,$$

$$b^k\left(x,\ t,\ \vec{\breve{u}},\ \nabla\vec{\breve{u}}\right) - b^k\left(x,\ t,\ \vec{\tilde{u}},\ \nabla\vec{\tilde{u}}\right) =$$

$$\nabla_j w^k \int_{[0,\,1]} \frac{\partial b^k\left(x,\ t,\ \tau\vec{\breve{u}} + (1-\tau)\,\vec{\tilde{u}},\ \tau\nabla\vec{\breve{u}} + (1-\tau)\,\nabla\vec{\tilde{u}}\right)}{\partial\nabla_j u^k} d\tau +$$

$$+w^k \int_{[0,\,1]} \frac{\partial b^k\left(x,\ t,\ \tau\vec{\breve{u}} + (1-\tau)\,\vec{\tilde{u}},\ \tau\nabla\vec{\breve{u}} + (1-\tau)\,\nabla\vec{\tilde{u}}\right)}{\partial u^k} d\tau =$$

$$\stackrel{def}{=} \left(\tilde{c}^k{}_i\right) \nabla_i w^k + \tilde{c}^k w^k$$

and $w^k = \breve{u}^k - \tilde{u}^k$. From the theory of linear systems we obtain that the linear system has a unique solution $\vec{w} = 0$.

References

[1] Adams, D. R. and Hedberg, L.I., *Function spaces and potential theory*, Springer-Verlag, Berlin, 1996.

[2] Agueh, M., *Gagliardo-Nirenberg inequalities involving the gradient L2 -norm*, C. R. Acad. Sci. Paris Ser., **346** (2008), 757–762.

[3] Amann, H., *Invariant sets and existence theorems for semilinear parabolic and elliptic systems*, J. Math. Anal. Appl., **65** (1978), pp. 432–467.

[4] Ambrosetti, A. and Rabinowitz, P., *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349-381.

[5] Jackson, J. and Zitkovic, G., *Existence, and uniqueness for non-Markovian triangular quadratic BSDEs*, 2021. M. F. Atiyah and I. M. Singer, The index of elliptic operators I, III, Ann. Math. **87** (1986), 484-530; 546-604.

[6] Bause, M. and Schwegler, K., *Analysis of stabilized higher-order finite element approximation of nonstationary and nonlinear convection-diffusion-reaction equations*, Comput. Methods Appl. Mech. Engrg., **209/212** (2012), 184–196.

[7] Cai, A.Q., Landman, K.A. and Hughes, B.D., *Multi-scale modeling of a wound-healing cell migration assay*, J. Theor. Biol., **245** (2007), 576–594.

[8] Hillen, T. and Painter, K.J., *A user's guide to PDE models for chemotaxis*, J. Math. Biol., **58** (2009), 183–217.

[9] John, V. and Schmeyer, E., *On finite element methods for 3D time-dependent convection-diffusion-reaction equations with small diffusion*, BAIL 2008 - Boundary and Interior Layers, Lect. Notes Comput. Sci. Eng., Springer, Berlin, **69** (2009), 173–181.

[10] Ladyzenskaja, O.A., Solonnikov, V.A. and Uralceva, N.N., *Linear and quasi-linear equations of parabolic type*, Translations of Mathematical Monographs, Amer. Math. Soc., Vol. **23**, 1968.

[11] Lee, J.M., Hillena, T. and Lewis, M.A., *Pattern formation in prey-taxis systems*, J. Biol. Dynamics, **3** (2009), 551–573.

[12] Berestycki, H. and Nirenberg, L., *Travelling fronts in cylinders*, Ann. IHP Anal. Nonlin. **9** (1992), 497-572.

[13] Berlyiand, A.G. and Semenov, Yu.A., *On the L_p-theory of Schrodinger semi-groups*, Siberian Math. J., (1990), V. **31**, P. 16 – 26.

[14] Bony, J.M., *Calcul symbolique et propagation des singularites pour les equations aux derivetes partielles nonlineaires*, Ann. Sc. Ec. Norm. Sup. **14** (1981), 209-246.

[15] Brezis, H., *Some variational problems with lack of compactness*, in Nonlinear Functional Analysis and Its Applications (F. Browder, Ed.), Proc. Symp. Pure Math., Vol. **45**, Am. Math. Soc., Providence, (1986).

[16] Brezis, H., Coron, J.M. and Lieb, E., *Harmonic maps with defects*, Comm. Math. Phys. **107** (1986), 649-705.

[17] Caffarelli, L., Kohn, R. and Nirenberg, L., *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math. **35** (1982), 771-831.

[18] Caffarelli, L., Nirenberg, L. and Spruck, J., *The Dirichlet problem for nonlinear second-order elliptic equations*, I: Monge-Ampere equation, Comm. Pure Appl. Math. **37** (1984), 369-402.

[19] Crandall, M. and Lions, P.L., *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983), 1-42.

[20] Dacorogna, B., *Weak continuity and weak lower semi-continuity of nonlinear functionals*, Springer-Verlag, Berlin, 1987.

[21] Davies, E.B., *Lp spectral theory of higher-order elliptic differential operators*, Bull. London Math. Soc. **29** (1997), 513-546.

[22] DeGiorgi, E., *Sulla differenziabilita e l'analiticita delle estremali degli integrali multipli regolari*, Mem. Accad Sc. Torino, C. Sc. Fis. Mat. Natur. **3** (1957), 25-43.

[23] DeGiorgi, E., *Un esempio di estremali discontinue per un problema variazionale di tipo ellitico*, Bull. UMI **4** (1968), 135-137.

[24] Dieudonne, J., *History of functional analysis*, North-Holland, Amsterdam, 1981.

[25] Dong, H., Kim, S. and Lee, S., *Estimates for fundamental solutions of parabolic equations in non-divergence form.* J. Differential Equations, **340**, 557–591, (2022).

[26] Hongjie, D., Escauriaza, L., Kim, S., *On C1/2,1, C1,2, and C0,0 estimates for linear parabolic operators.* J. Evol. Equ. **21** (2021), no. 4, 4641–4702.

[27] DiPerna, R. and Lions, P.L., *On the Cauchy problem for Boltzmann equations: global existence and weak stability*, Ann. Math. **130** (1989), 321-366.

[28] Evans, L.C., *Weak convergence methods for nonlinear partial differential equations*, CBMS Regional Conf. Ser. in Math. **74** (1988).

[29] Giaquinta, M., *Introduction to regularity theory for nonlinear elliptic systems*, Birkhauser, Basel, 1993.

[30] Gidas, B., Ni, W.M. and Nirenberg, L., *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **6** (1981), 883-901.

[31] Giusti, E., *Minimal surfaces and functions of bounded variation*, Birkhäuser, Basel, 1984.

[32] Gordon, C., Webb, L. and Wolpert, S., *One cannot hear the shape of a drum*, Bull. Amer. Math. Soc. **27** (1992), 134-138.

[33] Landes, R. and Mustonen, V., *On the pseudo-monotone operators and nonlinear noncoercive problems on unbounded domains*, Math. Ann. **248** (1980), 241-246.

[34] Mawhin, J. and Willem, M., *Critical point theory and Hamiltonian systems*, Springer-Verlag, Berlin, 1989.

[35] Nirenberg, L., *Partial differential equations in the first half of the century*, in "Development of Mathematics 1900-1950" (J. P. Pier, Ed.), Birkhäuser, Basel, 1994.

[36] Kim, S. and Lee, S., *Estimates for Green's functions of elliptic equations in non-divergence form with continuous coefficients*. Ann. Appl. Math. **37** (2021), no. 2, 111–130.

[37] Sylvester, J. and Uhlmann, G., *A global uniqueness theorem for an inverse boundary value problem*, Ann. Math. **125** (1987), 153-169.

[38] Taylor, M., *Partial differential equations*, I, II, III, Springer-Verlag, Berlin, 1996.

[39] Tao, Y. and Winkler, M., *Energy-type estimates and global solvability in a two-dimensional chemotaxis-haptotaxis model with the remodeling of non-diffusible attractant*, J. Diff. Eqns., **257** (2014), 784–815.

[40] Stepien, T.L., Rutter, E.M. and Kuang, Y., *A data-motivated density-dependent diffusion model of in vitro glioblastoma growth*, Mathematical Biosciences and Engineering, **12** (2015), 1157–1172.

[41] Tao, Y., *Global existence of classical solutions to a predator-prey model with nonlinear prey-taxis*, Nonl. Aanl.: RWA, 11 (2010), 2056–2064.

[42] Tao, Y. and Winkler, M., *Energy-type estimates and global solvability in a two-dimensional chemotaxis-haptotaxis model with the remodeling of non-diffusible attractant*, J. Diff. Eqns., **257** (2014), 784–815.

[43] Weissler, F.B., *Single-point blow-up for a semilinear initial value problem*, J. Diff. Eq. **55** (1984), 204-224.

[44] Wang, S., Teng, Y., and Perdikaris, P., *Understanding and mitigating gradient flow pathologies in physics-informed neural networks*, SIAM J. Sci. Comput. **43**, 5 (2021), A3055–A3081.

[45] Yaremenko, M.I., *The existence of a solution of evolution and elliptic equations with singular coefficients* , Asian Journal of Mathematics and Computer Research, Vol. **15** (2017), Issue.: 3. pp. 172- 204.

[46] Yaremenko, M.I., *Quasi-linear evolution and elliptic equations, Journal of Progressive Research in Mathematics*, **11**, (2017), no. 3., 1645-1669.

[47] Yaremenko, M.I., *The sequence of semigroups of nonlinear operators and their applications to study the Cauchy problem for parabolic equations*, Scientific Journal of the Ternopil National Technical University N. 4. (2016), pp. 149-160.

[48] Zhang, D., Guo, L., and Karniadakis, G. E., *Learning in modal space: Solving time-dependent stochastic PDEs using physics-informed neural networks.* SIAM J. Sci. Comput., **42** (2020), A639–A665.