Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 4(66), No. 2 - 2024, 251-256
https://doi.org/10.31926/but.mif.2024.4.66.2.16

# ON MARX-STROHHÄCKER TYPE RESULTS FOR MULTIVALENT FUNCTIONS AND THEIR $n^{th}$ ROOT

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*Dedicated to Professor Marin Marin on the occasion of his 70th anniversary*

**Abstract**

Two Marx-Strohhäcker type results for multivalent functions and their $n^{th}$ root are given. Both results provide a lower bound over the unit disk of $\Re\sqrt[n]{\frac{f(z)}{z^p}}$ for functions that are p-valent starlike of a given order and uniformly p-valent starlike of a given order, respectively. Connections with previous results are indicated.

*2020 Mathematics Subject Classification:* 30C45.
*Key words:* multivalent functions, p-valent starlike functions, Marx - Strohhäcker results.

## 1 Introduction

Denote by $H(\mathcal{U})$ the class of analytic functions $f$ which are defined in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

For $m \in \mathbb{N}$ and $a \in \mathbb{C}$, consider

$$\mathcal{H}[a, m] = \{f \in H(\mathcal{U}) : f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \dots\}$$

Let $\mathcal{A}_p$ $(p \in \mathbb{N})$ be the subclass of $H(\mathcal{U})$ consisting of normalized functions $f$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}. \tag{1}$$

Set $\mathcal{A} = \mathcal{A}_1$.

The classes of p-valent starlike functions of order $\alpha$ and p-valent convex of order $\alpha$ in $\mathcal{U}$ $(p \in \mathbb{N}, 0 \leq \alpha < p)$ are defined (see [3] ) by

$$S_p^*(\alpha) = \left\{ f \in \mathcal{A}_p : \Re\frac{zf'(z)}{f(z)} > \alpha, z \in \mathcal{U} \right\} \tag{2}$$

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and

$$K_p(\alpha) = \left\{ f \in \mathcal{A}_p : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, z \in \mathcal{U} \right\}. \tag{3}$$

respectively. Note that $S_1^*(\alpha) = S^*(\alpha)$ and $K_1(\alpha) = K(\alpha)$ are, respectively, the classes of starlike and convex functions of order $\alpha$ in $\mathcal{U}$. Also, note that $S^*(0) = S^*$ and $K(0) = K$ are, respectively, the well-known classes of starlike and convex functions in $\mathcal{U}$. Moreover, in the case $\alpha = 0$ we are led to the classes $S_p^*(0) = S_p^*$ and $K_p(0) = K_p$ of p-valent starlike and p-valent convex functions in $\mathcal{U}$.

Further, let

$$US_p^*(\alpha) = \left\{ f \in \mathcal{A}_p : \Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \left|\frac{zf'(z)}{f(z)} - p\right|, z \in \mathcal{U} \right\}. \tag{4}$$

be the class of uniformly p-valent starlike functions of order $\alpha$ ($p \in \mathbb{N}, -1 \leq \alpha < p$) in $\mathcal{U}$. Replacing $f(z)$ by $\frac{zf'(z)}{p}$ in definition (4) we obtain the class

$$UCV_p(\alpha) = \left\{ f \in \mathcal{A}_p : \Re\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > \left|\frac{zf''(z)}{f'(z)} - (p-1)\right|, z \in \mathcal{U} \right\} \tag{5}$$

of uniformly p-valent convex functions of order $\alpha$ in $\mathcal{U}$. For more details of multivalent classes of functions see, for example [1], [2], [9], [15].

A classic result in geometric function theory, due to Marx [6] and Strohhäcker [16], asserts that, for $f \in \mathcal{A}$ and all $z \in \mathcal{U}$

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \Longrightarrow \Re\frac{zf'(z)}{f(z)} > \frac{1}{2} \Longrightarrow \Re\frac{f(z)}{z} > \frac{1}{2}. \tag{6}$$

The first implication in (6) shows that any convex function is starlike of order $\frac{1}{2}$.

Making use of the technique of differential subordination, Miller and Mocanu [7] gave a simple proof, than the original one, of implication (6). Moreover, using the same technique, the following implications of Marx-Strohhäcker type result can be proved (see [8])

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \Longrightarrow \Re\sqrt{f'(z)} > \frac{1}{2} \Longrightarrow \Re\frac{f(z)}{z} > \frac{1}{2} \tag{7}$$

for $f \in \mathcal{A}$ and all $z \in \mathcal{U}$.

In [12] Nunokawa et al. extended the differential implications (6) for multivalent functions $f \in \mathcal{A}_p$ ($p \geq 2$), by finding $\beta$ and $\gamma$ such that

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \Longrightarrow \Re\frac{zf'(z)}{f(z)} > \beta \Longrightarrow \Re\frac{f(z)}{z^p} > \gamma. \tag{8}$$

Further, the differential implications (7) were also extended, for multivalent functions, by Gupta et al. In [4] they found $\beta$ and $\gamma$ such that, for $f \in \mathcal{A}_p$ ($p \geq 2$) and all $z \in \mathcal{U}$

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \Longrightarrow \Re\sqrt{\frac{f'(z)}{pz^{p-1}}} > \beta \Longrightarrow \Re\frac{f(z)}{z^p} > \gamma. \tag{9}$$

Other results involving Marx-Strohhäcker type results for multivalent functions can be found in the earlier work by Srivastava et al. [10], [14].

In the present paper we obtain certain Marx-Strohhäcker type results for multivalent functions and their $n^{th}$ root. Connections with previous results are also indicated.

The next lemma from the theory of differential subordination will be needed to prove our main results.

**Lemma 1.1** (See [8]). *Let $\Omega \subset \mathbb{C}$ and suppose that the function $\Psi : \mathbb{C}^2 \times \mathcal{U} \to \mathbb{C}$ satisfies $\Psi(i\rho, \sigma; z) \notin \Omega$ for all $\rho \in \mathbb{R}$, $\sigma \leq -\frac{m}{2}(1+\rho^2)$, $m \in \mathbb{N}$ and $z \in \mathcal{U}$. If $q \in \mathcal{H}[1, m]$ and $\Psi(q(z), zq'(z); z) \in \Omega$ for all $z \in \mathcal{U}$, then $\Re q(z) > 0$ for all $z \in \mathcal{U}$.*

## 2 Marx-Strohhäcker type implications

In this section, we provide a lower bound of $\Re \sqrt[n]{\frac{f(z)}{z^p}}$ first for p-valent starlike functions of order $\alpha$ and then for uniformly p-valent starlike functions of order $\alpha$. In the sequence, whenever we discuss about $n^{th}$ root of a function, we choose its principal one.

**Theorem 2.1.** *For $p \in \mathbb{N}$ and $0 \leq \alpha < p$ let $n \in \mathbb{N}$ such that $n \geq 2(p-\alpha)$. If $f \in \mathcal{A}_p$ is a p-valent starlike function of order $\alpha$, then*

$$\Re \sqrt[n]{\frac{f(z)}{z^p}} > \frac{n}{n+2(p-\alpha)}, \quad z \in \mathcal{U}. \tag{10}$$

*Proof.* Let $f \in S_p^*(\alpha)$ and set

$$\sqrt[n]{\frac{f(z)}{z^p}} = (1-\beta)q(z) + \beta, \tag{11}$$

where $q \in \mathcal{H}[1, 1]$. We want to prove that $\Re q(z) > 0$ when $\beta = \frac{n}{n+2(p-\alpha)}$. From (11) by logarithmic differentiation, we obtain

$$\frac{zf'(z)}{f(z)} = p + n\frac{zq'(z)}{q(z) + \frac{\beta}{1-\beta}}.$$

Let $\Psi(r, s) = p - \alpha + n\frac{s}{r+\frac{\beta}{1-\beta}}$, $r, s \in \mathbb{C}$. In order to obtain $\Re q(z) > 0$, we shall prove that the function $\Psi$ satisfies the assumptions of Lemma 1.1.

For $\rho \in \mathbb{R}$ and $\sigma \in \mathbb{R}$ such that $\sigma \leq -(1+\rho^2)/2$, we have

$$\begin{aligned}
\Re\Psi(i\rho, \sigma) &= p - \alpha + n\sigma\frac{\frac{\beta}{1-\beta}}{\rho^2 + (\frac{\beta}{1-\beta})^2} \\
&\leq p - \alpha - \frac{n}{2}\frac{\beta}{1-\beta}\frac{\rho^2+1}{\rho^2 + (\frac{\beta}{1-\beta})^2} \\
&= \frac{\rho^2\left[2(p-\alpha) - n\frac{\beta}{1-\beta}\right] + \frac{\beta}{1-\beta}\left[2(p-\alpha)\frac{\beta}{1-\beta} - n\right]}{2\left[\rho^2 + (\frac{\beta}{1-\beta})^2\right]}.
\end{aligned}$$

Since the quantities in the numerator, for $\beta = n/(n+2(p-\alpha))$, become

$$2(p-\alpha) - n\frac{\beta}{1-\beta} \leq 0 \ \text{ and } \ 2(p-\alpha)\frac{\beta}{1-\beta} - n = 0,$$

it follows that $\Re\Psi(i\rho, \sigma) \leq 0$. Therefore, by Lemma 1.1 with $\Omega = \{w \in \mathbb{C} : \Re w > 0\}$, we obtain $\Re q(z) > 0$ which leads to

$$\Re \sqrt[n]{\frac{f(z)}{z^p}} = \Re[(1-\beta)q(z) + \beta] > \beta = \frac{n}{n+2(p-\alpha)}.$$

This evidently complete the proof of our theorem. □

**Remark 2.1.**

*(i) If $p = 1$ and $n = 2$, then Theorem 2.1 reduces to a result obtained in [11].*

*(ii) For $p = 1$ and $\alpha = 0$, Theorem 2.1 yields the implication given in [13].*

It is known that a function $f$ is in the class $K_p(\alpha)$ if and only if $zf'(z)/p$ is in $S_p^*(\alpha)$. In view of this duality result, Theorem 2.1 can be rewritten in the following form:

**Corollary 2.1.** *For $p \in \mathbb{N}$ and $0 \leq \alpha < p$ let $n \in \mathbb{N}$ such that $n \geq 2(p-\alpha)$. If $f \in \mathcal{A}_p$ is a p-valent convex function of order $\alpha$, then*

$$\Re \sqrt[n]{\frac{f'(z)}{pz^{p-1}}} > \frac{n}{n+2(p-\alpha)}, \quad z \in \mathcal{U}. \tag{12}$$

**Remark 2.2.**

*(i) For $n = 2, p = 1$ and $\alpha = 0$ in Corollary 2.1, we obtain the Marx - Strohhäcker's result as in the first implication (7).*

*(ii) If $n = 2$, then Corollary 2.1 reduces to the case when $1/2 \leq \beta < 1$ in [4].*

The next result is a Marx-Strohhäcker type implication for uniformly p-valent starlike functions of order $\alpha$.

**Theorem 2.2.** *For $p \in \mathbb{N}$ and $-1 \leq \alpha < p$ let $n \in \mathbb{N}$ such that $n \geq p - \alpha$. If $f \in \mathcal{A}_p$ is a uniformly p-valent starlike function of order $\alpha$, then*

$$\Re \sqrt[n]{\frac{f(z)}{z^p}} > \frac{n}{n+p-\alpha}, \quad z \in \mathcal{U}. \tag{13}$$

*Proof.* Let $f \in US_p^*(\alpha)$ and consider $q \in \mathcal{H}[1,1]$ such that

$$\sqrt[n]{\frac{f(z)}{z^p}} = (1-\beta)q(z) + \beta.$$

We want to show that $\Re q(z) > 0$ when $\beta = n/(n+p-\alpha)$. We have

$$\frac{zf'(z)}{f(z)} = p + n\frac{zq'(z)}{q(z)+\frac{\beta}{1-\beta}}.$$

Define

$$\Psi(r,s) = p-\alpha+n\frac{s}{r+\frac{\beta}{1-\beta}} - \left|n\frac{s}{r+\frac{\beta}{1-\beta}}\right|,\ r,s\in\mathbb{C}.$$

In order to get $\Re q(z) > 0$, we shall prove that the function $\Psi$ satisfies the assumptions of Lemma 1.1. For $\rho\in\mathbb{R}$ and $\sigma\in\mathbb{R}$ such that $\sigma \le -(1+\rho^2)/2$, we get

$$\begin{aligned}
\Re\Psi(i\rho,\sigma) &= p-\alpha+n\sigma\frac{\frac{\beta}{1-\beta}}{\rho^2+(\frac{\beta}{1-\beta})^2} + n\frac{\sigma}{\sqrt{\rho^2+(\frac{\beta}{1-\beta})^2}}\\
&\le p-\alpha-\frac{n}{2}\frac{\beta}{1-\beta}\frac{\rho^2+1}{\rho^2+(\frac{\beta}{1-\beta})^2} - \frac{n}{2}\frac{\rho^2+1}{\sqrt{\rho^2+(\frac{\beta}{1-\beta})^2}}\\
&= p-\alpha-\frac{n}{2}\frac{\rho^2+1}{\rho^2+(\frac{\beta}{1-\beta})^2}\left[\frac{\beta}{1-\beta}+\sqrt{\rho^2+(\frac{\beta}{1-\beta})^2}\right]\\
&\le p-\alpha-n\frac{\beta}{1-\beta}\frac{\rho^2+1}{\rho^2+(\frac{\beta}{1-\beta})^2}\\
&= \frac{\rho^2\left[p-\alpha-n\frac{\beta}{1-\beta}\right]+\frac{\beta}{1-\beta}\left[(p-\alpha)\frac{\beta}{1-\beta}-n\right]}{\rho^2+(\frac{\beta}{1-\beta})^2}.
\end{aligned}$$

For $\beta = n/(n+p-\alpha)$, the quantities in the numerator become

$$p-\alpha-n\frac{\beta}{1-\beta} \le 0 \ \text{ and } \ (p-\alpha)\frac{\beta}{1-\beta} = 0.$$

It follows that $\Re\Psi(i\rho,\sigma)\le 0$. Making use of Lemma 1.1 with $\Omega = \{w\in\mathbb{C} : \Re w > 0\}$, we have $\Re q(z) > 0$ which leads to

$$\Re\sqrt[n]{\frac{f(z)}{z^p}} = \Re[(1-\beta)q(z)+\beta] > \beta = \frac{n}{n+p-\alpha}.$$

This evidently complete the proof of our theorem. □

**Remark 2.3.** *For $p=1$, $n=2$ and $\alpha=0$, Theorem 2.2 yields to a result obtained in [5].*

In view of the relationship between the classes $US_p^*(\alpha)$ and $UCV_p(\alpha)$ we have the next result:

**Corollary 2.2.** *For $p\in\mathbb{N}$ and $-1\le\alpha<p$ let $n\in\mathbb{N}$ such that $n\ge p-\alpha$. If $f\in\mathcal{A}_p$ is a uniformly p-valent convex function of order $\alpha$, then*

$$\Re\sqrt[n]{\frac{f'(z)}{pz^{p-1}}} > \frac{n}{n+p-\alpha},\ \ z\in\mathfrak{U}.$$

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