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# A SIMPLIFIED TEACHING APPROACH TO THE CLASSICAL LAMINATE THEORY OF LAYERED COMPOSITE MATERIALS

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*Dedicated to Professor Marin Marin on the occasion of his 70th anniversary.*

**Abstract**

The classical laminate theory is a common engineering approach to investigate the mechanical response of layered composite materials. This two-dimensional approach and the underlying continuum mechanical modeling might be very challenging for some students, particularly at universities of applied sciences. Thus, a reduced approach, the so-called simplified classical laminate theory, has been developed. The idea is to use solely isotropic one-dimensional elements, i.e., a superposition of bar and beam elements, to introduce the major calculation steps of the classical laminate theory. Understanding this simplified theory is much easier and the final step it to highlight the differences when moving to the general two-dimensional case.

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*Key words:* structured materials, multiphase materials, continuum mechanics, laminates, fibers and matrix.

## 1 Introduction

Composite materials refer to a wide class of advanced materials which are composed of different materials. The major idea is to obtain in total superior properties than a single component for itself could provide. Typical particularities include matrices (e.g. polymers, metals, or ceramic materials) which contain a second material in the form of particles or fibers as the reinforcing elements (see Fig. 1 for a schematic representation) or materials which are shaped in a particular way such as cellular materials (e.g. metals foams [2] or hollow sphere structures [10]). In the case of cellular materials, the macroscopic properties are defined by the base material as well as the shape of the cells (voids, cavities, free space, air) and their arrangement (e.g. periodic or stochastic).

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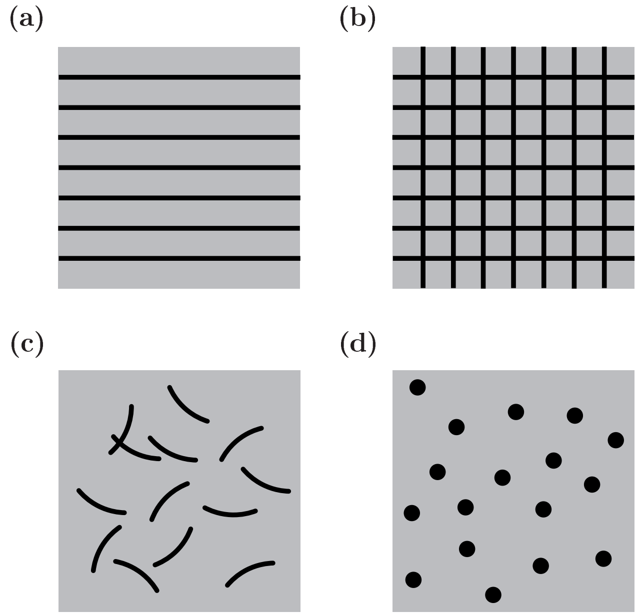


Figure 1: Different types of composite materials (matrix = gray, reinforcement = black): **(a)** unidirectional fibers, **(b)** woven fibers, **(c)** short fibers, and **(d)** particles

For the particular group of fiber-reinforced materials, one may distinguish between classical fibers such as glass, carbon and boron [3], natural fibers [8], and nanofibers (e.g. carbon nanotubes) [14]. The following sections will focus on an important configuration, i.e., unidirectional fiber-reinforced thin layers. Such single layers or plies, also called a lamina, provide superior mechanical properties compared to short fibers with random arrangement. The common approach is to stack several of such lamina with different orientations with respect to some reference direction to form a so-called laminate, see Fig. 2 for an example. Thus, the physical properties are dependent on the matrix material, the fibers, and the number/orientation/sequence (lay-up) of the plies which allows to tailor the macroscopic properties by adjusting the different parameters. This allows a much greater flexibility to adjust properties compared to classical engineering materials.

## 2 Classical laminate theory

This section summarizes the so-called classical laminate theory (CLT) for laminates (see Fig. 2) [11, 5]. It aims to provide a stress and a subsequent strength/-failure analysis without the solution of the system of coupled differential equations for the unknown displacements. This theory provides the solution of a statically indeterminate system based on a generalized stress-strain relationship under consideration of the constitutive relationship and the definition of the so-called stress resultants or generalized stresses and strains.

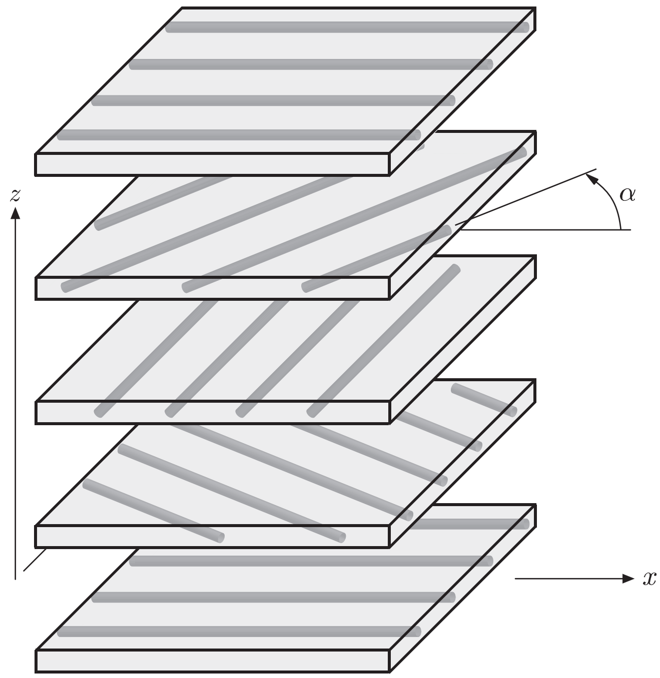


Figure 2: Unbonded view of single laminae forming a 5-layer laminate

The common assumptions of the CLT can be summarized as follows (e.g. [4]):

1. Each lamina is considered quasi-homogeneous and orthotropic (in general, the properties can range from isotropic to anisotropic).
2. Only unidirectional and flat lamina are considered in the following.
3. The laminate consists of perfectly bonded laminae and the bond lines are infinitesimally thin as well as non-shear-deformable.
4. The laminate is thin, i.e., the thickness is small compared to the lateral dimensions, and represents a state of plane stress.
5. Displacements (in thickness and lateral directions) are small compared to the thickness of the laminate.
6. Displacements are continuous throughout the laminate (non-shear-deformable bond lines).
7. In-plane displacements are linear functions of the thickness.
8. Shear strains in planes perpendicular to the middle surface are negligible.
9. Assumptions 7 and 8 imply that a line originally *straight* and *perpendicular* to the laminate middle surface remains so after deformation (Kirchhoff hypothesis of classical plate theory).

10. Kinematics and constitutive relations are linear.

11. Normal distances from the middle surface remain constant. Thus, the transverse normal strain is negligible compared to the in-plane normal strains (Kirchhoff hypothesis of classical plate theory).

Based on these assumptions, it is most suitable to describe a single lamina based on a combination of two-dimensional structural elements, i.e., a plane elasticity and a classical plate element.

## 2.1 Macromechanics of a lamina

Let us consider in the following a single unidirectional lamina as schematically shown in the Fig. 2. In addition to the global $(x, y, z)$ coordinate system, we introduce a local $(\mathit{1}, \mathit{2}, \mathit{3})$ coordinate system for each lamina, which is connected to the fiber direction. The *1*-axis is aligned to the fibers, the *2*-axis is perpendicular to the fibers in the plane and the *3*-axis indicates the thickness direction.

The kinematics or strain-displacement relations for a combination of a plane elasticity and a classical plate element are given as

$$\begin{bmatrix} \varepsilon_{\mathit{1}} \\ \varepsilon_{\mathit{2}} \\ \gamma_{\mathit{12}} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \mathit{1}} & 0 \\ 0 & \frac{\partial}{\partial \mathit{2}} \\ \frac{\partial}{\partial \mathit{2}} & \frac{\partial}{\partial \mathit{1}} \end{bmatrix} \begin{bmatrix} u_{\mathit{1}} \\ u_{\mathit{2}} \end{bmatrix} - \mathit{3} \begin{bmatrix} \frac{\partial^2}{\partial \mathit{1}^2} \\ \frac{\partial^2}{\partial \mathit{2}^2} \\ \frac{2\partial^2}{\partial \mathit{1} \partial \mathit{2}} \end{bmatrix} u_{\mathit{3}} \tag{1}$$

$$= \begin{bmatrix} \frac{\partial}{\partial \mathit{1}} & 0 \\ 0 & \frac{\partial}{\partial \mathit{2}} \\ \frac{\partial}{\partial \mathit{2}} & \frac{\partial}{\partial \mathit{1}} \end{bmatrix} \begin{bmatrix} u_{\mathit{1}} \\ u_{\mathit{2}} \end{bmatrix} + \mathit{3} \begin{bmatrix} \kappa_{\mathit{1}} \\ \kappa_{\mathit{2}} \\ \kappa_{\mathit{12}} \end{bmatrix}, \tag{2}$$

or symbolically as

$$\boldsymbol{\varepsilon} = \boldsymbol{\mathcal{L}}_1 \boldsymbol{u}_{\mathit{1,2}} - \mathit{3} \boldsymbol{\mathcal{L}}_2 u_{\mathit{3}} \tag{3}$$

$$= \underbrace{\boldsymbol{\mathcal{L}}_1 \boldsymbol{u}_{\mathit{1,2}}}_{\boldsymbol{\varepsilon}^0} + \mathit{3} \boldsymbol{\kappa}, \tag{4}$$

where $\boldsymbol{\varepsilon}^0$ collects the middle-surface strains. Alternatively, one may collect the single components of the plane elasticity and the plate contribution in matrix form as follows:

$$\begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \gamma_{12}^0 \\ \hline \kappa_1 \\ \kappa_2 \\ \kappa_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial 1} & 0 & 0 \\ 0 & \frac{\partial}{\partial 2} & 0 \\ \frac{\partial}{\partial 1} & \frac{\partial}{\partial 2} & 0 \\ \hline 0 & 0 & -\frac{\partial^2}{\partial 1^2} \\ 0 & 0 & -\frac{\partial^2}{\partial 2^2} \\ 0 & 0 & -\frac{2\partial^2}{\partial 1 \partial 2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \hline u_3 \end{bmatrix}, \tag{5}$$

or symbolically as

$$\boldsymbol{e} = \boldsymbol{\mathcal{L}}'\boldsymbol{u}\,. \tag{6}$$

The column matrix of generalized strains can be also expressend as:

$$\boldsymbol{e} = \begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{bmatrix}. \tag{7}$$

The constitutive relationship for the combined element can be stated in its compliance form, i.e.,

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 2\,\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{44} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{bmatrix}, \tag{8}$$

or in matrix notation as

$$\boldsymbol{\varepsilon} = \boldsymbol{D}\boldsymbol{\sigma}\,, \tag{9}$$

where $\boldsymbol{D} = \boldsymbol{C}^{-1}$ is the so-called elastic compliance matrix. The constant $G$ in Eq. (8) is the shear modulus in the *1-2* plane. It should be noted that in Eq. (8) the identity

$$-\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2} \tag{10}$$

holds and we can conclude that four independent elastic constants are required to describe a plane orthotropic material. The stress-strain relationship can be obtained from Eq. (8) by inverting the compliance matrix to give:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \dfrac{E_1}{1-\nu_{12}\nu_{21}} & \dfrac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \dfrac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \dfrac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 2\,\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 2\,\varepsilon_{12} \end{bmatrix}, \tag{11}$$

or in matrix notation as

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}\,, \tag{12}$$

where $\boldsymbol{C} = \boldsymbol{D}^{-1}$ is the so-called elasticity matrix.

Let us consider in the following the rotation of the coordinate system and the corresponding transformations of stresses and strains, see Fig. 3. This is important in the case of laminates with different laminae at different orientations (see Fig. 2).

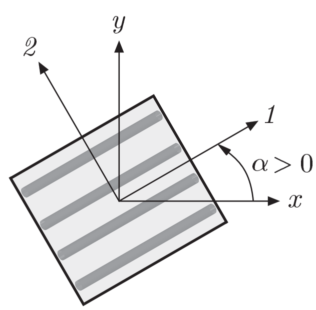


Figure 3: Rotated lamina: (*1*, *2*) principal material coordinates and $(x, y)$ arbitrary coordinates. The rotational angle $\alpha$ is from the $x$-axis to the *1*-axis (counterclockwise positive for the sketched coordinate systems)

The transformations of the stresses between these two coordinate systems is obtained as

$$\boldsymbol{\sigma}_{1,2} = \boldsymbol{T}_\sigma \boldsymbol{\sigma}_{x,y}\,, \tag{13}$$

$$\boldsymbol{\sigma}_{x,y} = \boldsymbol{T}_\sigma^{-1} \boldsymbol{\sigma}_{1,2}\,, \tag{14}$$

where the stress transformation matrix $\boldsymbol{T}_\sigma$ contains the sin and cos functions [1]. The corresponding transformations for the strain field read:

$$\boldsymbol{\varepsilon}_{1,2} = \boldsymbol{T}_\varepsilon \boldsymbol{\varepsilon}_{x,y}\,, \tag{15}$$

$$\boldsymbol{\varepsilon}_{x,y} = \boldsymbol{T}_\varepsilon^{-1} \boldsymbol{\varepsilon}_{1,2}\,, \tag{16}$$

where the strain transformation matrix $\boldsymbol{T}_\varepsilon$ is again given in classical textbooks [1]. Based on these transformations of the stresses and strains, the stress-strain relationship in the arbitrary $x$-$y$ coordinate system can be obtained as

$$\boldsymbol{\sigma}_{x,y} = \underbrace{\boldsymbol{T}_\sigma^{-1}\boldsymbol{C}\boldsymbol{T}_\varepsilon}_{\overline{\boldsymbol{C}}}\boldsymbol{\varepsilon}_{x,y}\,, \tag{17}$$

where the transformed elasticity matrix $\overline{\boldsymbol{C}}$ can be obtained from a triple matrix product. In a similar way, the strain-stress relationship in the $x$-$y$ coordinate system can be obtained as

$$\boldsymbol{\varepsilon}_{x,y} = \underbrace{\boldsymbol{T}_\varepsilon^{-1}\boldsymbol{D}\boldsymbol{T}_\sigma}_{\overline{\boldsymbol{D}}}\boldsymbol{\sigma}_{x,y}\,. \tag{18}$$

The single matrix entries $\overline{C}_{ij}$ or $\overline{D}_{ij}$ can be found again in classical textbooks [1].

The equilibrium equations, which relate the external loads to the corresponding internal reactions, reads for the combined element:

$$\begin{bmatrix} \frac{\partial}{\partial 1} & 0 & \frac{\partial}{\partial 2} & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial 2} & \frac{\partial}{\partial 1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial^2}{\partial 1^2} & \frac{\partial^2}{\partial 2^2} & \frac{2\partial^2}{\partial 1 \partial 2} \end{bmatrix} \begin{bmatrix} N_1^{\mathrm{n}} \\ N_2^{\mathrm{n}} \\ N_{12}^{\mathrm{n}} \\ M_1^{\mathrm{n}} \\ M_2^{\mathrm{n}} \\ M_{12}^{\mathrm{n}} \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{19}$$

or in symbolic notation:

$$\boldsymbol{\mathcal{L}}^{\mathrm{T}}\boldsymbol{s}^{\mathrm{n}} + \boldsymbol{q} = \boldsymbol{0}\,, \tag{20}$$

where $\boldsymbol{s}^{\mathrm{n}}$ is the column matrix of the stress resultants per unit length (generalized stresses per unit length). The derivation of the stress resultants $N_i^{\mathrm{n}}$ and $M_i^{\mathrm{n}}$ gives

$$\begin{bmatrix} N_1^{\mathrm{n}} \\ N_2^{\mathrm{n}} \\ N_{12}^{\mathrm{n}} \end{bmatrix} = \int_{-t/2}^{t/2} \boldsymbol{\sigma}\,\mathrm{d}3 = t\boldsymbol{C}\boldsymbol{\varepsilon}^0 + \frac{t^2}{4}\boldsymbol{C}\boldsymbol{\kappa}\,. \tag{21}$$

A corresponding derivation for the internal bending moments gives:

$$\begin{bmatrix} M_1^{\mathrm{n}} \\ M_2^{\mathrm{n}} \\ M_{12}^{\mathrm{n}} \end{bmatrix} = \int_{-t/2}^{t/2} 3\boldsymbol{\sigma}\,\mathrm{d}3 = \frac{t^2}{4}\boldsymbol{C}\boldsymbol{\varepsilon}^0 + \frac{t^2 3}{12}\boldsymbol{C}\boldsymbol{\kappa}\,. \tag{22}$$

Equations (21) and (22) can be combined to obtain a single matrix representation:

$$\begin{bmatrix} N_1^{\mathrm{n}} \\ N_2^{\mathrm{n}} \\ N_{12}^{\mathrm{n}} \\ M_1^{\mathrm{n}} \\ M_2^{\mathrm{n}} \\ M_{12}^{\mathrm{n}} \end{bmatrix} = \begin{bmatrix} \underbrace{t\boldsymbol{C}}_{\boldsymbol{A}} & \underbrace{\frac{t^2}{4}\boldsymbol{C}}_{\boldsymbol{B}} \\ \underbrace{\frac{t^2}{4}\boldsymbol{C}}_{\boldsymbol{B}} & \underbrace{\frac{t^3}{12}\boldsymbol{C}}_{\boldsymbol{D}} \end{bmatrix} \begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \gamma_{12}^0 \\ \kappa_1 \\ \kappa_2 \\ \kappa_{12} \end{bmatrix}, \tag{23}$$

or in symbolic notation:

$$\boldsymbol{s}^{\mathrm{n}} = \boldsymbol{C}^{*}\boldsymbol{e}\,, \tag{24}$$

where $\boldsymbol{C}^*$ is the generalized elasticity matrix.

## 2.2 Macromechanics of a laminate

Let us consider in the following a laminate, which is composed of $n$ layers, see Fig. 4. Each layer $k$ is a single lamina with its own orientation expressed in a lamina-specific coordinate system $(1_k, 2_k, 3_k)$. Thus, the global coordinate system $(x, y, z)$ is used to describe the orientation of the laminate.

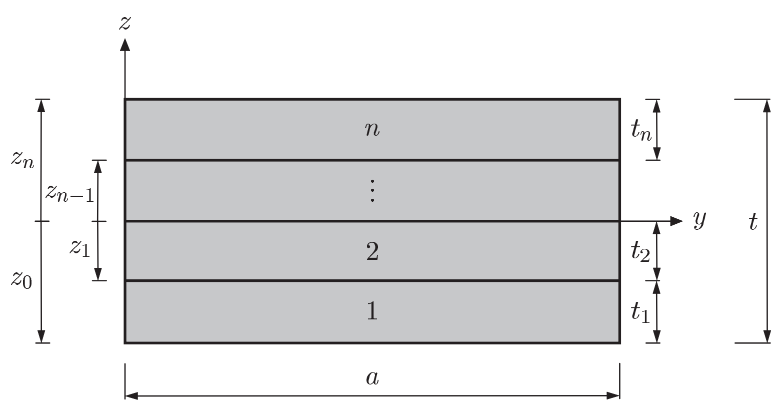


Figure 4: Geoemtry of a laminate with $n$ layers

The height of a layer $k$ ($1 \leq k \leq n$) can be expressed based on the thickness coordinate as $t_k = z_k - z_{k-1}$ and the total height of the laminate results as $t = \sum_{k=1}^{n} t_k$.

Let us focus in the following on the evaluation on the stress resultants as introduced in Eqs. (21) and (22) for a single lamina. The internal normal forces can be expressed in the laminate-specific coordinate system $(x, y, z)$ as

$$\begin{bmatrix} N_x^{\mathrm{n}} \\ N_y^{\mathrm{n}} \\ N_{xy}^{\mathrm{n}} \end{bmatrix} = \int\limits_{-t/2}^{t/2} \boldsymbol{\sigma} \mathrm{d}z = \sum_{k=1}^{n} \int\limits_{z_{k-1}}^{z_k} \boldsymbol{\sigma}_k \mathrm{d}\hat{z} = \sum_{k=1}^{n} \int\limits_{z_{k-1}}^{z_k} \overline{\boldsymbol{C}}_k \boldsymbol{\varepsilon}_k \mathrm{d}\hat{z} \tag{25}$$

$$= \sum_{k=1}^{n} \Bigg( \underbrace{\overline{\boldsymbol{C}}_k \left(z_k - z_{k-1}\right)}_{\boldsymbol{A}_k} \boldsymbol{\varepsilon}^0 + \underbrace{\overline{\boldsymbol{C}}_k \frac{1}{2} \left((z_k)^2 - (z_{k-1})^2\right)}_{\boldsymbol{B}_k} \boldsymbol{\kappa} \Bigg), \tag{26}$$

and the corresponding derivation for the internal bending moments:

$$\begin{bmatrix} M_x^{\mathrm{n}} \\ M_y^{\mathrm{n}} \\ M_{xy}^{\mathrm{n}} \end{bmatrix} = \int\limits_{-t/2}^{t/2} z\boldsymbol{\sigma} \mathrm{d}z = \sum_{k=1}^{n} \int\limits_{z_{k-1}}^{z_k} \boldsymbol{\sigma}_k \hat{z} \mathrm{d}\hat{z} = \sum_{k=1}^{n} \int\limits_{z_{k-1}}^{z_k} \overline{\boldsymbol{C}}_k \boldsymbol{\varepsilon}_k \hat{z} \mathrm{d}\hat{z} \tag{27}$$

$$= \sum_{k=1}^{n} \Bigg( \underbrace{\overline{\boldsymbol{C}}_k \frac{1}{2} \left((z_k)^2 - (z_{k-1})^2\right)}_{\boldsymbol{B}_k} \boldsymbol{\varepsilon}^0 + \underbrace{\overline{\boldsymbol{C}}_k \frac{1}{3} \left((z_k)^3 - (z_{k-1})^3\right)}_{\boldsymbol{D}_k} \boldsymbol{\kappa} \Bigg). \tag{28}$$

Equations (26) and (28) can be combined in a single matrix form to give

tension-shear coupling; tension-twist, bending-shear coupling

$$\left[\begin{array}{c} N_x^{\mathrm{n}} \\ N_y^{\mathrm{n}} \\ N_{xy}^{\mathrm{n}} \\ \hline M_x^{\mathrm{n}} \\ M_y^{\mathrm{n}} \\ M_{xy}^{\mathrm{n}} \end{array}\right] = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{14} & B_{11} & B_{12} & B_{14} \\ A_{12} & A_{22} & A_{24} & B_{12} & B_{22} & B_{24} \\ A_{14} & A_{24} & A_{44} & B_{44} & B_{24} & B_{44} \\ \hline B_{11} & B_{12} & B_{14} & D_{11} & D_{12} & D_{14} \\ B_{12} & B_{22} & B_{24} & D_{12} & D_{22} & D_{24} \\ B_{14} & B_{24} & B_{44} & D_{14} & D_{24} & D_{44} \end{array}\right] \left[\begin{array}{c} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \hline \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{array}\right], \tag{29}$$

bending-tension coupling; bending-twist coupling

or more symbolically as

$$\underbrace{\begin{bmatrix} \boldsymbol{N}^{\mathrm{n}} \\ \boldsymbol{M}^{\mathrm{n}} \end{bmatrix}}_{\boldsymbol{s}} \underbrace{\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{bmatrix}}_{\boldsymbol{C}^*} = \underbrace{\begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{bmatrix}}_{\boldsymbol{e}}, \tag{30}$$

where $\boldsymbol{s}$ is the column matrix of stress resultants (generalized stresses), $\boldsymbol{C}^*$ is the generalized elasticity matrix, and $\boldsymbol{e}$ is the column matrix of generalized strains. The corresponding submatrices are given as follows:

$$\boldsymbol{A} = \sum_{k=1}^{n} \boldsymbol{A}_k = \sum_{k=1}^{n} \overline{\boldsymbol{C}}_k \left(z_k - z_{k-1}\right) , \tag{31}$$

$$\boldsymbol{B} = \sum_{k=1}^{n} \boldsymbol{B}_k = \frac{1}{2}\sum_{k=1}^{n} \overline{\boldsymbol{C}}_k \left((z_k)^2 - (z_{k-1})^2\right) , \tag{32}$$

$$\boldsymbol{D} = \sum_{k=1}^{n} \boldsymbol{D}_k = \frac{1}{3}\sum_{k=1}^{n} \overline{\boldsymbol{C}}_k \left((z_k)^3 - (z_{k-1})^3\right) . \tag{33}$$

It should be noted here that $\boldsymbol{A}$ is called the extensional submatrix, $\boldsymbol{D}$ the bending submatrix, and $\boldsymbol{B}$ the coupling submatrix. Equation (30) can be inverted to obtain the strains and curvatures as function of the generalized stresses as

$$\underbrace{\begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{bmatrix}}_{\boldsymbol{e}} = \left(\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{bmatrix}\right)^{-1} \begin{bmatrix} \boldsymbol{N}^{\mathrm{n}} \\ \boldsymbol{M}^{\mathrm{n}} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{A}' & \boldsymbol{B}' \\ (\boldsymbol{B}')^{\mathrm{T}} & \boldsymbol{D}' \end{bmatrix}}_{(\boldsymbol{C}^*)^{-1}} \underbrace{\begin{bmatrix} \boldsymbol{N}^{\mathrm{n}} \\ \boldsymbol{M}^{\mathrm{n}} \end{bmatrix}}_{\boldsymbol{s}}, \tag{34}$$

where the submatrices are given as follows [3, 1]:

$$\boldsymbol{A}' = \boldsymbol{A}^{-1} + \left(-\boldsymbol{A}^{-1}\boldsymbol{B}\right)\left(\boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1}\left(-\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{\mathrm{T}} , \tag{35}$$

$$\boldsymbol{B}' = \left(-\boldsymbol{A}^{-1}\boldsymbol{B}\right)\left(\boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1} , \tag{36}$$

$$\boldsymbol{D}' = \left(\boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1} . \tag{37}$$

As can be concluded from Eq. (29), there is a coupling between different deformation modes. Based on the generalized strains obtained from Eq. (34), it is now possible to calculate the stresses in each layer $k$ expressed in the $x$-$y$ coordinate system:

$$\boldsymbol{\sigma}^k_{x,y}(z) = \overline{\boldsymbol{C}}_k \left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right) , \tag{38}$$

where the vertical coordinate $z$ ranges for the $k$th layer in the following boundaries: $z_{k-1} \leq z \leq z_k$. The stresses may be evaluated at the bottom ($z = z_{k-1}$), middle ($z = (z_k + z_{k-1})/2$) or top ($z = z_k$) of each layer. Based on relation (13), we can transform the stress values into the *1-2* coordinate system (see Fig. 3):

$$\boldsymbol{\sigma}^k_{\mathit{1,2}} = \boldsymbol{T}^k_{\sigma}\boldsymbol{\sigma}^k_{x,y} . \tag{39}$$

The obtained stress values may serve for a subsequent failure analysis of layer $k$ according to different criteria [6, 12, 13, 7, 1].

## 3 Simplified classical laminate theory

The simplified approach, i.e., the simplified classical laminate theory (SCLT) [9], will still consider a layer-wise composition of a mechanical member but is based on the following, see also Fig. 5:

- The two-dimensional combination of a plane elasticity and classical plate element is replaced by a superposition of a *one-dimensional* bar (tension and compression) and thin beam (bending) element. Bending occurs only in a single plane (here: $x$-$z$ plane).

- Each single layer $k$ is considered as isotropic and homogeneous. The constitutive description is based one the one-dimensional Hooke's law.

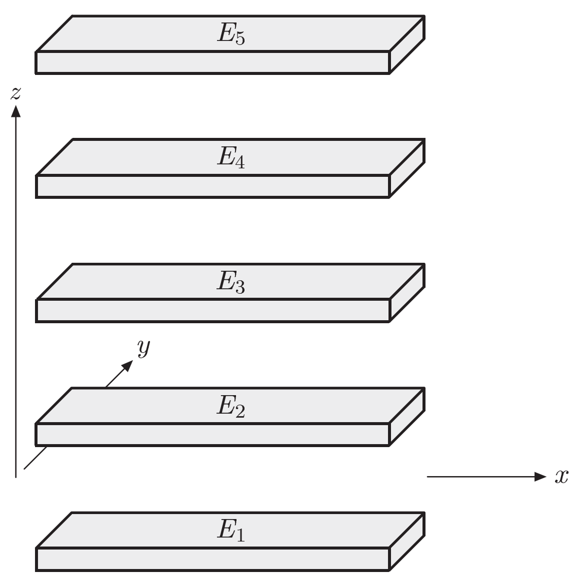


Figure 5: Unbonded view of single bar/beam elements forming a 5-layer composite

It is obvious that this simplified approach disregards the transformations of stresses, strains and stiffnesses between the $(1, 2)$ and $(x, y)$ coordinate systems. Nevertheless, the continuum mechanical modeling as well as the composition of a composite element by layers is well included.

The kinematics relations for a combination of bar and a thin beam element are given as

$$\varepsilon_x(x) = \mathcal{L}_1 u_x(x) - z\mathcal{L}_2 u_z(x) \tag{40}$$

$$= \underbrace{\mathcal{L}_1 u_x}_{\varepsilon_x^0} + z\kappa_y \,. \tag{41}$$

The last two equations can be differently arranged in matrix form as

$$\begin{bmatrix} \varepsilon_x^0 \\ \kappa_y \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x} & 0 \\ 0 & -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \end{bmatrix} \begin{bmatrix} u_x \\ u_z \end{bmatrix}, \tag{42}$$

or symbolically as

$$\boldsymbol{e} = \boldsymbol{\mathcal{L}}' \boldsymbol{u}\,, \tag{43}$$

where the column matrix of generalized strains is expressed as:

$$\boldsymbol{e} = \begin{bmatrix} \varepsilon_x^0 \\ \kappa_y \end{bmatrix}. \tag{44}$$

The constitutive equation is the same for the bar and beam element and represented by 1D Hooke's law:

$$\sigma_x(x) = E\varepsilon_x(x) = C\varepsilon_x(x)\,. \tag{45}$$

The equilibrium relation for the rod and the thin beam can be combined in a single matrix equation to obtain:

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x} & 0 \\ 0 & \frac{\mathrm{d}^2}{\mathrm{d}x^2} \end{bmatrix} \begin{bmatrix} N_x(x) \\ M_y(x) \end{bmatrix} + \begin{bmatrix} p_x(x) \\ q_y(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{46}$$

As in the case of two-dimensional composite layers, we define the internal reactions per unit length (normalized with the corresponding side length of the element). The normalized (superscript'n') internal reactions are obtained as:

$$N_x^{\mathrm{n}}(x) = \frac{N_x(x)}{a}, \tag{47}$$

$$M_y^{\mathrm{n}}(x) = \frac{M_y(x)}{a}. \tag{48}$$

Thus, the combined equilibrium equation can be written as

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x} & 0 \\ 0 & \frac{\mathrm{d}^2}{\mathrm{d}x^2} \end{bmatrix} \begin{bmatrix} N_x^{\mathrm{n}}(x) \\ M_y^{\mathrm{n}}(x) \end{bmatrix} + \begin{bmatrix} \frac{p_x(x)}{a} \\ \frac{q_y(x)}{a} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{49}$$

or in symbolic notation:

$$\boldsymbol{\mathcal{L}}^{\mathrm{T}}\boldsymbol{s}^{\mathrm{n}} + \boldsymbol{q} = \boldsymbol{0}\,. \tag{50}$$

Let us note here that the derivation of the internal reactions or stress resultants, i.e., $N_x(x)$ and $M_y(x)$, must consider the total strain in the form $\varepsilon_x(x) = \varepsilon^0 + z\kappa$. The derivation for a rectangular cross session (see Fig. 6) reads

$$N_x(x) = \int\limits_A \sigma_x(x)\mathrm{d}A = \int\limits_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_x(x)a\mathrm{d}z = \int\limits_{-\frac{t}{2}}^{\frac{t}{2}} aC\varepsilon_x(x)\mathrm{d}z \tag{51}$$

$$= taC\varepsilon_x^0 + \frac{t^2}{4}aC\kappa_y\,. \tag{52}$$

In the same way, the evaluation of the internal bending moment gives:

$$M_y(x) = \int\limits_A z\sigma_x(x)\mathrm{d}A = \int\limits_{-\frac{t}{2}}^{\frac{t}{2}} z\sigma_x(x)a\mathrm{d}z = \int\limits_{-\frac{t}{2}}^{\frac{t}{2}} aCz\varepsilon_x(x)\mathrm{d}z \tag{53}$$

$$= \frac{t^2}{4}aC\varepsilon_x^0 + \frac{t^3}{12}aC\kappa_y\,. \tag{54}$$

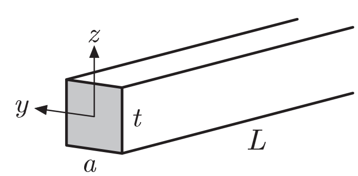


Figure 6: Particular configuration of a combined bar/beam element

Equations (52) and (54) can be combined to obtain a single matrix representation:

$$\begin{bmatrix} N_x(x) \\ M_y(x) \end{bmatrix} = \begin{bmatrix} \underbrace{ta\,C}_{\overline{A}} & \underbrace{\frac{t^2}{4}aC}_{\overline{B}} \\ \underbrace{\frac{t^2}{4}aC}_{\overline{B}} & \underbrace{\frac{t^3}{12}aC}_{\overline{D}} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \kappa_y \end{bmatrix}, \tag{55}$$

or based on the internal reactions per unit length (normalized with the corresponding side length of the plate element, see Eqs. (47) and (48))

$$\begin{bmatrix} N_x^{\mathrm{n}}(x) \\ M_y^{\mathrm{n}}(x) \end{bmatrix} = \begin{bmatrix} \underbrace{t\,C}_{A} & \underbrace{\frac{t^2}{4}C}_{B} \\ \underbrace{\frac{t^2}{4}C}_{B} & \underbrace{\frac{t^3}{12}C}_{D} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \kappa_y \end{bmatrix}, \tag{56}$$

or in symbolic notation:

$$\boldsymbol{s}^{\mathrm{n}} = \boldsymbol{C}^{*}\boldsymbol{e}\,, \tag{57}$$

where $\boldsymbol{C}^{*}$ is the generalized elasticity matrix. In the following, we consider a composite bear/beam element, which is composed of $n$ layers, see Fig. 4. In our simplified approach, each layer $k$ is considered as an isotropic and homogeneous material. However, the properties can vary from layer to layer. The global coordinate system $(x, y, z)$ is used to describe the entire composite bar/beam element.

Let us focus in the following on the evaluation on the stress resultants as introduced in Eqs. (51) and (53) for a single layer. The internal normal force can be expressed in the laminate-specific coordinate system $(x, y, z)$ as

$$N_x = a\int_{-t/2}^{t/2} \sigma_x \mathrm{d}z = a\sum_{k=1}^{n}\int_{z_{k-1}}^{z_k} \sigma_{x,k}\mathrm{d}\hat{z} = a\sum_{k=1}^{n}\int_{z_{k-1}}^{z_k} C_k\varepsilon_{x,k}\mathrm{d}\hat{z} \tag{58}$$

$$= a\sum_{k=1}^{n}\left(\underbrace{C_k\left(z_k - z_{k-1}\right)}_{A_k}\varepsilon_x^0 + \underbrace{C_k\frac{1}{2}\left((z_k)^2 - (z_{k-1})^2\right)}_{B_k}\kappa_y\right), \tag{59}$$

or with $N_x^{\mathrm{n}} = N_x/a$ as

$$N_x^{\mathrm{n}}(x) = \sum_{k=1}^{n}\left(\underbrace{C_k\left(z_k - z_{k-1}\right)}_{A_k}\varepsilon_x^0 + \underbrace{C_k\frac{1}{2}\left((z_k)^2 - (z_{k-1})^2\right)}_{B_k}\kappa_y\right). \tag{60}$$

The corresponding derivation for the internal bending moment gives:

$$M_y = a\int_{-t/2}^{t/2} z\sigma_x \mathrm{d}z = a\sum_{k=1}^{n}\int_{z_{k-1}}^{z_k} \sigma_{x,k}\hat{z}\mathrm{d}\hat{z} = a\sum_{k=1}^{n}\int_{z_{k-1}}^{z_k} C_k\varepsilon_{x,k}\hat{z}\mathrm{d}\hat{z} \tag{61}$$

$$= a\sum_{k=1}^{n}\left(\underbrace{C_k\frac{1}{2}\left((z_k)^2 - (z_{k-1})^2\right)}_{B_k}\varepsilon_x^0 + \underbrace{C_k\frac{1}{3}\left((z_k)^3 - (z_{k-1})^3\right)}_{D_k}\kappa_y\right), \tag{62}$$

or with $M_y^{\mathrm{n}} = M_y/a$ as

$$M_y^{\mathrm{n}}(x) = \sum_{k=1}^{n} \left( \underbrace{C_k \frac{1}{2} \left( (z_k)^2 - (z_{k-1})^2 \right)}_{B_k} \varepsilon_x^0 + \underbrace{C_k \frac{1}{3} \left( (z_k)^3 - (z_{k-1})^3 \right)}_{D_k} \kappa_y \right) . \tag{63}$$

Equations (60) and (63) can be combined in a single matrix form to give

$$\underbrace{\begin{bmatrix} N_x^{\mathrm{n}} \\ M_y^{\mathrm{n}} \end{bmatrix}}_{\boldsymbol{s}} \underbrace{\begin{bmatrix} A & B \\ B & D \end{bmatrix}}_{\boldsymbol{C}^*} = \underbrace{\begin{bmatrix} \varepsilon_x^0 \\ \kappa_y \end{bmatrix}}_{\boldsymbol{e}} , \tag{64}$$

where $\boldsymbol{s}$ is the column matrix of stress resultants (generalized stresses), $\boldsymbol{C}^*$ is the generalized elasticity matrix, and $\boldsymbol{e}$ is the column matrix of generalized strains. The corresponding scalar elements of $\boldsymbol{C}^*$ are given as follows:

$$A = \sum_{k=1}^{n} A_k = \sum_{k=1}^{n} C_k \left( z_k - z_{k-1} \right) , \tag{65}$$

$$B = \sum_{k=1}^{n} B_k = \frac{1}{2} \sum_{k=1}^{n} C_k \left( (z_k)^2 - (z_{k-1})^2 \right) , \tag{66}$$

$$D = \sum_{k=1}^{n} D_k = \frac{1}{3} \sum_{k=1}^{n} C_k \left( (z_k)^3 - (z_{k-1})^3 \right) . \tag{67}$$

It should be noted here that the matrix element $B$ represents a bending-tension coupling. Equation (64) can be inverted to obtain the strains and curvatures (generalized strains) as a function of the generalized stresses as

$$\underbrace{\begin{bmatrix} \varepsilon_x^0 \\ \kappa_y \end{bmatrix}}_{\boldsymbol{e}} = \left( \begin{bmatrix} A & B \\ B & D \end{bmatrix} \right)^{-1} \frac{\begin{bmatrix} N_x^{\mathrm{n}} \end{bmatrix}}{\begin{bmatrix} M_y^{\mathrm{n}} \end{bmatrix}} = \underbrace{\begin{bmatrix} A' & B' \\ B' & D' \end{bmatrix}}_{(\boldsymbol{C}^*)^{-1}} \underbrace{\begin{bmatrix} N_x^{\mathrm{n}} \\ M_y^{\mathrm{n}} \end{bmatrix}}_{\boldsymbol{s}} , \tag{68}$$

where the elements of the inverse are simply given for a $2 \times 2$ matrix as follows:

$$A' = \frac{D}{AD - BB} , B' = \frac{-B}{AD - BB} , D' = \frac{A}{AD - BB} . \tag{69}$$

Based on the generalized strains obtained from Eq. (68), it is now possible to calculate the stresses in each layer $k$ expressed in the $x$-$y$ coordinate system:

$$\sigma_{x,k}(z) = C_k \left( \varepsilon_x^0 + z \kappa_y \right) , \tag{70}$$

where the vertical coordinate $z$ ranges for the $k$th layer in the following boundaries: $z_{k-1} \leq z \leq z_k$. The stresses may be evaluated at the bottom ($z = z_{k-1}$), middle ($z = (z_k + z_{k-1})/2$) or top ($z = z_k$) of each layer. The obtained stress values may serve for a subsequent failure analysis of layer $k$.

Table 1: Recommended steps for a calculation according to the classical laminate theory. Case: internal forces $N_i^{\mathrm{n}}$ and internal moments $M_i^{\mathrm{n}}$ given

| Nr. | Steps to perform |
|---|---|
| ① | Define for each lamina $k$ the material $(E_{1,k}; E_{2,k}; \nu_{12,k}; G_{12,k})$ and the geometrical $(z_k; z_{k-1}; \alpha_k)$ properties. |
| ② | Define the column matrix of generalized stresses: $\boldsymbol{s} = \left[N_x^{\mathrm{n}} \quad N_y^{\mathrm{n}} \quad N_{xy}^{\mathrm{n}} \qquad M_x^{\mathrm{n}} \quad M_y^{\mathrm{n}} \quad M_{xy}^{\mathrm{n}}\right]^{\mathrm{T}}$. |
| ③* | Calculate for each layer $k$ the elasticity matrix $\boldsymbol{C}_k$ in the *1-2* lamina system according to Eq. (11). Transform each matrix according to Eq. (17) to obtain the elasticity matrix $\overline{\boldsymbol{C}}_k$ in the $x$-$y$ laminate system. |
| ④ | Calculate the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{D}$ according to Eqs. (31)–(33). Assemble the generalized elasticity matrix $\boldsymbol{C}^*$ according to Eq. (30). |
| ⑤ | Calculate the generalized compliance matrix $(\boldsymbol{C}^*)^{-1}$ based on Eqs. (35)–(37). |
| ⑥ | Calculate the generalized strains $\boldsymbol{e} = \left[\boldsymbol{\varepsilon}^0 \quad \boldsymbol{\kappa}\right]^{\mathrm{T}}$ according to Eq. (34). |
| ⑦ | Calculate the stresses in each layer $k$ according to Eq. (38), $\boldsymbol{\sigma}_{xy,k}(z) = \overline{\boldsymbol{C}}_k\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$, in the $x$-$y$ laminate system $(z_{k-1} \leq z \leq z_k)$. The strains in the $x$-$y$ laminate system are obtained from $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$. |
| ⑧* | Transform the stresses in each layer to the *1-2* lamina system according to Eq. (39): $\boldsymbol{\sigma}_{12,k} = \boldsymbol{T}_{\sigma,k}\boldsymbol{\sigma}_{xy,k}$. The strains in the *1-2* lamina system are obtained from Eq. (8): $\boldsymbol{\varepsilon}_{12,k} = (\boldsymbol{C}_k)^{-1}\boldsymbol{\sigma}_{12,k}$. |
| ⑨ | Perform the failure analysis for each layer $k$, e.g. [6, 12, 7, 1]. |

## 4 Comparison of the approaches

Let us summarize now the recommended steps for a calculation according to the two-dimensional classical laminate theory, see Table 1 for the case that the internal forces and the internal moments are given and Table 2 for the case that the generalized strains are given. The steps marked with "$*$" are additional steps, which do not occur in the case of the *simplified* laminate theory.

Table 2: Recommended steps for a calculation according to the classical laminate theory. Case: generalized strains $\boldsymbol{e} = \left[\varepsilon_x^0 \quad \varepsilon_y^0 \quad \gamma_{xy}^0 \quad \kappa_x \quad \kappa_y \quad \kappa_{xy}\right]^{\mathrm{T}}$ given

| Nr. | Steps to perform |
|---|---|
| ① | Define for each lamina $k$ the material $(E_{1,k}; E_{2,k}; \nu_{12,k}; G_{12,k})$ and the geometrical $(z_k; z_{k-1}; \alpha_k)$ properties. |
| ② | Define the column matrix of generalized strains: $\boldsymbol{e} = \left[\varepsilon_x^0 \quad \varepsilon_y^0 \quad \gamma_{xy}^0 \quad \kappa_x \quad \kappa_y \quad \kappa_{xy}\right]^{\mathrm{T}}$. |
| ③* | Calculate for each layer $k$ the elasticity matrix $\boldsymbol{C}_k$ in the *1-2* lamina system according to Eq. (11). Transform each matrix according to Eq. (17) to obtain the elasticity matrix $\overline{\boldsymbol{C}}_k$ in the $x$-$y$ laminate system. |
| ④ | Calculate the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ according to Eqs. (31)–(33). Assemble the generalized elasticity matrix $\boldsymbol{C}^*$ according to Eq. (30). |
| ⑤ | Calculate the generalized stresses $\boldsymbol{s} = \left[\boldsymbol{N}^{\mathrm{n}} \quad \boldsymbol{M}^{\mathrm{n}}\right]$ according to Eq. (29). |
| (⑥) | In case that the stresses and strains in each layer are required (e.g. for a subsequent failure analysis), go to steps ⑦–⑨ in Table 1. |

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