Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 4(66), No. 2 - 2024, 225-232
https://doi.org/10.31926/but.mif.2024.4.66.2.14

SHARP BOUNDS OF LOGARITHMIC COEFFICIENTS FOR A CLASS OF UNIVALENT FUNCTIONS

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Dedicated to Professor Marin Marin on the occasion of his 70th anniversary

Abstract

Let $\mathcal{U}(\alpha,\lambda)$, $0<\alpha<1$, $0<\lambda<1$, be the class of functions $f(z)=z+a_2z^2+a_3z^3+\cdots$ satisfying

$$\left|\left(\frac{z}{f(z)}\right)^{1+\alpha}f'(z)-1\right|<\lambda$$

in the unit disc $\mathbb{D}$. For $f\in\mathcal{U}(\alpha,\lambda)$ we give sharp bounds of its initial logarithmic coefficients γ_1, γ_2, γ_3.

2020 Mathematics Subject Classification: 30C45, 30C50.
Key words: univalent functions, logarithmic coefficients, sharp bounds.

1 Introduction and definitions

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disc $\mathbb{D}=\{z:|z|<1\}$ of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots, \tag{1}$$

and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{D}$.

For a function $f\in\mathcal{S}$ we define its logarithmic coefficients, γ_n, $n=1,2,\ldots$, by

$$\log\frac{f(z)}{z}=2\sum_{n=1}^{\infty}\gamma_n z^n. \tag{2}$$

Relatively little exact information is known about those coefficients. The natural conjecture $|\gamma_n|\le 1/n$, inspired by the Koebe function (whose logarithmic

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coefficients are $1/n$) is false even in order of magnitude (see Duren [3]). For the class $\mathcal{S}$ the sharp estimates of single logarithmic coefficients are known only for γ_1 and γ_2, namely,

$$|\gamma_1| \leq 1 \quad \text{and} \quad |\gamma_2| \leq \frac{1}{2} + \frac{1}{e} = 0.635\ldots,$$

and are unknown for $n \geq 3$. For the subclasses of univalent functions the situation is not a great deal better. Only the estimates of the initial logarithmic coefficients are available. For details see [1, 2, 4, 7].

In the 1998 the class $\mathcal{U}(\alpha, \lambda)$ $(0 < \alpha < 1,\ 0 < \lambda < 1)$ of functions $f \in \mathcal{A}$ was introduced by the first author with the condition

$$\left| \left(\frac{z}{f(z)} \right)^{1+\alpha} f'(z) - 1 \right| < \lambda, \quad z \in \mathbb{D}. \tag{3}$$

There is shown that functions from $\mathcal{U}(\alpha, \lambda)$ are starlike, i.e., belong to the class $\mathcal{S}^\star$ of functions that map the unit disk onto a starlike domain, if

$$0 < \lambda \leq \frac{1-\alpha}{\sqrt{(1-\alpha)^2 + \alpha^2}} \equiv \lambda_\star. \tag{4}$$

In the limiting cases when $\lambda = 1$, and either $\alpha = 0$ or $\alpha = 1$, functions in the classes $\mathcal{U}(0, 1)$ and $\mathcal{U}(1, 1)$ satisfy

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1, \quad \text{and} \quad \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < 1,$$

respectively. The former is a subclass of $\mathcal{S}^\star$ since the analytical characterisation of starlike functions is $\operatorname{Re} \frac{zf'(z)}{f(z)} > 0$ $(z \in \mathbb{D})$, while functions in the latter class are univalent.

In this paper we consider estimates of three initial logarithmic coefficients for the class $\mathcal{U}(\alpha, \lambda)$, where $0 < \alpha < 1$, $0 < \lambda \leq \lambda_\star$ and $\lambda_\star$ is defined by (4).

For our consideration we need the next lemma.

Lemma 1. *[5] Let $f \in \mathcal{U}(\alpha, \lambda)$, $0 < \alpha < 1$, $0 < \lambda < 1$. Then there exists a function ω, analytic in $\mathbb{D}$, such that $\omega(0) = 0$, $|\omega(z)| < 1$ for all $z \in \mathbb{D}$, and*

$$\left[\frac{z}{f(z)} \right]^\alpha = 1 - \alpha\lambda z^\alpha \int_0^z \frac{\omega(t)}{t^{\alpha+1}} dt. \tag{5}$$

By Ω we denote the class of analytic functions in $\mathbb{D}$:

$$\omega(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \tag{6}$$

with $\omega(0) = 0$, and $|\omega(z)| < 1$ for all $z \in \mathbb{D}$.

In their paper [6] Prokhorov and Szynal obtained sharp estimates on the functional

$$\Psi(\omega) = |c_3 + \mu c_1 c_2 + \nu c_1^3|$$

within the class of all $\omega \in \Omega$. For our application we need only a part of those results.

Lemma 2. *[6] Let $\omega(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots \in \Omega$. For μ and ν real numbers, let*

$$\Psi(\omega) = \left|c_3 + \mu c_1 c_2 + \nu c_1^3\right|,$$

and

$$\begin{aligned} D_1 &= \left\{(\mu,\nu) : |\mu| \le \frac{1}{2}, |\nu| \le 1\right\}, \\ D_2 &= \left\{(\mu,\nu) : \frac{1}{2} \le |\mu| \le 2, \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \le \nu \le 1\right\}, \\ D_3 &= \{(\mu,\nu) : |\mu| \le 2, |\nu| \ge 1\}. \end{aligned}$$

Then, the sharp estimate $\Psi(\omega) \le \Phi(\mu,\nu)$ holds, where

$$\Phi(\mu,\nu) = \begin{cases} 1, & (\mu,\nu) \in D_1 \cup D_2 \cup \{(2,1)\}; \\ |\nu|, & (\mu,\nu) \in D_3. \end{cases}$$

2 Main results

Theorem 1. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ belongs to the class $\mathcal{U}(\alpha,\lambda)$ and $\lambda_\star$ is defined by (4). Then the following results are best possible.*

(i) $|\gamma_1| \le \frac{\lambda}{2(1-\alpha)}$ *when* $0 < \lambda \le \lambda_\star$ *and* $0 < \alpha < 1$.

(ii) *Let $\lambda_1 = \frac{2(1-\alpha)^2}{\alpha(2-\alpha)}$ and let $\alpha_1 = 0.4825\ldots$ be the unique real root of the equation*

$$7\alpha^4 - 20\alpha^3 + 24\alpha^2 - 16\alpha + 4 = 0$$

on the interval $(0,1)$. Then

$$|\gamma_2| \le \frac{\lambda}{2(2-\alpha)} \quad \text{if} \quad 0 < \lambda \le \begin{cases} \lambda_1, \ \alpha \in [\alpha_1, 1), \\ \lambda_\star, \ \alpha \in (0, \alpha_1], \end{cases}$$

and

$$|\gamma_2| \le \frac{\alpha\lambda^2}{4(1-\alpha)^2} \quad \text{if} \quad \lambda_1 \le \lambda \le \lambda_\star, \ \alpha \in [\alpha_1, 1).$$

(iii) *Let $\lambda_{1/2} = \frac{(1-\alpha)(2-\alpha)}{2\alpha(3-\alpha)}$, $\lambda_\nu = \sqrt{\frac{3(1-\alpha)^3}{\alpha^2(3-\alpha)}}$ and $\alpha_{1/2} = 0.2512\ldots$ and $\alpha_\nu = 0.5337\ldots$ are the unique roots of equations*

$$4 - 12\alpha - 19\alpha^2 + 14\alpha^3 - 2\alpha^4 = 0$$

and

$$3 - 9\alpha + 9\alpha^2 - 5\alpha^3 = 0,$$

on the interval $(0,1)$, respectively. Then

$$|\gamma_3| \le \frac{\lambda}{2(3-\alpha)} \quad \text{if} \quad 0 < \lambda \le \begin{cases} \lambda_\star, & \alpha \in (0, \alpha_\nu], \\ \lambda_{1/2}, & \alpha \in [\alpha_\nu, \alpha_2], \\ \lambda_\nu, & \alpha \in [\alpha_2, 1), \end{cases}$$

where $\alpha_2 = 0.9555\ldots$ *is the unique real root of equation* $11\alpha^2 - 44\alpha + 32 = 0$ *on* $(0,1)$. *Also,*

$$|\gamma_3| \le \frac{\alpha^2\lambda^3}{6(1-\alpha)^3} \quad \text{if} \quad \lambda_\nu \le \lambda \le \lambda_\star,\ \alpha \in [\alpha_\nu, 1).$$

Proof. Let $f \in \mathcal{U}(\alpha,\lambda)$ and $\omega \in \Omega$ are given by (1) and (6), respectively. Then, from (5), upon integration, we have

$$\left[\frac{z}{f(z)}\right]^\alpha = 1 - \alpha\lambda \sum_{n=1}^{\infty} \frac{c_n}{n-\alpha} z^n,$$

that is,

$$\frac{f(z)}{z} = \left(1 - \alpha\lambda \sum_{n=1}^{\infty} \frac{c_n}{n-\alpha} z^n\right)^{-\frac{1}{\alpha}} \tag{7}$$

(the principal value is used here). Further, from (7), having in mind that

$$(1-\alpha z)^{-1/\alpha} = 1 + z + \frac{1+\alpha}{2} z^2 + \frac{(1+\alpha)(1+2\alpha)}{6} z^3 + \cdots,$$

after some calculations, we obtained

$$\sum_{n=1}^{\infty} a_{n+1} z^n = \sum_{n=1}^{\infty} \frac{\lambda c_n}{n-\alpha} z^n + \frac{1+\alpha}{2}\left(\sum_{n=1}^{\infty} \frac{\lambda c_n}{n-\alpha} z^n\right)^2 + \frac{(1+\alpha)(1+2\alpha)}{6}\left(\sum_{n=1}^{\infty} \frac{\lambda c_n}{n-\alpha} z^n\right)^3 + \cdots.$$

By comparing the coefficients we receive

$$\begin{aligned} a_2 &= \frac{\lambda}{1-\alpha} c_1, \\ a_3 &= \frac{\lambda}{2-\alpha} c_2 + \frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2} c_1^2, \\ a_4 &= \frac{\lambda}{3-\alpha} c_3 + \frac{(1+\alpha)\lambda^2}{(1-\alpha)(2-\alpha)} c_1 c_2 + \frac{(1+\alpha)(1+2\alpha)\lambda^3}{6(1-\alpha)^3} c_1^3. \end{aligned} \tag{8}$$

On the other hand, by comparing the coefficients in the relation (2), for the logarithmic coefficients we obtain

$$\gamma_1 = \frac{1}{2} a_2, \quad \gamma_2 = \frac{1}{4}(2a_3 - a_2^2), \quad \gamma_3 = \frac{1}{2}\left(a_4 - a_2 a_3 + \frac{1}{3} a_2^3\right). \tag{9}$$

Using the relations (8) and (9), after some calculations, we have

$$\begin{aligned} \gamma_1 &= \frac{\lambda}{2(1-\alpha)} c_1, \\ \gamma_2 &= \frac{1}{4}\left[\frac{2\lambda}{2-\alpha} c_2 + \frac{\alpha\lambda^2}{(1-\alpha)^2} c_1^2\right], \\ \gamma_3 &= \frac{\lambda}{2(3-\alpha)}\left(c_3 + \mu c_1 c_2 + \nu c_1^3\right), \end{aligned} \tag{10}$$

where

$$\mu = \frac{\alpha(3-\alpha)\lambda}{(1-\alpha)(2-\alpha)} \quad \text{and} \quad \nu = \frac{\alpha^2(3-\alpha)\lambda^2}{3(1-\alpha)^3}. \tag{11}$$

Since logarithmic coefficients are defined for univalent functions, in order to guarantee univalence of f in all cases we need $0 < \lambda \leq \lambda_\star$, where $\lambda_\star$ is defined in (4).

(*i*) From (10) we have $|\gamma_1| \leq \frac{\lambda}{2(1-\alpha)}$, where $0 < \lambda \leq \lambda_\star$ and $0 < \alpha < 1$. The result is the best possible as the function f_1 defined by

$$f_1(z) = z\left(1 - \frac{\alpha\lambda}{1-\alpha}z\right)^{-1/\alpha} = z + \frac{\lambda}{1-\alpha}z^2 + \dots$$

shows.

(*ii*) Using the inequalities $|c_1| \leq 1$, $|c_2| \leq 1 - |c_1|^2$ for $\omega \in \Omega$ and (10), we have

$$\begin{aligned} |\gamma_2| &\leq \frac{1}{4}\left[\frac{2\lambda}{2-\alpha}|c_2| + \frac{\alpha\lambda^2}{(1-\alpha)^2}|c_1|^2\right] \\ &\leq \frac{1}{4}\left[\frac{2\lambda}{2-\alpha}(1-|c_1|^2) + \frac{\alpha\lambda^2}{(1-\alpha)^2}|c_1|^2\right] \\ &\leq \frac{1}{4}\left[\frac{2\lambda}{2-\alpha} + \left(\frac{\alpha\lambda^2}{(1-\alpha)^2} - \frac{2\lambda}{2-\alpha}\right)|c_1|^2\right] \equiv H_1(|c_1|). \end{aligned}$$

If $\frac{\alpha\lambda^2}{(1-\alpha)^2} - \frac{2\lambda}{2-\alpha} \leq 0$, or equivalently,

$$\lambda \leq \frac{2(1-\alpha)^2}{\alpha(2-\alpha)} \equiv \lambda_1,$$

then $|\gamma_2| \leq H_1(0) = \frac{\lambda}{2(2-\alpha)}$. It is also necessary that

$$\lambda \leq \lambda_\star = \frac{1-\alpha}{\sqrt{(1-\alpha)^2+\alpha^2}}.$$

The last inequality will hold if $\lambda_1 \leq \lambda_\star$, or equivalently, if

$$7\alpha^4 - 20\alpha^3 + 24\alpha^2 - 16\alpha + 4 \leq 0,$$

i.e., if $\alpha \in [\alpha_1, 1)$, where $\alpha_1 = 0.4825\dots$ is the unique real root of equation

$$7\alpha^4 - 20\alpha^3 + 24\alpha^2 - 16\alpha + 4 = 0$$

on the interval $(0,1)$. If $\alpha \in (0, \alpha_1]$, then $\lambda_1 \geq \lambda_\star$ and we have that $0 < \lambda \leq \lambda_\star$ will imply the same result.

Finally, if $\alpha \in [\alpha_1, 1)$, i.e., $\lambda_1 \le \lambda_\star$, and $\lambda_1 \le \lambda \le \lambda_\star$, then, from the previous consideration we obtain

$$|\gamma_2| \le H_1(1) = \frac{\alpha\lambda^2}{4(1-\alpha)^2}.$$

Those results are the best possible as the functions

$$f_1(z) = z\left(1 - \frac{\alpha\lambda}{1-\alpha}z\right)^{-1/\alpha} \quad \text{and} \quad f_2(z) = z\left(1 - \frac{\alpha\lambda}{2-\alpha}z^2\right)^{-1/\alpha},$$

given by (7) for $c_2 = 1$ ($c_1 = c_3 = \cdots = 0$) or for $c_1 = 1$ ($c_2 = c_3 = \cdots = 0$), show.

(iii) From (10) we have

$$|\gamma_3| \le \frac{\lambda}{2(3-\alpha)}\left|c_3 + \mu c_1 c_2 + \nu c_1^3\right| = \frac{\lambda}{2(3-\alpha)}\Psi(\omega), \tag{12}$$

where μ and ν are given by (11).

Next, we want to apply the results of Lemma 2, and for that we need to distinguish the cases in the definitions of the sets D_1, D_2, and D_3.

First, we note that μ and ν are both positive.

Further, $\mu = \frac{\alpha(3-\alpha)\lambda}{(1-\alpha)(2-\alpha)} \le \frac{1}{2}$ is equivalent to

$$0 < \lambda \le \frac{(1-\alpha)(2-\alpha)}{2\alpha(3-\alpha)} \equiv \lambda_{1/2}.$$

It is necessary that $\lambda \le \lambda_\star$, where $\lambda_\star$ is defined by (4). After some calculations, $\lambda_{1/2} \le \lambda_\star$ is equivalent to

$$4 - 12\alpha - 19\alpha^2 + 14\alpha^3 - 2\alpha^4 \le 0,$$

i.e., to $\alpha \in [\alpha_{1/2}, 1)$, where $\alpha_{1/2} = 0.2512\ldots$ is the unique real root of the equation

$$4 - 12\alpha - 19\alpha^2 + 14\alpha^3 - 2\alpha^4 = 0$$

on the interval $(0, 1)$. In that sense we have

$$0 < \mu \le \frac{1}{2} \quad \Leftrightarrow \quad \lambda \le \begin{cases} \lambda_{1/2}, & \alpha \in [\alpha_{1/2}, 1), \\ \lambda_\star, & \alpha \in (0, \alpha_{1/2}]. \end{cases} \tag{13}$$

On the other hand, by (11), $\nu = \frac{\alpha^2(3-\alpha)\lambda^2}{3(1-\alpha)^3} \le 1$ is equivalent to

$$0 < \lambda \le \sqrt{\frac{3(1-\alpha)^3}{\alpha^2(3-\alpha)}} \equiv \lambda_\nu.$$

It is again necessary that $\lambda \le \lambda_\star$.

Next, $\lambda_\nu \le \lambda_\star$ after some calculations is equivalent to

$$3 - 9\alpha + 9\alpha^2 - 5\alpha^3 \le 0,$$

which is true when $\alpha \in [\alpha_\nu, 1)$, where $\alpha_\nu = 0.5337\ldots$ is the unique real root of equation

$$3 - 9\alpha + 9\alpha^2 - 5\alpha^3 = 0.$$

It means that

$$0 < \nu \le 1 \quad \Leftrightarrow \quad \lambda \le \begin{cases} \lambda_\nu, & \alpha \in [\alpha_\nu, 1), \\ \lambda_\star, & \alpha \in (0, \alpha_\nu]. \end{cases} \tag{14}$$

Also, $\lambda_{1/2} \le \lambda_\nu$ is equivalent to $11\alpha^2 - 44\alpha + 32 \ge 0$, i.e., to $\alpha \in (0, \alpha_2]$, where $\alpha_2 = 0.9555\ldots$ is the unique real root of equation

$$11\alpha^2 - 44\alpha + 32 = 0$$

on the interval $(0, 1)$.

Using all those previous facts, we can conclude that if

$$0 < \lambda \le \begin{cases} \lambda_\star, & \alpha \in (0, \alpha_\nu], \\ \lambda_{1/2}, & \alpha \in [\alpha_\nu, \alpha_2], \\ \lambda_\nu, & \alpha \in [\alpha_2, 1), \end{cases}$$

then $0 < \mu \le \frac{1}{2}$ and $0 < \nu \le 1$. By Lemma 2 (case D_1) it means that $\Psi(\omega) \le 1$ and so, by (12):

$$|\gamma_3| \le \frac{\lambda}{2(3-\alpha)}.$$

The result is best possible as the function $f_3(z) = z\left(1 - \frac{\alpha\lambda}{3-\alpha} z^3\right)^{-1/\alpha}$ obtained for $c_3 = 1$ ($c_1 = c_2 = c_4 = \cdots = 0$) in (7) shows.

If $\lambda_{1/2} \le \lambda \le \lambda_\nu$ $\alpha_\nu \le \alpha \le \alpha_2$, then $0 < \nu \le 1$ and

$$\begin{aligned} \frac{1}{2} \le \mu = \frac{\alpha(3-\alpha)\lambda}{(1-\alpha)(2-\alpha)} &\le \frac{\alpha(3-\alpha)\lambda_\nu}{(1-\alpha)(2-\alpha)} \\ = \frac{\sqrt{3(1-\alpha)(3-\alpha)}}{2-\alpha} &\le 1.2667\ldots. \end{aligned}$$

The last is obtained for $\alpha = \alpha_\nu = 0.5337\ldots$ since $\frac{\sqrt{3(1-\alpha)(3-\alpha)}}{2-\alpha}$ is a decreasing function on (α_ν, α_2).

For the study of the set D_2, we note that the function

$$\phi(\mu) \equiv \frac{4}{27}(1+\mu)^3 - (1+\mu)$$

is an increasing function for $\frac{1}{2} \le \mu \le 2$, and

$$\phi(\mu) \le \phi(1.2667\ldots) = -0,541\ldots < 0 < \nu \le 1.$$

That implies $\Psi(\omega) \le 1$ (by Lemma 2, case D_2), and follows the same sharp estimate $|\gamma_3| \le \frac{\lambda}{2(3-\alpha)}$ as in previous case.

Finally, since for all $0 < \lambda \le \lambda_\star$ we have $0 < \mu \le 2$ (easy to check) and if $\lambda_\nu \le \lambda \le \lambda_\star$, $\alpha \in [\alpha_\nu, 1)$, then $\nu \ge 1$, and by Lemma 2 (case D_3): $\Psi(\omega) \le \nu$, which by (12) implies

$$|\gamma_3| \le \frac{\lambda}{2(3-\alpha)} \frac{\alpha^2(3-\alpha)\lambda^2}{3(1-\alpha)^3} = \frac{\alpha^2\lambda^3}{6(1-\alpha)^3}.$$

The result is the best possible as the function $f_1(z) = z\left(1 - \frac{\alpha\lambda}{1-\alpha}z\right)^{-1/\alpha}$ given by (7) and $c_1 = 1$ $(c_2 = c_3 = \cdots = 0)$ shows.

□

References

[1] Cho, N.E., Kowalczyk, B., Kwon, O.S. et al., *On the third logarithmic coefficient in some subclasses of close-to-convex functions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., **114** (2020), art. no. 52.

[2] Deng, Q., *On the logarithmic coefficients of Bazilevič functions*, Appl. Math. Comput., **217** (2011), no. 12, 5889-5894.

[3] Duren, P.L., *Univalent function*, Springer-Verlag, New York, 1983.

[4] Lecko, A. and Partyka, D., *Successive logarithmic coefficients of univalent functions*, Comput. Methods Funct. Theory (2023).

[5] Obradović, M., *A class of univalent functions. II*, Hokkaido Math. J., **28** (1999), no. 3, 557–562.

[6] Prokhorov, D.V. and Szynal, J., *Inverse coefficients for (α, β)-convex functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **35** (1981), 125–143.

[7] Thomas, D.K., *The logarithmic coefficients of close-to convex functions*, Proc. Amer. Math. Soc., **144** (2016), no. 2, 1681-1687.