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ON THE CONVERGENCE OF THE NEWTON-RAPHSON METHOD AND SOME OF ITS GENERALIZATIONS

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Dedicated to Professor Marin Marin on the occasion of his 70th anniversary

Abstract

The article aims to improve the solving algorithms and methods of increasing the speed of convergence of the solution of nonlinear algebraic systems or even ill-conditioned, as well as some generalizations of the proposed method, in the sense of broadening the conditions of its application. These algebraic systems come, for example, from discretization with the method of finite elements of some boundary value problems that also contain nondifferentiable terms given by some boundary conditions and which are smoothed using the regularization methods. To solve these algebraic systems, an incremental-iterative algorithm is chosen, which involved a great computational effort, but which proved to be useful. These proposed algorithms can simulate the evolution in time of some processes, such as quasistatic or dynamic cases, and the article proves that the use of the Newton-Raphson method and generalizations they lead to an increase in the speed of convergence with a decrease in the calculation effort and to broadening the conditions of applicability.

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1 Introduction

Solving nonlinear algebraic equations is a complex problem, whose approach is based on the concept of approximation. Perhaps the most well-known and used approximation method is that of successive approximations due to the French mathematician Émile Picard [1].

The notion of approximation is based on that of convergence, and the solution of an equation, expressed through an iterative method, leads to the consideration

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of a series of approximations and it is usually hoped that, by demonstrating the convergence of the method, the solution will be obtained through the crossing operation at the limit in this string or in its substrings, which generally makes sense in the presence of a topology.

An important class of topological spaces in which the notion of convergence can be introduced is that of metric spaces. They were obtained around the year 1900 through the efforts of several mathematicians (see e.g. [5]) who managed to make a clear distinction between metric and topological properties. A good understanding of approximation implies knowledge of these notions, such as those exposed in [7] or [8].

In a topological space, a special role in the meaning of our discussion about approximation is played by the notion of compactness. At the beginning, this was conceived in the sense that today we call sequential compactness (Frechet, 1909) and then it was moved to a more general concept, initially called bicompactness (Tikhonov and Alexandrov, 1935) and today we call it compactness. For our purposes, it is enough to mention only the first concept: X a topological space is sequentially compact if any string contains a convergent substring; on $\mathbb{R}^n$ these two notions (on spaces that verify certain countability axioms) are equivalent.

It follows from here that, if we want to ensure the existence of a limit point of a string (which can in particular be formed by successive approximations of the solution of an equation), it is enough to know that it is in a compact set. (We consider only separate Hausdorff spaces).

Regardless of this, however, we must prove the convergence of the string or a certain substring, so let's use Cauchy's criterion or a condition that ensures this fact. In this sense, the competitiveness of the space is essential; finally, the Cauchy criterion may not be verified for the string, but it may be verified for the substring.

So we have to consider a context in which we can formulate these: a metric space in which we have compactness criteria. It is worth mentioning that the compactness actually depends on the topology (therefore on the metric). The case of Banach spaces is sufficient for most applications.

From the point of view of numerical applications, (see [9], [10]), a very important problem is that of the speed of convergence ([15], [16]), which again makes sense only in the previous context.

Now let X be a Banach space and $T : X \to X$ a nonlinear operator (we can admit more generally $T : X \to Y$ where Y is also a $B-$space). Our goal is to solve the equation

$$T(x) = 0. \tag{1}$$

By an iterative process, we mean a string of the form $x_n = f(x_{n-1}), n \geq 1, n \in \mathbb{N}$, where $f : X \to X$ is an application called iteration or process iteration. An iteration approximates the equation (1) if:

(i) there is a limit $x_0 = \lim\limits_{n} x_n$;

(ii) x_0 satisfies the equation (1).

We will say, after [6], that f is an iteration of order p if the string $\{y_n\}$ satisfies the inequalities:

$$||y_{n+1} - y_n|| = ||f(y_n) - f(y_{n-1})|| \leq q\, ||y_n - y_{n-1}||^p,$$

where $q > 0$ is a priori given (independent of $n \in \mathbb{N}$). and f satisfies the last inequality, we say it satisfies a Lipschtz condition of order p.

Obviously, for $p = 1$ and $0 < q < 1$ hold

$$||y_{n+m} - y_n|| \leq \frac{q^n}{1-q} ||y_1 - y_0||,$$

which corresponds to Banach's contraction principle, which in the accepted context leads to the existence of a $y_0 \in X$ such that

$$||y_0 - y_n|| \leq \frac{q^n}{1-q} ||y_1 - y_0||.$$

Conditions of this type are in fact a generalization of those used by Leonid Vasilievich Kantorovici in demonstrating the convergence of the Newton-Raphson method ([8]) which applies to first-order iterations, i.e. $f(y) = y + \lambda T(y)$ where $T : Y \to X$ with $Y \subset X$ is a bounded and linear transformation (operator). In the hypotheses that will be presented below, the string $\{y_n\}$ is defined by

$$y_1 = f(y_0),$$
$$y_n = f(y_{n-1}) = y_{n-1} + \lambda T(y_{n-1})$$

and this converges to a solution y_0 of $T(y) = 0$.

Observation 1.1. *In fact, in the paper [8], L. V. Kantorovici generalized Newton's tangent method, that is, he approximated the solution of the equation* $g(y) = 0$*, where* $g : \mathbb{R} \to \mathbb{R}$*, using the tangent to the graph (obviously it is assumed that* g *is "with tangent"), because in this case we have*

$$y_{n+1} - y_n = -\frac{g(y_n)}{g'(y_n)}, \quad n \in \mathbb{N}$$

as well as other methods of this type due to Cauchy ([4]).

Observation 1.2. *If the linear operator* T *satisfies a Lipshitz type condition of order* 1 *în the sphere* $B(y_0, r)$ *with* $r > 0$ *i.e.* $\forall\, y_1, y_2 \in B(y_0, r)$ *we have* $||Ty_1 - Ty_2|| \leq L||y_1 - y_2||$ *and* $\exists\, T^{-1}$ *for any* $y \in B(y_0, r)$ *then we can still apply Kantorovici's method to an iteration of the form*

$$f(y) = y + \lambda T(y).$$

We further propose two generalizations, in the sense of weakening the assumptions, on Newton-Raphson's method:

1. Presentation of Bartle's results which show that we can apply the Newton-Raphson method without imposing conditions on the second-order derivative [2].

2. Presentation of Altman's result which under certain conditions solves the problem if $[T'(y_0)]^{-1}$ *does not exist at the initial point* y_0 *[3].*

2 Convergence of the Newton-Raphson method without imposing conditions on the second-order derivative (Bartle's Theorem)

Let Ω be a domain, that is $\Omega = \overset{0}{\Omega} \subset X$, and X is a Banach space and $f \in C^1(\Omega)$ means $f : \Omega \to Y$ where Y is a another Banach space, enjoys the property that $\forall x \in \Omega\ \exists f'(x)$ and the application $x \to f'(x)$ is continuous from Ω in $\mathcal{L}(X, Y)$. Here, obviously $\mathcal{L}(X, Y)$ is the space of linear and continuous (bounded) operators normed with the operational norm (of uniform convergence).

For $f \in C^1(\Omega)$ we understand by the continuity module, the number $\delta > 0$ that depends on x_0 and ε, a priori given so that immediately what the $||x - x_0|| \leq \delta(x_0, \varepsilon)$, to result

$$||f'(x) - f'(x_0)|| < \varepsilon.$$

The following statements are almost obvious:

Proposition 2.1. *Let $f \in C^1(B(x_0, \alpha))$ and $x_1, x_2 \in X$ with the property that $||x_i - x_0|| \leq \delta(x_0, \varepsilon)$ for $i = 1, 2$ then, it results*

$$||f(x_1) - f(x_2) - f'(x_0)(x_1 - x_2)|| \leq \varepsilon ||x_1 - x_2||.$$

Indeed, with the average theorem (Lagrange's theorem in fact, in the case of $B-$spaces) written for x_i and x_0 and the definition of the continuity module, the statement of this proposition results.

Proposition 2.2. *If $f \in C^1(B(x_0, \alpha))$ and $f'(x_0)$ is invertible with a continuous inverse (that is, $f'(x_0)$ is a linear homeomorphism of of X on Y). Then $\forall M > ||[f'(x_0)]^{-1}||\ \exists\ \ \beta \in (0, min(1, \alpha))$, and*

a) since $||x - x_0|| \leq \beta$, $f'(x)$ is invertible and $||[f'(x)]^{-1}|| \leq M$;

b) if $||x_i - x_0|| \leq \beta$ for $i \in \{1, 2, 3\}$ then we have the estimate

$$||f(x_1) - f(x_2) - f'(x_3)(x_1 - x_2)|| \leq \frac{1}{2M}||x_1 - x_2||.$$

Indeed, the first point uses the well-known fact that the set of invertible operators is open in $\mathcal{L}(X, Y)$ (with the topology induced by the uniform norm, obviously) and the second is a consequence of the first proposition. With this we have the data needed to prove Bartle's Theorem.

Theorem 2.3. *Let $f : B(x_0, \alpha) \to Y, f \in C^1(B(x_0, \alpha))$ şi $[f'(x_0)]^{-1}$ exist with $||[f'(x_0)]^{-1}|| < M < \infty$. If $\beta = min\{1, \alpha, \delta(x_0, \frac{1}{4M})\}$ and $||f(x_0)|| < \frac{\beta}{2M}$ also $\{z_n\}_{n \in \mathbb{N}}$ is a string subject to the condition $||z_n - x_0|| < \beta$ otherwise arbitrary, then the string $\{z_n\}_{n \in \mathbb{N}}$ defined by iteration*

$$x_{n+1} = x_n - [f'(z_n)]^{-1} f(x_n),$$

converges to a solution $\bar{x}$ of the equation $f(x) = 0$.
Moreover, $||\bar{x} - x_0| \leq \beta$ and $\bar{x}$ is the only solution of the indicated equation in the neighborhood of x_0.
In addition, the speed of convergence is exponential:

$$||x_n - \bar{x}|| \leq \frac{\beta}{2^n}, \qquad n \in \mathbb{N}.$$

Demonstration. Let's choose $\delta_1 > 0$ so that, immediately what

$$||x - x_0|| < \delta_1 \Rightarrow ||f'(x) - f'(x_0)|| \leq \frac{1}{4M},$$

(from $f \in C^1$, obviously).

Then let $\alpha_1 \leq min\{\alpha, \delta_1\} = min\{\alpha, \delta(x_0, \frac{1}{4M})\}$ such that $||x - x_0|| \leq \alpha_1 \Rightarrow ||[f'(x)]^{-1}|| < M$ (which is possible from Proposition 2.2 a)).
Now we will take over the relations at our disposal in order to prove the theorem by complete induction.
We have from the definition of the approximating string

$$x_1 = x_0 - [f'(z_0)]^{-1} f(x_0), \tag{2}$$

wherefrom

$$||x_1 - x_0|| \leq ||[f'(z_0)]^{-1}|| \, ||f(x_0)|| < M||f(x_0)||M \frac{\beta}{2M} = \frac{\beta}{2},$$

because we have from Proposition 2.2 a)) $||[f'(z_0)]^{-1}|| < M$. On the other hand, also from eq.(2) results equivalently

$$f'(z_0)(x_1 - x_0) = -f(x_0),$$

so still

$$f(x_1) = f(x_1) - f(x_0) - f'(z_0)(x_1 - x_0)$$

and so

$$||f(x_1)|| = ||f(x_1) - f(x_0) - f'(z_0)(x_1 - x_0)|| \leq \frac{1}{2M}||x_1 - x_0||,$$

(from Proposition 2.2 b)), obviously.
With these, the "verification" stage is completed. Let's admit the induction hypothesis, that is, the relationships

A) $||x_k - x_0|| < \alpha$,

B) $||x_k - x_{k-1}|| \leq M||f(x_{k-1})||$,

C) $||f(x_k)|| \leq \frac{1}{2M}||x_k - x_{k-1}||$, fulfilled for $k = 1, 2, \ldots, n$ and show that they hold for $k = n + 1$.

• We show that relation B) holds for $k = n + 1$. we start with

$$x_{n+1} - x_n = -[f'(z_n)]^{-1} f(x_n),$$

from where with Proposition 2.2 a) it follows

$$||x_{n+1} - x_n|| \leq ||[f'(z_n)]^{-1}||\, ||f(x_n)|| \leq M||f(x_n)||.$$

• We show that relation A) is satisfied; starting from relation B):

$$||x_{n+1} - x_n|| \leq M||f(x_n)| \leq M\frac{1}{2M}||x_n - x_{n-1}|| = \frac{1}{2}||x_n - x_{n-1}||,$$

where we used the relation C) with $k = n$. Hence, iteratively for $k = 1, 2, \cdots n$

$$||x_{n+1} - x_n|| \leq \sum_{k=0}^{n} \frac{1}{2^k}||x_1 - x_0|| < \left(1 - \frac{1}{2^{n+1}}\right)\beta,$$

so relation A) takes place.
• We show that relation C) also occurs. Resuming the calculations made at the beginning for x_n, we have

$$f(x_{n+1}) = f(x_{n+1}) - f(x_n) - f'(z_n)(x_{n+1} - x_n),$$

so with Proposition 2.2 b) we have

$$||f(x_{n+1})|| \leq \frac{1}{2M}||x_{n+1} - x_n||,$$

which shows that relationship C) takes place. From here, we have for everything $n, p \in \mathbb{N}$

$$\begin{aligned} ||x_{n+p} - x_n|| &\leq \sum_{k=1}^{p} ||x_{n+k} - x_{n+k-1}|| \leq ||x_{n+1} - x_n|| \sum_{k=0}^{p-1} \frac{1}{2^k} \leq \\ &\leq M||f(x_n)|| \sum_{k=0}^{p-1} \frac{1}{2^k} \leq M||f(x_0)||2^{-n} \sum_{k=0}^{p-1} \frac{1}{2^k} < \\ &< \frac{\beta}{2^n} \end{aligned} \tag{3}$$

and therefore $\{x_n\}$ is a Cauchy sequence in a B-space, so $\exists\ \bar{x}$ s. t. $x_n \to \bar{x}$, and from condition A) results in $||\bar{x} - x_0|| \leq \beta$ so $\bar{x}$ is in the neighborhood of x_0.

Taking now $p = 1$ in eq.(2) we obtain for any n

$$0 \leq M||f(x_n)|| \leq \frac{\beta}{2^n} \to 0,$$

so $f(\bar{x}) = 0$.

We prove the local uniqueness (in the neighborhood of the radius β of x_0) of $\bar{x}$. If $x^* \neq \bar{x}$ s. t.

$$||\bar{x} - x^*|| \leq \beta$$

and

$$f(\bar{x}) = f(x^*) = 0,$$

so, we will get

$$||\bar{x} - x^*|| = ||[f'(x_0)]^{-1} f'(x_0)(\bar{x} - x^*)|| \leq M||f'(x_0)(\bar{x} - x^*)||$$

and with Proposition 2.1

$$||f'(x_0)(\bar{x} - x^*)|| \leq \frac{1}{2M}||\bar{x} - x^*||,$$

choosing $M > 1$ is obtained

$$||\bar{x} - x^*|| \leq \frac{1}{2}||\bar{x} - x^*||,$$

which is a contradiction, so the theorem is proved. □

Other concerns for demonstrating the convergence of the Newton-Raphson method can be found in [12], [13], [17], [18].

Observation 2.4. An example of operator for which does not exist $[T'(x_0)]^{-1}$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ given through

$$T(x_1, x_2) = (x_1^2 + x_1 + x_2 + 1.5,\ x_1^2 + x_1 - x_2 - 1).$$

We

$$T'(x_1, x_2) = \begin{pmatrix} 2x_1 + 1 & 1 \\ 2x_1 + 1 & -1 \end{pmatrix}$$

for which

$$det\, T'(x_1, x_2) = -2x_1 - 1 - 2x_1 - 1 = -2(2x_1 + 1),$$

now taking $x_1 = -0.5$ and $x_2 = -1$ results that $T'(-0.5, -1)$ is not invertible. So in a simple case the situation can be degenerate; it follows that a theorem is needed to overcome such a difficulty.

The price paid is the restriction to the case of Hilbert spaces (or reflexive Banach at most).

The case is solved by Altman's Theorem, presented in the following section:

3 Convergence of the Newton-Raphson method if $[T'(x_0)]^{-1}$ does not exist at the initial point x_0 (Altman's Theorem)

Let H be a Hilbert space and $T : B(x_0, r) \to H$, differentiable in the Fréchet sense. We consider Newtonian iterations.

$$x_{n+1} = x_n - \frac{||T(x_n)||}{2||Q(x_n)||} Q(x_n), \quad n = 1, 2, \ldots$$

where $Q(x) = T'^*(x)T(x)$ (with T'^* the adjunct of T'). If $Q'(x)$ exists in $B(x_0, r)$ (and is bounded s. t. $||Q'(x)|| \le k, \quad \forall x \in B(x_0, r)$) then the conditions

a) $Q(x_0) = T'^*(x_0) \neq 0$,

b) $||Q'(x)|| < k, \quad \forall x \in B(x_0, r)$,

c) $\dfrac{||T(x_0)||^2}{||Q(x_0)||^2} k \le 2h_0 \le 1$,

where

$$r = \frac{1 - \sqrt{1 - 2h_0}}{h_0} \eta_0, \quad h_0 = B_0 \eta_0 k < \frac{1}{2}, \quad \frac{1}{||T(x_0)||} \le B_0,$$

$$||x_1 - x_0|| \le \frac{||T(x_0)||^2}{2||Q(x_0)||^2} \le \eta_0,$$

for B_0, η_0 subject to these conditions, implies

1) $T(x) = 0$ has a solution $\bar{x} \in B(x_0, r)$;

2) The string x_n converges to $\bar{x}$;

3) The speed of convergence is of the form

$$||x_n - \bar{x}|| \le \frac{1}{2^{n-1}} (2h_0)^{n-1} \eta_0 = h_0^{n-1} \eta_0.$$

For a complete demonstration, see [3]. □

Similar concerns of generalization and weakening of the imposed conditions can be found in [11], [14]

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