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ON M - PROJECTIVE CURVATURE TENSOR OF LORENTZIAN β - KENMOTSU MANIFOLD

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Dedicated to Professor Marin Marin on the occasion of his 70th anniversary

Abstract

In this paper, we explore the characteristics of Lorentzian β - Kenmotsu manifolds admitting M - projective curvature tensor. We demonstrate that M - projectively flat and irrotational M - projective curvature tensor of Lorentzian β - Kenmotsu manifolds are locally isometric to hyperbolic space $H^n(c)$, where $c = -\beta^2$. Further, we deal with the M - projectively flat Lorentzian β - Kenmotsu manifold satisfies the condition $R(X,Y) \cdot S = 0$. The Lorentzian β - Kenmostu manifold with conservative M - projective curvature tensor is the subject of our next analysis. Finally, we obtain certain geometrical facts if the Lorentzian β - Kenmotsu manifold satisfying the relation $M(X,Y) \cdot R = 0$.

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1 Introduction

If a differentiable manifold has a Lorentzian metric g, i.e., a symmetric non-degenerate (0,2) tensor field of index 1, then it is called a Lorentzian manifold.

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The notion of Lorentzian manifold was initially proposed by Matsumoto [14] in 1989. The same notion was independently studied by Mihai and Rosca [11]. Since then several geometers studied Lorentzian manifold and obtained a number of significant characteristics. Our present note deals with a special kind of manifold, i.e., Lorentzian β - Kenmotsu manifold. First, we provide an overview of the evolution of such manifold.

In [24], S. Tano categorised connected metric manifolds whose automorphisms groups possess the maximum dimension. The sectional curvature of a plane section containing such a manifold is a constant, let's say c. He demonstrated that they can be categorised into three cases: (I). Homogeneous normal contact Riemannian manifolds with $c > 0$: (II). Global Riemannian product of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature of $c = 0$ and: (III). A warped product space $\mathbf{R} \times_f \mathbf{C}$ if $c > 0$. It is know that the manifolds of class (I) are characterized by admitting a Sasakian structure. In the Gray-Hervella classification of almost Hermition manifolds which are closely related to locally conformal Kaehler manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [12, 13], if the product manifold $M\mathbf{R}$ belongs to the class W_4. The class $C_6 \otimes C_5$ [2] coincides with the class of the trans-Sasakian structures of type (α, β) in fact, in [12]. Local nature of the two subclasses, namely, C_5 and C_6, structures of trans-Sasakian structures are characterized completely. We note that trans-Sasakian structure of type $(0,0)$, $(0,\beta)$ and $(\alpha,0)$ are cosymplectic, β - Kenmotsu [19], and α-Sasakian [9,22], respectively. In [12], it is proved that trans-Sasakian structures are generalized quasi-Sasakian. Thus, trans-Sasakian structures also provide a large class of generalized quasi-Sasakian structures.

An almost contact metric manifold with structure tensors (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ a vector field, η a 1-form and g is a Riemannian metric on M is called trans-Sasakian structure [12, 13] if $(M \times \mathbf{R}, J, G)$ belongs to the class W_4 [8, 12] of the Hermitian structure, where J is the almost complex structure on $M \times \mathbf{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)f\frac{d}{dt}),$$

for every vector field X on M and smooth function f on $M \times \mathbf{R}$, and G is the product metric on $M \times \mathbf{R}$. This can come across in the condition

$$(\nabla_X \phi)Y = \alpha\left\{g(X,Y)\xi - \eta(Y)X\right\} + \beta\left\{g(\phi X,Y)\xi - \eta(Y)\phi X\right\},$$

for some smooth functions α and β on M, and we express that the trans-Sasakian structure is of type (α, β).

Theorem 1 (See [1]). *A trans-Sasakian structure of type (α, β) with β a non-zero constant is always β - Kenmotsu.*

In this scenario β becomes a constant. If $\beta = 1$, then β - Kenmotsu manifold is Kenmotsu.

Definition 1. *The M - projective curvature tensor of Riemannian manifold M^n was defined by Pokhariyal and Mishra [18] is of the following form:*

$$M(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY], \quad (1)$$

where, Q is the Ricci operator defined on $S(X,Y) = g(QX,Y)$.

A space form (i.e., a complete simply connected Riemannian manifold of constant curvature) is said to be elliptic, hyperbolic or Euclidean according as the sectional curvature tensor is positive, negative or zero [5]. The authors extensively studied the properties of M - projective curvature tensor on the various manifolds (see, [4, 6, 10, 16, 17, 20, 21, 25–27]). In this paper, we have studied some special properties of Lorentzian β - Kenmotsu manifold.

The purpose of this paper is to study the properties of M - projective curvature tensor in Lorentzian β - Kenmotsu manifolds. The paper is organized as follows: Section 2 is concerned with preliminaries of Lorentzian β - Kenmotsu manifolds. In section 3, we study the M - projectively flat of Lorentzian β - Kenmotsu manifold. Section 4 deals with the M - projectively flat Lorentzian β - Kenmotsu manifold satisfies the condition $R(X,Y)\cdot S = 0$. In section 5, we study conservative M - projective curvature tensor of Lorentzian β - Kenmotsu manifold. In section 6, irrotational M - projective curvature tensor of Lorentzian β - Kenmotsu manifold are studied. Section 7 is devoted with study of Lorentzian β - Kenmotsu manifold satisfies the condition $M(X,Y) \cdot R = 0$.

2 Preliminaries

In this section, we briefly recall some general definitions of Lorentzian β - Kenmotsu manifold:

A $(2n+1)$ - dimensional differentiable manifold M is called Lorentzian β - Kenmotsu manifold if it admits a (1, 1)-tensor field ϕ, a vector field ξ, a 1-form η and Lorentzian metric g which satisfy the conditions

$$\phi^2 X = X + \eta(X)\xi, \qquad g(X,\xi) = \eta(X), \quad (2)$$

$$\eta\xi = -1, \qquad \phi\xi = 0, \qquad \eta(\phi X) = 0, \quad (3)$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \quad (4)$$

for all $X, Y \in \chi(M^n)$, where $\chi(M^n)$ is the Lie algebra of smooth vector fields on M^n. Also a Lorentzian β - Kenmotsu manifold M is satisfying

$$\nabla_X \xi = \beta[X - \eta(X)\xi], \quad (5)$$

$$(\nabla_X \eta)(Y) = \beta[g(X,Y) - \eta(X)\eta(Y)], \quad (6)$$

$$(\nabla_X \phi)(Y) = \beta[g(\phi X, Y)\xi - \eta(Y)\phi X], \quad (7)$$

where, ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. Further, on a Lorentzian β - Kenmotsu manifold M the following relations hold (See [1, 19])

$$\eta(R(X,Y)Z) = \beta^2[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)], \tag{8}$$

$$R(X,Y)Z = \beta^2[g(X,Z)Y - g(Y,Z)X], \tag{9}$$

$$R(\xi,X)Y = \beta^2(\eta(Y)X - g(X,Y)\xi), \tag{10}$$

$$R(X,Y)\xi = \beta^2(\eta(X)Y - \eta(Y)X), \tag{11}$$

$$S(X,\xi) = -(n-1)\beta^2\eta(X), \tag{12}$$

$$QX = -(n-1)\beta^2 X, \quad Q\xi = -(n-1)\beta^2\xi, \tag{13}$$

$$S(\xi,\xi) = (n-1)\beta^2, \tag{14}$$

$$g(QX,Y) = S(X,Y) = -(n-1)\beta^2 g(X,Y), \tag{15}$$

$$S(\phi X,\phi Y) = S(X,Y) - (n-1)\beta^2\eta(X)\eta(Y), \tag{16}$$

for any vector fields X, Y, Z on M, where R, S and Q denotes the curvature tensor, Ricci tensor and Ricci operator on M.

Definition 2. *A Lorentzian β - Kenmotsu manifold M is said to be η-Einstein manifold if it's Ricci tensor S is of the form*

$$S(X,Y) = \lambda_1 g(X,Y) + \lambda_2\eta(X)\eta(Y), \tag{17}$$

for any vector fields X, Y, where λ_1, λ_2 are smooth functions on M.

If $\lambda_2 = 0$, then M is an Einstein manifold.

In view of (2) and (17), we have

$$QX = \lambda_1 X + \lambda_2\eta(X)\xi. \tag{18}$$

Let us consider Lorentzian β - Kenmotsu manifold. Then putting $X = Y = e_i$ in (17), $i = 1, 2, ..., n$ and taking summation for $1 \leq i \leq n$, we have

$$r = n\lambda_1 - \lambda_2. \tag{19}$$

Now, putting $X = Y = \xi$ in (17) and using (2), (3) and (12), we obtain

$$\lambda_2 - \lambda_1 = (n-1)\beta^2. \tag{20}$$

From the condition (19) and (20), we have

$$\lambda_1 = \frac{r}{(n-1)} + \beta^2 \quad \text{and} \quad \lambda_2 = \frac{r}{(n-1)} + n\beta^2, \tag{21}$$

where, r is the scalar curvature.

In view of (8)-(11), it can be easily constructed that in n - dimensional Lorentzian β - Kenmotsu manifold M^n, the M - projective curvature tensor satisfies the following condition from (1.1):

$$M(X,Y)\xi = \frac{\beta^2}{2}\left\{\eta(X)Y - \eta(Y)X\right\} - \frac{1}{2(n-1)}\left\{\eta(Y)QX - \eta(X)QY\right\}, \tag{22}$$

$$M(\xi, X)Y = \frac{\beta^2}{2}\{\eta(Y)X - g(X,Y)\xi\} - \frac{1}{2(n-1)}\{S(X,Y)\xi - \eta(Y)QX\}, \quad (23)$$

$$\eta(M(X,Y)Z) = \frac{\beta^2}{2}\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\} - \frac{1}{2(n-1)}\{S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\}. \quad (24)$$

The facts above will be applied in a subsequent section.

3 M - projectively flat Lorentzian β - Kenmotsu manifold

In this section, we study M - projectively - flat in Lorentzian β - Kenmotsu manifold.

Definition 3. *A Lorentzian β - Kenmotsu manifold M^n is said to be M - projectively flat if*

$$M(X,Y)Z = 0,$$

for any vector fields X, Y, Z on M^n.

By virtue of Definition 3 in (1), we have

$$R(X,Y)Z = \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY], \quad (25)$$

Taking $Z = \xi$ in (25) and using (3), (11) and (12), we obtain

$$\beta^2[\eta(X)Y - \eta(Y)X] = \frac{1}{n-1}[\eta(Y)QX - \eta(X)QY]. \quad (26)$$

Again, putting $Y = \xi$ in (26) and using relation (2), (3) in (12), we get

$$QX = -(n-1)\beta^2 X,$$

which on simplification gives,

$$S(X,Y) = -(n-1)\beta^2 g(X,Y). \quad (27)$$

which yields,

$$r = -n(n-1)\beta^2. \quad (28)$$

Hence, we state the following theorem:

Theorem 2. *If an n - dimensional Lorentzian β - Kenmotsu manifold M^n is M - Projectively flat, then it is an Einstein manifold and Ricci tensor of M has the form $S(X,Y) = -(n-1)\beta^2 g(X,Y)$.*

In consequence of (27), (25) becomes

$$R(X,Y)Z = -\beta^2 \{g(Y,Z)X - g(X,Z)Y\}. \tag{29}$$

A space form is said to be hyperbolic if the sectional curvature tensor is negative [5]. Thus, we can state

Theorem 3. *If an n - dimensional Lorentzian β - Kenmotsu manifold M^n is M - Projectively flat, then it is either locally isometric to the hyperbolic space $H^n(c)$, where $c = -\beta^2$ or M^n has constant scalar curvature $-n(n-1)\beta^2$.*

4 M - projectively flat Lorentzian β - Kenmotsu manifold satisfying $R(X,Y)\cdot S = 0$ condition

In this section, we consider that M^n is an M - projectively flat Lorentzian β - Kenmotsu manifold (M^n, g) satisfying the condition $R(X,Y)\cdot S = 0$. Thus we have

$$S(R(X,Y)Z,U) + S(Z,R(X,Y)U) = 0. \tag{30}$$

In view of (25) in (30), we have

$$\frac{1}{2(n-1)}[S(QX,U)g(Y,Z) - S(QY,U)g(X,Z) + S(QX,Z)g(Y,U) - S(QY,Z)g(X,U)] = 0. \tag{31}$$

Putting $Y = Z = \xi$ in (31) and using (2), (3) and (12), we obtain

$$[S(QX,U) + \eta(X)S(Q\xi,U) - \eta(U)S(QX,\xi) + g(X,U)S(Q\xi,\xi)] = 0. \tag{32}$$

Again, using (12) in (32), we have

$$-S(QX,U) - (n-1)^2\beta^4\eta(U)\eta(X) + \eta(U)S(QX,\xi) + (n-1)^2\beta^4 g(X,U) = 0. \tag{33}$$

Let λ be the eigen value of the endomorphism Q corresponding to an eigen vector X. Then putting $QX = \lambda X$ in (33) and using relation $g(QX,Y) = S(X,Y)$, we find that

$$-\lambda^2 g(X,U) - (n-1)\lambda\beta^2\eta(X)\eta(U) - (n-1)^2\beta^4\eta(U)\eta(X) + (n-1)^2\beta^4 g(X,U) = 0. \tag{34}$$

Now, putting $U = \xi$ in (34), we get

$$[\lambda^2 - (n-1)\beta^2\lambda - 2(n-1)^2\beta^4]\eta(X) = 0. \tag{35}$$

In this case, since $\eta(X) \neq 0$, the relation (35) gives that

$$\lambda^2 - (n-1)\beta^2\lambda - 2(n-1)^2\beta^4 = 0. \tag{36}$$

From above equation it follows that the endomorphism Q has two different non-zero eigen values, namely, $-(n-1)\beta^2$ and $2(n-1)\beta^2$.

Hence, we state the following theorem:

Theorem 4. *Let M^n be an n - dimensional M - Projectively flat Lorentzian β - Kenmotsu manifold satisfies $R(X,Y)\cdot S=0$, then symmetric endomorphism Q of the tangent space corresponding to S has two different non-zero eigen values.*

5 Conservative M - projective curvature tensor on Lorentzian β - Kenmotsu manifold

Definition 4. *Lorentzian β - Kenmotsu Manifold (M^n, g) is said to be M - projective conservative if*

$$divM = 0, \tag{37}$$

where div denotes the divergence.

Taking the covariant derivative of (1), we get

$$(\nabla_U M)(X,Y)Z = (\nabla_U R)(X,Y)Z - \frac{1}{2(n-1)}[(\nabla_U S)(Y,Z)X - (\nabla_U S)(X,Z)Y + g(Y,Z)(\nabla_U Q)X - g(X,Z)(\nabla_U Q)Y]. \tag{38}$$

Contracting with respect to U in (38), we obtain

$$(divM)(X,Y)Z = (divR)(X,Y)Z - \frac{1}{2(n-1)}[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) + g(Y,Z)divQX - g(X,Z)divQY]. \tag{39}$$

We know that

$$divQ(X) = \frac{1}{2}\nabla_X r. \tag{40}$$

By virtue of (40) in (39), we obtain

$$(divM)(X,Y)Z = (divR)(X,Y)Z - \frac{1}{2(n-1)}[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) + \frac{1}{2}g(Y,Z)\nabla_X r - \frac{1}{2}g(X,Z)\nabla_Y r]. \tag{41}$$

But from [7], we have

$$divR = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \tag{42}$$

Again, by virtue of (37) and (42) in (41), it reduces to

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{2(2n-3)}\left\{g(Y,Z)\nabla_X r - g(X,Z)\nabla_Y r\right\}. \tag{43}$$

Putting $X=\xi$ in (43), we get

$$(\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi,Z) = \frac{1}{2(2n-3)}\left\{g(Y,Z)\nabla_\xi r - g(\xi,Z)\nabla_Y r\right\}. \tag{44}$$

Further, we know that

$$\begin{aligned}(\nabla_\xi S)(X,Y) &= \xi S(X,Y) - S(\nabla_\xi X,Y) - S(X,\nabla_\xi Y)\\ &= \xi S(X,Y) - S([\xi,X]+\nabla_X\xi,Y) - S(X,[\xi,Y]+\nabla_Y\xi)\\ &= \xi S(X,Y) - S([\xi,X],Y) - S(\nabla_X\xi,Y)\\ &\qquad\qquad - S(X,[\xi,Y]) - S(X,\nabla_Y\xi)\\ &= (\pounds_\xi S)(X,Y) - S(\nabla_X\xi,Y) - S(X,\nabla_Y\xi).\end{aligned} \tag{45}$$

The Lie derivative of metric g along with vector field X is

$$(\pounds_X g)(Y,Z) = \pounds_X g(Y,Z) - g(\pounds_X Y,Z) - g(Y,\pounds_X Z). \tag{46}$$

Putting $X=\xi$ in (46) and using (5), we obtain

$$(\pounds_\xi g)(Y,Z) = 2\beta[g(Y,Z) - \eta(Y)\eta(Z)]. \tag{47}$$

Notice that $g(QX,Y)=S(X,Y)$ and using relation (47), we get

$$(\pounds_\xi S)(Y,Z) = 2\beta[S(Y,Z) + (n-1)\beta^2\eta(Y)\eta(Z)]. \tag{48}$$

Making use of (5) and (48) in (45), we get

$$(\nabla_\xi S)(Y,Z) = 0, \tag{49}$$

which yields

$$\nabla_\xi r = 0. \tag{50}$$

In view of (44) and making use of (3), (4), (5), (49) and (50), we obtain

$$S(Y,Z) + (n-1)\beta^2 g(Y,Z) = \frac{-1}{2\beta(2n-3)}\eta(Z)dr(Y). \tag{51}$$

Interchanging Z by ϕZ in (51) and using (3), (4) and (16), we get

$$S(Y,Z) = -(n-1)\beta^2 g(Y,Z). \tag{52}$$

Contracting the equation (52), we have

$$r = -n(n-1)\beta^2. \tag{53}$$

Hence, we state the following:

Theorem 5. *Let M^n be an n - dimensional M - Projective curvature tensor of Lorentzian β - Kenmotsu manifold is conservative, then M^n is an Einstein manifold and Ricci tensor of M has the form $S(Y,Z) = -(n-1)\beta^2 g(Y,Z)$.*

6 Irrotational M - projective curvature tensor of Lorentzian β - Kenmotsu manifold

Definition 5. *The rotation (curl) of M - projective curvature tensor on a Lorentzian β - Kenmotsu manifold M^n is defined as*

$$RotM = (\nabla_U M)(X,Y)Z + (\nabla_X M)(U,Y)Z + (\nabla_Y M)(X,U)Z - (\nabla_Z M)(X,Y)U. \quad (54)$$

In consequence of Bianchi second identity for Riemannian connection ∇, (54) becomes

$$RotM = -(\nabla_Z M)(X,Y)U. \quad (55)$$

If the M - projective curvature tensor is irrotational, then curl M= 0 and so by (55), we get

$$(\nabla_Z M)(X,Y)U = 0,$$

which gives

$$\nabla_Z(M(X,Y)U) = M(\nabla_Z X,Y)U + M(X,\nabla_Z Y)U + M(X,Y)\nabla_Z U. \quad (56)$$

Putting $U = \xi$ in (56), we obtain

$$\nabla_Z(M(X,Y)\xi) = M(\nabla_Z X,Y)\xi + M(X,\nabla_Z Y)\xi + M(X,Y)\nabla_Z \xi. \quad (57)$$

Now, substituting $Z = \xi$ in (1) and using (2), (3), (11), (12) and (18), we obtain

$$M(X,Y)\xi = \lambda[\eta(Y)X - \eta(X)Y], \quad (58)$$

where,

$$\lambda = \frac{1}{2(n-1)^2}[n(n-1)\beta^2 + r]. \quad (59)$$

By virtue of (58) and (5) in (57), we have

$$M(X,Y)Z = \lambda[g(Z,X)Y - g(Z,Y)X)]. \quad (60)$$

In view of (1) and (60), we have

$$\lambda[g(Z,X)Y - g(Z,Y)X] = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \quad (61)$$

Contracting (61) over X and using (59), we get

$$S(Y,Z) = -(n-1)\beta^2 g(Y,Z). \quad (62)$$

From (62), we find that

$$r = -n(n-1)\beta^2. \quad (63)$$

Thus, we state the following theorem:

Theorem 6. *If the M - projective curvature tensor in a Lorentzian β - Kenmotsu manifold M^n is irrotational, then the manifold M^n is an Einstein manifold with constant scalar curvature $-n(n-1)\beta^2$.*

7 Lorentzian β - Kenmotsu manifold satisfying $M \cdot R = 0$ condition

Consider an Lorentzian β - Kenmotsu manifold satisfying the condition

$$M \cdot R = 0. \quad (64)$$

Then from the relation (64), it follows that

$$M(\xi, X)R(Y, Z)U - R(M(\xi, X)Y, Z)U - R(Y, M(\xi, X)Z)U - R(Y, Z)M(\xi, X)U = 0. \quad (65)$$

In view of (23), it follows from (65) that

$$\frac{\beta^2}{2}[\eta(R(Y, Z)U)X - g(X, R(Y, Z)U)\xi - \eta(Y)R(X, Z)U + g(X, Y)R(\xi, Z)U - \eta(Z)R(Y, X)U + g(X, Z)R(Y, \xi)U - \eta(U)R(Y, Z)X + g(X, U)R(Y, Z)\xi] - \frac{1}{2(n-1)}[g(R(Y, Z)U, QX)\xi - \eta(R(Y, Z)U)QX - S(X, Y)R(\xi, Z)U + \eta(Y)R(QX, Z)U - S(X, Z)R(Y, \xi)U + \eta(Z)R(Y, QX)U - S(X, U)R(Y, Z)\xi + \eta(U)R(Y, Z)QX] = 0. \quad (66)$$

Taking the inner product of (66) with ξ and on simplification, we obtain

$$\frac{\beta^2}{2}R'(Y, Z, U, X) - \frac{\beta^2}{2}[\eta(Y)\eta(R(X, Z)U) - g(X, Y)\eta(R(\xi, Z)U) + \eta(Z)\eta(R(Y, X)U) - g(X, Z)\eta(R(Y, \xi)U) + \eta(U)\eta(R(Y, Z)X) - g(X, U)\eta(R(Y, Z)\xi)] + \frac{1}{2(n-1)}R'(Y, Z, U, QX) + \frac{1}{2(n-1)}[S(X, Y)\eta(R(\xi, Z)U) - \eta(Y)\eta(R(QX, Z)U) + S(X, Z)\eta(R(Y, \xi)U) - \eta(Z)\eta(R(Y, QX)U) + S(X, U)\eta(R(Y, Z)\xi) - \eta(U)\eta(R(Y, Z)QX)] = 0. \quad (67)$$

Making use of (8), (10) and (11) in (67), we have

$$\frac{\beta^2}{2}R'(Y, Z, U, X) + \frac{1}{2(n-1)}R'(Y, Z, U, QX) + \frac{\beta^4}{2}[g(X, Y)g(Z, U) - g(X, Z)g(Y, U)] + \frac{\beta^2}{2(n-1)}[S(X, Y)g(Z, U) - S(X, Z)g(Y, U)] = 0. \quad (68)$$

Let $\{e_i : i = 1, 2, ..., n\}$ be an orthonormal frame field at any point of the manifold. Then contracting $Z = U = e_i$ in (68) and on simplification, we have

$$S^2(X, Y) = -(n-1)\beta^2[2S(X, Y) + (n-1)\beta^2 g(X, Y)]. \quad (69)$$

Therefore, the S^2 of the Ricci tensor S is the linear combination of the Ricci tensor and the metric tensor g. Here the $(0,2)$-tensor S^2 is defined by $S^2(X,Y) = S(QX,Y)$.

Hence, we state the following:

Theorem 7. *Let M be an n - dimensional Lorentzian β - Kenmotsu manifold is satisfying the condition $M \cdot R = 0$. Then the S^2 of the Ricci tensor S is the linear combination of the Ricci tensor and the metric tensor g has the form* $S^2(X,Y) = -(n-1)\beta^2 \left\{2S(X,Y) + (n-1)\beta^2 g(X,Y)\right\}$.

It is well known that:

Lemma 1 (See [23])**.** *If $\theta = g \wedge A$ be the Kulkarni-Nomizu product of g and A, where g being Riemannian metric and A be a symmetric tensor of type $(0,2)$ at point x of a semi-Riemannian manifold (M^n, g). Then relation $\theta \cdot \theta = \rho Q(g,\theta), \rho \in \mathbb{R}$ is true at x if and only if the condition $A^2 = \rho A + \mu g, \mu \in \mathbb{R}$ holds at x.*

In consequence of Theorem 7 and Lemma 1, we have the following Corollary:

Corollary 1. *Let M be an n - dimensional Lorentzian β - Kenmotsu manifold satisfying the condition $M \cdot R = 0$, then $\theta \cdot \theta = \rho Q(g,\theta), \rho \in \mathbb{R}$, where $\theta = g \wedge S$ and $\rho = -2(n-1)\beta^2$.*

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