Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 4(66), No. 2 - 2024, 185-200
https://doi.org/10.31926/but.mif.2024.4.66.2.11

MATRIX REPRESENTATION OF $(d, k)-$FIBONACCI POLYNOMIALS

Bahar KULOĞLU[1] **and Engin ÖZKAN** [*,2]

Dedicated to Professor Marin Marin on the occasion of his 70th anniversary

Abstract

In this study, we define the $(d, k)-$ Fibonacci polynomial and examine its properties. We give the generating function, characteristic equation, and matrix representation of this polynomial. Then we get the infinite sum for the $(d, k)-$ Fibonacci polynomials. We give the relationship between $(d, k)-$ Fibonacci polynomial and $d-$ Fibonacci polynomial. Also, with the help of $(d, k)-$ Fibonacci polynomial matrix representation and the Riordan matrix, the factorization of the Pascal matrix in two different ways is given. In addition, we define the infinite $(d, k)-$ Fibonacci polynomial matrix and give their inverses. The Riordan arrays linked here help us understand patterns of number concepts and prove many theorems, as well as helping us make an intuitive connection for solving combinatorial problems. Among our main goals is to combine Riordan arrays with the Fibonacci number sequence, which is the most important of the number sequences, and to expand this study to the k-Fibonacci number sequence, which is the general form of Fibonacci number sequences. Based on the information given above, Riordan array and Pascal matrices, which have an important place in matrix theory and combinatorics studies also it derived an encoding of Pascal's triangle in matrix form, were discussed in this study and a very different generalization of the Fibonacci number sequence was studied.

2020 Mathematics Subject Classification: 11B39, 15A09, 15A15, 11B83, 11C08 .

Key words: $k-$ Fibonacci polynomials, generating function, Pascal matrix, Riordan matrix. .

[1]Hatuniye Vocational and Technical Anatolian High School, Ministry of Education, Mardin, Turkey e-mail: bahar_kuloglu@hotmail.com

[2*] *Corresponding author* Marmara University, Faculty of Sciences, Department of Mathematics, İstanbul, TÜRKİYE , e-mail: engin.ozkan@marmara.edu.tr

1 Introduction

The importance of number sequences, their contribution to science and studies in this field have been the subject of study for scientists with ongoing interest for years. Although there are many number sequences that have been the subject of lately, as it is known, the most famous of them is the Fibonacci number sequence.

These numbers, which were written on the basis of the reproduction of rabbits, which Fibonacci gave her name, and obtained by adding the two terms before the next term, have almost revolutionized the history of science.

Because of their importance, many authors have done many studies on both these numbers and their generalization [13, 14, 15, 12, 19, 21, 5, 8, 9].

We know that the Fibonacci numbers F_n are given as follows

$$F_n = F_{n-1} + F_{n-2}, n \geq 2$$

with $F_0 = 0$ and $F_1 = 1$ [9].

Riordan arrays are formed by lower triangular matrices, each column of which is generated by formal power series and formed by two functions. It is known that the first studies on Riordan arrays started with John Riordan. It is known that John Riordan was the pioneer of combinatorial studies.

Riordan arrays show themselves in many fields of science together with known matrix studies. Riordan arrays and Riordan matrices, which are the structural combination of matrices that play a leading role in coding and decoding studies, have been a pioneering work in many fields from analysis, which is many sub-fields of computation, to error verification codes and wireless communication, since the 1990s. In addition, Riordan arrays have taken an active role in fields such as molecular biology, RNA structure and chemistry by pushing the boundaries of mathematics in many different fields of science beyond these studies.

Among our main goals is to combine Riordan arrays with the Fibonacci number sequence, which is the most important of the number sequences, and to expand this study to the k-Fibonacci number sequence, which is the general form of Fibonacci number sequences.

Studies in this field have been discussed by many authors as it has an important place in the scientific field.

Tian-Xiao [23] examined the Riordan array characterization of matrices formed as two matrix characterizations, P-matrix and A-matrix. Wang et al. [24] emphasized that generalized Riordan arrays have the same properties as classical Riordan arrays. Kılıç [8] obtained the products of infinite generalized Pascal matrices with the Riordan group approximation of the arbitrary binary polynomial sequence taken. Shapiro et al. [21] created q-analogues of Riordan arrays and named them q-Riordan arrays. Alp and Koçer [1] have developed a different approach to number sequences by dealing with Leonardo and Hyper-Leonardo numbers through Riordan arrays, and studies in many other fields show themselves together with Riordan matrix and group structures in [19, 20, 3].

Fibonacci numbers have great importance in many areas such as mathematics, physics, biology, statistics, etc. Falcon et al. [5, 6] presented a general Fibonacci sequence.

In [13], Nalli and Haukkanen defined $h(x)-$ Fibonacci and Lucas polynomials.

Created with elements taken in complex numbers, Consider the infinite matrix S and B is generating function for matrix S;

$$S = (s_{ij})_{i,j>0}$$

$$B_i(x) = \sum_{n\geq 0}^{\infty} s_{ij} x^n$$

and

$$B_i(x) = m(x)[n(x)]^i$$

$$m(x) = 1 + m_1 x + m_2 x^2 + m_3 x^3 + \ldots$$

and

$$n(x) = x + n_2 x^2 + n_3 x^3 + \ldots.$$

Here, if we write it as $S = (m(x), n(x))$, we call $(m(x), n(x))$ a Riordan matrix. See [20] for more details.

Riordan matrices $F(x) = (g(x), f(x))$ which are considered from the set of matrices and whose elements are complex numbers were first discussed by Shapiro in 1991 [20].

The Riordan groups are given as the set of Riordan matrices where $g(z) = \sum_{n=0}^{\infty} g_n z^n$, $f(z) = \sum_{n=0}^{\infty} f_n z^n$with $g_0 \neq 0$ and $f_1 \neq 1$ [20].

Whereas Riordan groups consist of infinite lower triangular matrices defined by two functions f and g, Double Riordan groups have two generating functions such as f_1 and f_2 and they also have generating functions f_1 and f_2 in columns starting from the left. The important point to emphasize here is that Double Riordan groups are not a generalization of Riordan groups, but a generalization of the checkerboard subgroup. Davenport et al., Cameron et al., Donaghey and Shapiro have done many effective studies in this area [2, 3, 4].

There are many areas that can be expanded using Riordan groups. determinant sequences, division properties, $Z(x) = \sum_{n=0}^{\infty} z_n x^n / d_n$ form for various d_n sequences and many more examples can be given to these fields.

If we accept $d_n = n!$ as [18], Roman has mentioned many examples that can be extended to Riordan groups.

In the following years, such studies were discussed in many different series.

If we mention some of them, Özkan et al. [11, 16, 17] examined d-Gauss Pell, d-Gauss Fibonacci and d-Gauss Lucas polynomials and d-Jacobsthal, d-Jacobsthal Lucas polynomials dealt with Riordan groups and matrices in polynomials containing complex numbers. In addition, Kuloğlu [10] generalized these studies a little more and added a different dimension to the studies by dealing with the Tribonacci sequences, which are the number sequence with three recurrences.

2 A generalization of Fibonacci polynomials

Definition 1. $(d,k)-$ *Fibonacci polynomial is given in [19]*

$$F_{k,n}(x) = q_1(x)F_{k,n-1}(x) + q_2(x)F_{k,n-2}(x) + \cdots + q_{d+1}(x)F_{k,n-d-1}(x) \quad (1)$$

with $F_{k,1}(x) = k,\ F_{k,n}(x) = 0$ *for* $n \leq 0$, *where* $q_i(x)$ *is a polynomial with real coefficient,* $1 \leq i \leq d+1$.

A few terms for these polynomials:

n	$F_{k,n}(x)$
0	$F_{k,0}(x) = 0$
1	$F_{k,1}(x) = k$
2	$F_{k,2}(x) = q_1(x)k$
3	$F_{k,3}(x) = q_1^2(x)k + q_2(x)k$
4	$F_{k,4}(x) = q_1^3(x)k + 2q_1(x)q_2(x)k + q_3(x)k$

From equation (1), the characteristic equation;

$$s^{d+1} - q_1(x)s^d - \cdots - q_{d+1}(x) = 0$$

where $\delta_1(x), \delta_2(x), \ldots, \delta_{d+1}(x)$ are roots of the equation.

Theorem 1. *Generating function of* $F_{k,n}(x)$ *is*

$$F_k(x,s) = \sum_{n=0}^{\infty} F_{k,n}(x)s^n = \frac{ks}{(1 - q_1(x)s - \cdots - q_{d+1}(x)s^{d+1})}.$$

Proof. We have

$$F_k(x,s) = F_{k,0}(x) + F_{k,1}(x)s + F_{k,2}(x)s^2 + \ldots \quad (2)$$

Multiply Eq. (2) by $q_1(x)s,\ q_2(x)s^2,\ \ldots\ , q_{d+1}(x)s^{d+1}$, respectively. We obtain

$$\begin{aligned}
q_1(x)sF_k(x,s) &= q_1(x)sF_{k,0}(x) + q_1(x)s^2F_{k,1}(x) + \ldots \\
q_2(x)s^2F_k(x,s) &= q_2(x)s^2F_{k,0}(x) + q_2(x)s^3F_{k,1}(x) + \ldots \\
&\vdots \\
q_{d+1}(x)s^{d+1}F_k(x,s) &= q_{d+1}(x)s^{d+1}F_{k,0}(x) + q_{d+1}(x)s^{d+2}F_{k,1}(x) + \ldots
\end{aligned}$$

If the necessary calculations are made, then

$$\begin{aligned}
F_k(x,s)\left[1 - q_1(x)s - \cdots - q_{d+1}(x)s^{d+1}\right] \\
&= F_{k,0}(x) + s(F_{k,1}(x) - q_1(x)F_{k,0}(x)) \\
&\quad + s^2(F_{k,2}(x) - q_1(x)F_{k,1}(x) - q_2(x)F_{k,0}(x)) + 0 + \ldots
\end{aligned}$$

$$F_k\left(x,s\right)=\frac{ks}{\left(1-q_1\left(x\right)s-\cdots-q_{d+1}\left(x\right)s^{d+1}\right)}.$$

We know that Binet formula similarly in [19] for $F_{k,n}(x)$ has the form

$$F_{k,n}(x)=\sum_{i=1}^{d+1}K_i(x)(\delta_i(x))^n.$$

We get the following equation for each value of n.

$$\begin{aligned}
F_{k,0}(x)&=\sum_{i=1}^{d+1}K_i(x)\\
F_{k,1}(x)&=\sum_{i=1}^{d+1}K_i(x)(\delta_i(x))^1\\
&\vdots\\
F_{k,n}(x)&=\sum_{i=1}^{d+1}K_i(x)(\delta_i(x))^n
\end{aligned}$$

So, we can write the following equations

$$\begin{aligned}
F_{k,0}(x)&=\sum_{i=1}^{d+1}K_i(x)\\
sF_{k,1}(x)&=\sum_{i=1}^{d+1}K_i(x)(\delta_i(x))^1 s\\
&\vdots\\
s^nF_{k,n}(x)&=\sum_{i=1}^{d+1}K_i(x)(\delta_i(x))^n s^n
\end{aligned}$$

The sum of the left-hand sides of above equations

$$\frac{ks}{\left(1-q_1\left(x\right)s-\cdots-q_{d+1}\left(x\right)s^{d+1}\right)}.$$

The sum of the right-hand sides of above equations

$$\sum_{i=1}^{d+1}K_i\left(x\right)\left[1+(\delta_i(x))^1 s+\cdots+(\delta_i(x))^n s^n\right]=\sum_{i=1}^{d+1}K_i(x)\left(\frac{1}{1-\delta_i(x)s}\right).$$

Thus we have

$$\frac{ks}{\left(1-q_1\left(x\right)s-\cdots-q_{d+1}\left(x\right)s^{d+1}\right)}=\sum_{i=1}^{d+1}\left(\frac{K_i(x)}{1-\delta_i(x)s}\right). \qquad \square$$

Theorem 2. *For $n \geq 0$, we have*

$$F_{k,n}(x) = k\sum_{n_1+2n_2+\cdots+(d+1)n_{d+1}=n} \binom{n_1+n_2+\cdots+\ n_{d+1}}{n_1, n_2, \ldots,\ n_{d+1}} {q_1}^{n_1}(x)\, {q_2}^{n_2}(x) \ldots {q_{d+1}}^{n_{d+1}}(x).$$

Proof. Generating function for $(d,k)-$ Fibonacci polynomials are found with the help of multinomial coefficient for the number. For any positive integer m and any non-negative integer n, the multinomial formula describes how a sum with m terms expands when raised to an arbitrary power n:

$$\sum_{n=0}^{\infty} F_{k,n+1}(x)s^n = \frac{k}{(1 - q_1(x)\, s - \cdots - q_{d+1}(x)\, s^{d+1})}$$

$$= k\sum_{n=0}^{\infty}\left(q_1(x)\, s + \cdots + q_{d+1}(x)\, s^{d+1}\right)^{n+2}$$

$$= k\sum_{n=0}^{\infty}\left(\sum_{n_1+n_2+\cdots+\ n_{d+1}=n}^{\infty}\left[\binom{n}{n_1, n_2, \ldots,\ n_{d+1}} {q_1}^{n_1}(x) \ldots {q_{d+1}}^{n_{d+1}}(x)\right] s^{n_1+2n_2+\cdots+(d+1)n_{d+1}}\right)$$

$$= k\sum_{n=0}^{\infty}\left(\sum_{n_1+2n_2+\cdots+(d+1)n_{d+1}=n}\binom{n_1+n_2+\cdots+\ n_{d+1}}{n_1, n_2, \ldots,\ n_{d+1}} {q_1}^{n_1}(x) \ldots {q_{d+1}}^{n_{d+1}}(x)\right) s^n$$

as desired. $\square$

Theorem 3. *The sum $SF_{k,n}(x)$ of the $(d,k)-$ Fibonacci polynomials is as follows*

$$SF_{k,n}(x) = \sum_{n=0}^{\infty} F_{k,n}(x) = \frac{k}{1 - q_1(x) - \cdots - q_{d+1}(x)}.$$

Proof. We have

$$SF_{k,n}(x) = \sum_{n=0}^{\infty} F_{k,n}(x).$$

Then we obtain

$$\begin{aligned}
q_1(x)\ SF_{k,n}(x) &= q_1(x)\, F_{k,0}(x) + \ldots + q_1(x)\, F_{k,n}(x) + \ldots\\
q_2(x)\, SF_{k,n}(x) &= q_2(x)\, F_{k,0}(x) + \ldots + q_2(x)\, F_{k,n}(x) + \ldots\\
&\vdots\\
q_{d+1}(x)\, SF_{k,n}(x) &= q_{d+1}(x)\, F_{k,0}(x) + \ldots + q_{d+1}(x)\, F_{k,n}(x) + \ldots.
\end{aligned}$$

From the last equations, we get

$$SF_{k,n}(x)\,(1 - q_1(x) - \cdots - q_{d+1}(x)) = k$$

Thus, we have

$$SF_{k,n}(x)=\sum_{n=0}^{\infty}F_{k,n}(x)=\frac{k}{1-q_1(x)-\cdots-q_{d+1}(x)}.$$

From [19], matrix Q_d for the $d-$ Fibonacci polynomial is defined by

$$Q_d=\begin{pmatrix} q_1(x) & q_2(x) & \cdots & q_{d+1}(x)\\ 1 & 0 & \cdots & 0\\ 0 & \ddots & & \vdots\\ \vdots & & \ddots & \vdots\\ 0 & 0 & 1 & 0\end{pmatrix} \tag{3}$$

Then we have

$$kQ_d=F_d=\begin{pmatrix} q_1(x)k & q_2(x)k & \cdots & q_{d+1}(x)k\\ k & 0 & \cdots & 0\\ 0 & \ddots & & \vdots\\ \vdots & & \ddots & \vdots\\ 0 & 0 & k & 0\end{pmatrix}$$

where $detQ_d=(-1)^d q_{d+1}(x)$. □

Theorem 4. *Matrix representation of* $F_{k,n}(x)$ *is as follows*

$$F_d{}^n=\begin{pmatrix} F_{k,n+1}(x) & q_2(x)F_{k,n}(x)+\cdots+q_{d+1}(x)F_{k,n-d+1}(x) & \dots & q_{d+1}(x)F_{k,n}(x)\\ F_{k,n}(x) & q_2(x)F_{k,n-1}(x)+\cdots+q_{d+1}F_{k,n-d}(x) & \dots & q_{d+1}(x)F_{k,n-1}(x)\\ \vdots & \vdots & \ddots & \vdots\\ F_{k,n-d+1}(x) & q_2(x)F_{k,n-d}(x)+\cdots+q_{d+1}(x)F_{k,n-2d+1}(x) & \dots & q_{d+1}(x)F_{k,n-d}(x)\end{pmatrix} \tag{4}$$

where $F_d{}^n=F_d{}^{n-1}Q_d$.

Proof. Let's apply the induction on n.

For $n=1$,

$$F_d{}^1=\begin{pmatrix} F_{k,2}(x) & q_2(x)F_{k,1}(x) & \dots & q_{d+1}(x)F_{k,1}(x)\\ F_{k,1}(x) & q_2(x)F_{k,0}(x) & \dots & q_{d+1}(x)F_{k,0}(x)\\ \vdots & \vdots & \ddots & \vdots\\ F_{k,2-d}(x) & q_2(x)F_{k,1-d}(x) & \dots & q_{d+1}(x)F_{k,1-d}(x)\end{pmatrix}$$

$$=\begin{pmatrix} q_1(x)k & q_2(x)k & \dots & q_{d+1}(x)k\\ k & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \dots & k & 0\end{pmatrix} \tag{5}$$

From the definition of $F_{k,n}(x)$, we obtain that the matrices in (3) and (5) are the same.

Suppose that the result satisfies for n. So, we get

$$F_d{}^n = \begin{pmatrix} F_{k,n+1}(x) & q_2(x)F_{k,n}(x)+\cdots+q_{d+1}(x)F_{k,n-d+1}(x) & \dots & q_{d+1}(x)F_{k,n}(x) \\ F_{k,n}(x) & q_2(x)F_{k,n-1}(x)+\cdots+q_{d+1}F_{k,n-d}(x) & \dots & q_{d+1}(x)F_{k,n-1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ F_{k,n-d+1}(x) & q_2(x)F_{k,n-d}(x)+\cdots+q_{d+1}(x)F_{k,n-2d+1}(x) & \dots & q_{d+1}(x)F_{k,n-d}(x) \end{pmatrix}$$

To prove the theorem, it remains to show that the result is true for $n+1$.

$$F_d{}^{n+1} = F_d{}^n F_d{}^1$$

$$= \begin{pmatrix} F_{k,n+1}(x) & q_2(x)F_{k,n}(x)+\cdots+q_{d+1}(x)F_{k,n-d+1}(x) & \dots & q_{d+1}(x)F_{k,n}(x) \\ F_{k,n}(x) & q_2(x)F_{k,n-1}(x)+\cdots+q_{d+1}F_{k,n-d}(x) & \dots & q_{d+1}(x)F_{k,n-1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ F_{k,n-d+1}(x) & q_2(x)F_{k,n-d}(x)+\cdots+q_{d+1}(x)F_{k,n-2d+1}(x) & \dots & q_{d+1}(x)F_{k,n-d}(x) \end{pmatrix}$$

$$\cdot \begin{pmatrix} q_1(x)k & q_2(x)k & \cdots & q_{d+1}(x)k \\ k & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & 0 & k & 0 \end{pmatrix}$$

$$= \begin{pmatrix} F_{k,n+2}(x) & q_2(x)F_{k,n+1}(x)+\cdots+q_{d+1}(x)F_{k,n-d+2}(x) & \dots & q_{d+1}(x)F_{k,n+1}(x) \\ F_{k,n+1}(x) & q_2(x)F_{k,n}(x)+\cdots+q_{d+1}F_{k,n-d+1}(x) & \dots & q_{d+1}(x)F_{k,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ F_{k,n-d+2}(x) & q_2(x)F_{k,n-d+1}(x)+\cdots+q_{d+1}(x)F_{k,n-2d+2}(x) & \dots & q_{d+1}(x)F_{k,n-d+1}(x) \end{pmatrix}$$

□

Corollary 1. *For $n, m \geq 0$, we have*

$$F_{k,n+m}(x) = F_{k,n}(x)F_{k,m}(x) + (q_2(x)F_{k,n-1}(x)F_{k,m-1}(x)+\cdots+ q_{d+1}(x)F_{k,n-d}(x)F_{k,m-1}(x)) + \cdots + q_{d+1}(x)F_{k,n-1}(x)F_{k,m-d}(x)$$

Proof. Since $F_d^n F_d^m = F_d^{n+m}$, the desired is the first row and the first column of matrix F_d^{n+m}. □

Lemma 1. *For $n \geq 1$, we have*

$$F_{k,n}(x) = kF_{\boldsymbol{n}}(x).$$

Proof. For $n=1$ equality is true so that$F_{k,1}(x) = kF_{\mathbf{1}}(x) = k$

Let the result be true for $n=k$. For $n=k+1$, we show that the equation is true.

$$F_{k,n+1}(x) = q_1(x)F_{k,n}(x)+\cdots+q_{d+1}(x)F_{k,n-d}(x)$$

$$= q_1(x)kF_n(x)+\cdots+q_{d+1}(x)kF_{n-d}(x) = kF_{n+1}(x).$$ □

Theorem 5. *For $d \geq 2$, $n \geq 0$,*

$$\sum_{\substack{n_1,\ n_2, \ldots, n_{d+1} \\ (d+1)\, n_1 + dn_2 + \cdots + n_{d+1} = n}} \binom{n_1 + n_2 + \cdots +\ n_{d+1}}{n_1, n_2, \ldots,\ n_{d+1}} {q_1}^{n_1}\,(x) \cdot \ldots \cdot$$

$$\ldots \cdot {q_{d+1}}^{n_{d+1}}\,(x)\, F_{k,n-(n_1+n_2+\cdots+n_{d+1})}\,(x)$$

$$= F_{k,n(d+1)}(x) \tag{6}$$

Proof. For $n = 1$, we have

$$F_{k,d+1}\,(x) = q_1\,(x)\, F_{k,d}\,(x) + \cdots + q_{d+1}(x) F_{k,0}\,(x)$$

Let's show the right-hand side of Eq. (6) with RH.
For $n \geq 0$, we get

$$RH = \sum_{\substack{n_1,\ n_2, \ldots, n_{d+1} \\ (d+1)\, n_1 + dn_2 + \cdots + n_{d+1} = n}} \binom{n_1 + n_2 + \cdots +\ n_{d+1}}{n_1, n_2, \ldots,\ n_{d+1}}.$$

$${q_1}^{n_1}\,(x) \ldots {q_{d+1}}^{n_{d+1}}\,(x) \left[\sum_{i=1}^{d+1} K_i(x) \delta_i(x)^{n-(n_1+n_2+\cdots+n_{d+1})}\right]$$

$$= \sum_{\substack{n_1,\ n_2, \ldots, n_{d+1} \\ (d+1)\, n_1 + dn_2 + \cdots + n_{d+1} = n}} \binom{n_1 + n_2 + \cdots +\ n_{d+1}}{n_1, n_2, \ldots,\ n_{d+1}}.$$

$${q_1}^{n_1}\,(x) \ldots {q_{d+1}}^{n_{d+1}}\,(x) \left[\sum_{i=1}^{d+1} K_i(x) \delta_i(x)^{(dn_1+(d-1)n_2+\cdots+n_{d+1})}\right]$$

$$K_1(x) \sum_{\substack{n_1,\ n_2, \ldots, n_{d+1} \\ (d+1)\, n_1 + dn_2 + \cdots + n_{d+1} = n}} \binom{n_1 + n_2 + \cdots +\ n_{d+1}}{n_1, n_2, \ldots,\ n_{d+1}}$$

$$\left(\delta_1^d(x) q_1(x)\right)^{n_1} \left(\delta_1^{d-1}(x) q_2(x)\right)^{n_2} \ldots \left(q_{d+1}\,(x)\right)^{n_{d+1}} + \cdots +$$

$$+ K_{d+1}(x) \sum_{\substack{n_1,\ n_2, \ldots, n_{d+1} \\ (d+1)\, n_1 + dn_2 + \cdots + n_{d+1} = n}} \binom{n_1 + n_2 + \cdots +\ n_{d+1}}{n_1, n_2, \ldots,\ n_{d+1}}$$

$$\left(\delta_1^d(x) q_1(x)\right)^{n_1} \left(\delta_1^{d-1}(x) q_2(x)\right)^{n_2} \ldots \left(q_{d+1}\,(x)\right)^{n_{d+1}}$$

$$= K_1(x)\left[\delta_1^d(x)\,q_1(x) + \cdots + q_{d+1}(x)\right]^n + \dots$$
$$\cdots + K_{d+1}(x)\left[\delta_1^d(x)\,q_1(x) + \cdots + q_{d+1}(x)\right]^n.$$

From the characteristic equation, we get

$$= \sum_{i=1}^{d+1} K_i(x)\left(\delta_i(x)^{d+1}\right)^n = F_{k,n(d+1)}(x)$$

as desired. □

Lemma 2. *For $n \geq 1$,*

$$F_{k,n}(x) = k(L_n(x) - F_{n+1}(x) + q_1(x)\,F_n(x)).$$

Proof. From [19], Eq. 4, we get

$$\begin{aligned} F_{k,n}(x) = kF_n(x) &= k(L_n(x) - q_2(x)\,F_{n-1}(x) - \cdots - q_{d+1}(x)F_{n-d}(x)) \\ &= k(L_n(x) - (q_2(x)\,F_{n-1}(x) + \cdots + q_{d+1}(x)F_{n-d}(x))) \\ &= k(L_n(x) - F_{n+1}(x) + q_1(x)\,F_n(x)). \end{aligned}$$
□

3 The infinite matrix of Fibonacci polynomials

The matrix of $(d,k)-$ Fibonacci polynomials is defined by

$$\mathcal{F}(x) = [\mathcal{F}_{k,(q_1,q_2,\dots,q_{d+1},i,j)}(x)]$$

and we have

$$\mathcal{F}(x) = \begin{pmatrix} k & 0 & \dots \\ q_1(x)\,k & k & \dots \\ {q_1}^2(x)\,k + q_2(x)\,k & q_1(x)\,k & \dots \\ s_1(x) & s_2(x) & \vdots \\ \vdots & \vdots & \dots \end{pmatrix} = \left(g_{\mathcal{F}(x)}(s), f_{\mathcal{F}(x)}(s)\right),$$

where $s_1(x) = {q_1}^3(x)\,k + 2q_1(x)\,q_2(x)\,k + q_3(x)\,k$, $s_2(x) = {q_1}^2(x)\,k + q_2(x)\,k$ and $s_3(x) = q_1(x)\,k$.

In other words, Fibonacci polynomial matrices can be written as follows:

$$\mathcal{F}(x) = \begin{pmatrix} F_{k,1}(x) & F_{k,0}(x) & 0 & \dots \\ F_{k,2}(x) & F_{k,1}(x) & F_{k,0}(x) & \dots \\ \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Theorem 6. *The first column of $\digamma(x)$ matrix is as follows*

$$\left(k, q_1(x)k, {q_1}^2(x)k + q_2(x)k, \ldots\right)^{\boldsymbol{T}}.$$

According to the theory on which Riordan groups are based, the generator function for the first column is

$$g_{\digamma(x)}(s) = \sum_{n=0}^{\infty} \digamma_{q_1,q_2,\ldots,q_{d+1},i,j}(x)s^n = \frac{k}{(1 - q_1(x)s - \cdots - q_{d+1}(x)s^{d+1})}.$$

Proof. For the first column, the generating function is

$$k + q_1(x)ks + \left({q_1}^2(x) + q_2(x)\right)ks^2 + \ldots.$$

Let's make necessary operations to find the generating function for $F_{k,n}(x)$. So, we have

$$g_{\digamma(x)}(s) = \frac{k}{(1 - q_1(x)s - \cdots - q_{d+1}(x)s^{d+1})}.$$

The desired expression is obtained. From the Riordan matrix, we have $f_{\digamma(x)}(s) = s$.

$$\digamma(x) = \left(g_{\digamma(x)}(s), f_{\digamma(x)}(s)\right) = \left(\frac{k}{(1 - q_1(x)s - \cdots - q_{d+1}(x)s^{d+1})}, s\right).$$

If the $(d, k)-$ Fiboonacci polynomials matrix $\digamma(x)$ is finite, then the matrix is

$$\digamma_f(x) = \begin{pmatrix} F_{k,1}(x) & F_{k,0}(x) & 0 & \ldots \\ F_{k,2}(x) & F_{k,1}(x) & F_{k,0}(x) & \ldots \\ \vdots & \vdots & \vdots & \ldots \\ F_{k,n}(x) & F_{k,n-1}(x) & F_{k,n-2}(x) & \ldots \end{pmatrix}$$

and $det\digamma_f(x) = |\digamma_f(x)| = (k)^n$.

We introduce two factorizations of Pascal Matrix including the $(d, k)-$ Fibonacci polynomials matrix.

Let's define a matrix $M(x) = k(m_{i,j}(x))$ such that

$$m_{i,j} = \binom{i-1}{j-1} - q_1(x)\binom{i-2}{j-1} - \cdots - q_{d+1}(x)\binom{i-d-2}{j-1}$$

So, we get

$$M(x) = \begin{pmatrix} k & 0 & \ldots \\ k - q_1(x)k & k & \ldots \\ k - q_1(x)k - q_2(x)k & 2k - q_1(x)k & \ldots \\ \ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots \end{pmatrix}. \qquad \square$$

Theorem 7. *The factorization is as follows*

$$P(x) = \mathcal{F}(x)\, M(x).$$

Proof. The generating function for the first column of matrix $M(x)$ is

$$\begin{aligned}
g_{M(x)}(s) &= k + (1 - q_1(x))\, ks + (1 - q_1(x) - q_2(x))ks^2 + \dots \\
&= k(1 + s + \dots) - q_1(x)\, k\left(s + s^2 + \dots\right) \\
&\qquad\qquad - q_2(x)\, k\left(s^2 + s^3 + \dots\right) + \dots + q_{d+1}k(s^{d+1} + s^{d+2} + \dots) \\
&= \frac{k}{1-s} - \frac{q_1 ks}{1-s} - \dots - \frac{q_{d+1}ks^{d+1}}{1-s} \\
&= \frac{k(1 - q_1 s - q_2 s^2 - \dots - q_{d+1}s^{d+1})}{1-s}.
\end{aligned}$$

By the definition of the Riordan matrix, we write $f_{M(x)}(s)$.

$$\begin{aligned}
f_{M(x)}(s) &= ks + (2 - q_1(x))\, ks^2 + (3 - 2q_1(x) - q_2(x))\, ks^3 + \dots \\
&= k\left(s + 2s^2 + 3s^3 + \dots\right) - q_1 ks\left(s + 2s^2 + 3s^3 + \dots\right) \\
&\qquad\qquad - \dots - q_{d+1}ks^{d+1}\left(s + 2s^2 + 3s^3 + \dots\right) \\
&= \frac{s}{1-s}\left(\frac{k(1 - q_1 s - q_2 s^2 - \dots - q_{d+1}s^{d+1})}{1-s}\right)
\end{aligned}$$

From definition of the Riordan array, $i\ th$ column generating function $g(x)(f(x))^i$ is as follows

$$f_{M(x)}(s) = \frac{s}{1-s}.$$

Then we get

$$M(x) = \left(g_{M(x)}(s), f_{M(x)}(s)\right) = \left(\frac{k(1 - q_1 s - \dots - q_{d+1}s^{d+1})}{1-s}, \frac{s}{1-s}\right).$$

From the matrices of Pascal and $(d,k)-$Fibonacci polynomials, the Riordan representations are as follows

$$P = \left(\frac{1}{1-s}, \frac{s}{1-s}\right),$$

$$\mathcal{F}(x) = \left(\frac{k}{(1 - q_1(x)\, s - \dots - q_{d+1}(x)\, s^{d+1})}, s\right).$$

The proof is completed if we use the Riordan matrix multiplication.

Now, we give other factorization for these matrices.

Let's consider an infinite $N(x) = k(n_{i,j}(x\ ))$ where

$$n_{i,j} = \binom{i-1}{j-1} - q_1(x)\binom{i-1}{j} - \dots - q_{d+1}(x)\binom{i-1}{j+d}$$

We give the infinite $N(x)$ by

$$N(x)=\begin{pmatrix} k & 0 & \dots \\ k-q_1(x)k & k & \dots \\ k-2q_1(x)k-q_2(x)k & 2k-q_1(x)k & \dots \\ \dots & \dots & \dots \\ \vdots & \vdots & \vdots \end{pmatrix}. \qquad \square$$

Now, we can give another factorization.

Theorem 8. *The factorization is*

$$P(x)=F(x)\,N(x).$$

Proof. The proof is similar to Theorem 7. $\square$

Now, let's give the inverse of $F(x)$ by using the definition of reverse element in Riordan group [20] with the following Corollary.

Corollary 2. *We can find easily the inverses of the matrices by using the Riordan representations of the given matrices as follows:*

$$F^{-1}(x)=\left(\frac{1-q_1s-\cdots-q_{d+1}s^{d+1}}{k},s\right)$$

4 Conclusions

In this paper, we defined the $(d,k)-$ Fibonacci polynomial and gave its properties. We found the generating function, characteristic equation, and matrix representation of this polynomial. Then we got the infinite sum for the $(d,k)-$ Fibonacci polynomials. We obtained the relationship between $(d,k)-$ Fibonacci polynomial and d- Fibonacci polynomial. Also, with the help of $(d,k)-$ Fibonacci polynomial matrix representation and the Riordan matrix, the factorization of the Pascal matrix in two different ways were given. This and similar studies in this field can be applied to different number sequences, as well as to number sequences with characteristic equations with more degrees, and it will help to reveal different and interesting properties.

References

[1] Alp, Y. and Koçer, E. G., *Leonardo and hyper-Leonardo numbers via Riordan arrays*, Ukrains'kyi Matematychnyi Zhurnal, **76** (2024), no. 3, 326-340.

[2] Cameron, N. T. and Matthews, L. S., *A Motzkin variation on the Tennis Ball Problem*, Ars Combinatoria, **114** (2014), 145-152.

[3] Davenport, D., Shapiro, L. W. and Woodson, L. C., *The double Riordan group*, Electron. J. Combin., **18** (2011), no. 2, 13pp.

[4] Donaghey, R. and Shapiro, L. W., *A survey of the Motzkin numbers*, J. Combin. Theory Ser. A, **23** (1977), 291-301.

[5] Falcón, S. and Plaza, Á., *On the Fibonacci k-numbers*, Chaos. Solitons & Fractals, **32** (2007), 1615-1624.

[6] Falcón, S.,Plaza, Á., *The k-Fibonacci sequence and the Pascal 2-triangle*, Chaos. Solitons & Fractals, **33** (2007), 38-49.

[7] Hoggatt Jr, V. E., Bicknell, M., *Roots of Fibonacci polynomials*, The Fibonacci Quart., **11** (1973), 271-274.

[8] Kılıç, E., *Riordan group approaches in matrix factorization*, Bull. Iranian Math. Soc., **38** (2012), 491-506.

[9] Koshy, T., *Fibonacci and Lucas numbers with applications*, John Wiley & Sons (2019).

[10] Kuloğlu, B., Özkan, E., *d-Tribonacci polynomials and their matrix representations*, WSEAS Trans. Math., **22** (1977), 204-212.

[11] Kuloğlu, B., Ozkan E. and Marin, M., *Fibonacci and Lucas polynomials in n-gon*, An. Ştiinţ. Univ. "Ovidius" Constanța Ser. Mat., **31** (2023), 127-140.

[12] Marin, M., An uniqueness result for a body with voids in linear thermoelasticity, Rend. Mat. Appl., 17 (1997), 103-113.

[13] Nalli, A. and Haukkanen, P., *On generalized Fibonacci and Lucas polynomials*, Chaos, Solitons & Fractals, **42** (2009), 3179-3186.

[14] Özkan, E., *3-step Fibonacci sequences in nilpotent groups*, Appl. Math. Comput. , **144** (2003), 517-527.

[15] Özkan, E., Aydın, H. and Dikici, R., *3-step Fibonacci series modulo m*, Appl. Math. Comput., **143** (2003), 165-172.

[16] Özkan, E. and Uysal, M., *D-Gaussian Jacobsthal, d-Gaussian Jacobsthal-Lucas polynomials and their matrix representations*, Electron. J. Math. Anal. Appl., **10** (2022), 124-140.

[17] Özkan, E. and Uysal, M., *d-Gaussian Fibonacci, d-Gaussian Lucas polynomials and their matrix representations*, Ukrainian Math. J., **75** (2023), no.4, 562-585.

[18] Roman, S., *The Umbral calculus*, Academic Press, New York, (1984).

[19] Sadaoui, B. and Krelifa, A., *d- Fibonacci and d- Lucas polynomials*, J. Math. Model., **9** (2021), 1-12.

[20] Shapiro, L. W., Getu, S., Woan, W. J., Woodson, L. C., *The Riordan group*, Discrete Appl. Math., **34** (1991), 229-239.

[21] Shapiro, L., Sprugnoli, R., Barry, P. and Wang, W., *q-analougs of Riordan Arrays*, In book: The Riordan group and applications, **8** (2022), 291-301.

[22] Taştan, M., Özkan, E and Shannon, A.G., *The generalized k-Fibonacci polynomials and generalized k-Lucas polynomials*, Notes Number Theory Discrete Math., **27** (2021), 148-158.

[23] Tian-Xiao, H., *Matrix characterizations of Riordan arrays*, Linear Algebra Appl., **465** (2015), 15-42.

[24] Weiping, W. and Tianming, W., *Generalized Riordan arrays*, Discrete Math., **308** (2008), 6466-6500.