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NORM ATTAINING MULTILINEAR FORMS ON $\mathbb{R}^n$ WITH THE ℓ_1-NORM

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Dedicated to Professor Marin Marin on the occasion of his 70th anniversary

Abstract

Let $n, m \in \mathbb{N}$ with $n, m \geq 2$. For given unit vectors $x_1, \cdots, x_n$ of a real Banach space E, we define

$$NA(\mathcal{L}(^nE))(x_1, \cdots, x_n) = \{T \in \mathcal{L}(^nE) : |T(x_1, \cdots, x_n)| = \|T\| = 1\},$$

where $\mathcal{L}(^nE)$ denotes the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1, 1\leq k\leq n} |T(x_1, \ldots, x_n)|$.

In this paper, we present a characterization of the elements in the set $NA(\mathcal{L}(^m\ell_1^n))(W_1, \cdots, W_m)$ for any given unit vectors $W_1, \ldots, W_m \in \ell_1^n$, where $\ell_1^n = \mathbb{R}^n$ with the ℓ_1-norm. This result generalizes the results from [7], and two particular cases for it are presented in full detail: the case $n = 2, m = 2$, and the case $n = 3, m = 2$.

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1 Introduction

Let $n, m \in \mathbb{N}, n, m \geq 2$. We write S_E for the unit sphere of a real Banach space E. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1, 1\leq k\leq n} |T(x_1, \ldots, x_n)|$. The subspace of all continuous symmetric n-linear forms on E is denoted by $\mathcal{L}_s(^nE)$. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists $T \in \mathcal{L}(^nE)$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of $T \in \mathcal{L}(^nE)$ if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|$. In this case, T is called *norm*

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attaining at $(x_1, \ldots, x_n)$. Similarly, an element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. In this case, P is called *norm attaining* at x. Let $X = \mathcal{L}(^nE)$ or $\mathcal{L}_s(^nE)$. For $x, x_1, \cdots, x_n \in S_E$, we define

$$NA(X)(x_1, \cdots, x_n) = \{T \in X : |T(x_1, \cdots, x_n)| = \|T\| = 1\}$$

and

$$NA(\mathcal{P}(^nE))(x) = \{P \in \mathcal{P}(^nE) : |P(x)| = \|P\| = 1\}.$$

Notice that

$$NA(\mathcal{L}(^nE))(x_1, \cdots, x_n) = NA(\mathcal{L}(^nE))(\pm x_1, \cdots, \pm x_n),$$

$$NA(\mathcal{L}_s(^nE))(x_1, \cdots, x_n) = NA(\mathcal{L}_s(^nE))(\pm x_{\sigma(1)}, \cdots, \pm x_{\sigma(n)})$$

and

$$NA(\mathcal{P}(^nE))(x) = NA(\mathcal{P})(^nE))(-x)$$

for all $x, x_1, \cdots, x_n \in S_E$ and for all permutation σ on $\{1, \ldots, n\}$.

Let us introduce a brief history of norm attaining multilinear forms and polynomials on Banach spaces. It was initiated in 1961 by Bishop and Phelps [2], who showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jiménez-Sevilla and Payá [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

It seems to be natural and interesting to study about

$$NA(\mathcal{L}(^nE))(x_1, \cdots, x_n), NA(\mathcal{L}_s(^nE))(x_1, \cdots, x_n) \text{ and } NA(\mathcal{P}(^nE))(x)$$

for $x, x_1, \cdots, x_n \in S_E$. Kim [6] classified $NA(\mathcal{P}(^2\ell_p^2))((x_1, x_2))$ for $(x_1, x_2) \in S_{\ell_p^2}$ and $p \in \{1, 2, \infty\}$, where $\ell_p^n = \mathbb{R}^n$ with the ℓ_p-norm. Recently, Kim [7] classified $NA(\mathcal{L}(^2\ell_1^2))((x_1, x_2))$ for $(x_1, x_2) \in S_{\ell_1^2}$.

In this paper, we characterize $NA(\mathcal{L}(^m\ell_1^n))(W_1, \cdots, W_m)$ for any given unit vectors $W_1, \ldots, W_m \in S_{\ell_1^n}$. This result generalizes the results from [7], and two particular cases for it are presented in full detail: the case $n = 2, m = 2$, and the case $n = 3, m = 2$.

2 Main results

Theorem 1. *Let $n, m \geq 2$. Let $T \in \mathcal{L}(^m\ell_1^n)$ with*

$$T\Big(\big(x_1^{(1)}, \ldots, x_n^{(1)}\big), \ldots, \big(x_1^{(m)}, \ldots, x_n^{(m)}\big)\Big) = \sum_{1\leq i_k\leq n,\ 1\leq k\leq m} a_{i_1\cdots i_m}\ x_{i_1}^{(1)} \cdots x_{i_m}^{(m)},$$

where $a_{i_1\cdots i_m} \in \mathbb{R}$ for all $1 \leq i_k \leq n, 1 \leq k \leq m$. Then

$$\|T\| = \max\Big\{|a_{i_1\cdots i_m}| : 1 \leq i_k \leq n,\ 1 \leq k \leq m\Big\}.$$

Proof. Let $M := \max\{|a_{i_1\cdots i_m}| : 1 \leq i_k \leq n,\ 1 \leq k \leq m\}$. Let $\big(x_1^{(k)}, \ldots, x_n^{(k)}\big) \in S_{\ell_1^n}$ for $1 \leq k \leq m$. It follows that

$$\begin{aligned}
& \Big|T\Big((x_1^{(1)}, \ldots, x_n^{(1)}), \ldots, (x_1^{(m)}, \ldots, x_n^{(m)})\Big)\Big| \\
\leq\ & \sum_{1\leq i_k\leq n,\ 1\leq k\leq m} |a_{i_1\cdots i_m}|\ |x_{i_1}^{(1)}| \cdots |x_{i_m}^{(m)}| \\
\leq\ & M \sum_{1\leq i_k\leq n,\ 1\leq k\leq m} |x_{i_1}^{(1)}| \cdots |x_{i_m}^{(m)}| \\
=\ & M\Big(\sum_{1\leq j\leq n} |x_j^{(1)}|\Big) \cdots \Big(\sum_{1\leq j\leq n} |x_j^{(m)}|\Big) = M \\
=\ & \max\Big\{\big|T(e_{i_1}, \ldots, e_{i_m})\big| : 1 \leq i_k \leq n,\ 1 \leq k \leq m\Big\} \\
\leq\ & \|T\|.
\end{aligned}$$

Therefore, $\|T\| = M$. □

Let $T \in \mathcal{L}(^m\ell_1^n)$ with

$$T\Big(\big(x_1^{(1)}, \ldots, x_n^{(1)}\big), \ldots, \big(x_1^{(m)}, \ldots, x_n^{(m)}\big)\Big) = \sum_{1\leq i_k\leq n,\ 1\leq k\leq m} a_{i_1\cdots i_m}\ x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}$$

for some $a_{i_1\cdots i_m} \in \mathbb{R}$. For simplicity we will denote $T = \big(a_{i_1\cdots i_m}\big)_{1\leq i_k\leq n}$.

Let $W_1, \ldots, W_m \in S_{\ell_1^n}$. Notice that $T \in NA(\mathcal{L}(^m\ell_1^n))(W_1, \cdots, W_m)$ if and only if $-T \in NA(\mathcal{L}(^m\ell_1^n))(W_1, \cdots, W_m)$.

We are in a position to prove the main result of this paper.

Theorem 2. *Let $n,\ m \geq 2$. Suppose that $W_1,\ \ldots,\ W_m \in S_{\ell_1^n}$ with $W_j = \big(t_1^{(j)},\ \ldots,\ t_n^{(j)}\big)$ for $1 \leq j \leq m$. Then the following assertions hold:*

Case 1. *If $t_{i_1}^{(1)} \cdots t_{i_m}^{(m)} \neq 0$ for all $1 \leq i_k \leq n, 1 \leq k \leq m$, then*

$$NA(\mathcal{L}(^m\ell_1^n))(W_1, \cdots, W_m) = \Big\{ \pm \Big(\operatorname{sign}\big(t_{i_1}^{(1)} \cdots t_{i_m}^{(m)}\big)\Big)_{1\leq i_k\leq n}\Big\}.$$

Case 2. *If* $t^{(1)}_{i'_1}\cdots t^{(m)}_{i'_m}=0$ *for some* $1\leq i'_k\leq n$, *then*

$$NA(\mathcal{L}(^m\ell_1^n))(W_1,\cdots,W_m)=\Big\{\pm\big(a_{i_1\cdots i_m}\big)_{1\leq i_k\leq n}: a_{i_1\cdots i_m}=\text{sign}\,\big(t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}\big)$$
$$\textit{if}\; t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}\neq 0 \;\textit{and}\; a_{i_1\cdots i_m}\in[-1,1]\;\textit{if}\; t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}=0\Big\}.$$

Proof. It suffices to show the case 2 because the proof of the case 1 is similar.

Let

$$\begin{aligned}\mathcal{F}: \;=\; &\Big\{\pm\big(a_{i_1\cdots i_m}\big)_{1\leq i_k\leq n}: a_{i_1}\cdots a_{i_m}=\text{sign}(t^{(1)}_{i_1}\cdots t^{(m)}_{i_m})\\ &\text{if } t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}\neq 0 \text{ and } a_{i_1\cdots i_m}\in[-1,1] \text{ if } t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}=0\Big\}.\end{aligned}$$

Claim 1. $\mathcal{F}\subseteq NA(\mathcal{L}(^m\ell_1^n))(W_1,\cdots,W_m)$.

Let $T\in\mathcal{F}$. Write $T=\big(a_{i_1\cdots i_m}\big)_{1\leq i_k\leq n}$ for some $a_{i_1\cdots i_m}\in\mathbb{R}$. By Theorem 1, $\|T\|=1$. It follows that

$$\begin{aligned}|T(W_1,\ldots,W_m)| &= \Big|\sum_{1\leq i_k\leq n,\;1\leq k\leq m} a_{i_1\cdots i_m}\; t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}\Big|\\ &= \Big|\sum_{t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}\neq 0} a_{i_1\cdots i_m}\; t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}+\sum_{t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}=0} a_{i_1\cdots i_m}\; t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}\Big|\\ &= \sum_{t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}\neq 0}|t^{(1)}_{i_1}|\cdots|t^{(m)}_{i_m}|+\sum_{t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}=0}|t^{(1)}_{i_1}|\cdots|t^{(m)}_{i_m}|\\ &= \Big(\sum_{1\leq i_1\leq n}|t^{(1)}_{i_1}|\Big)\cdots\Big(\sum_{1\leq i_m\leq n}|t^{(m)}_{i_m}|\Big)=1=\|T\|.\end{aligned}$$

Thus $T\in NA(\mathcal{L}(^m\ell_1^n))(W_1,\cdots,W_n)$. Claim 1 holds.

Claim 2. $NA(\mathcal{L}(^m\ell_1^n))(W_1,\cdots,W_m)\subseteq\mathcal{F}$.

Let $S\in NA(\mathcal{L}(^m\ell_1^n))(W_1,\cdots,W_m)$. Write $S=\big(b_{i_1\cdots i_m}\big)_{1\leq i_k\leq n}$ for some $b_{i_1\cdots i_m}\in\mathbb{R}$. Since $\|S\|=1$, $|b_{i_1\cdots i_m}|\leq 1$ for every $1\leq i_k\leq n$. It follows that

$$\begin{aligned}1 &= |S(W_1,\ldots,W_m)|=\Big|\sum_{1\leq i_k\leq n,\;1\leq k\leq m} b_{i_1\cdots i_m}\; t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}\Big|\\ &= \Big|\sum_{t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}\neq 0} b_{i_1\cdots i_m}\; t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}+\sum_{t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}=0} b_{i_1\cdots i_m}\; t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}\Big|\\ &= \sum_{t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}\neq 0}|b_{i_1\cdots i_m}|\;|t^{(1)}_{i_1}|\cdots|t^{(m)}_{i_m}|\\ &\leq \sum_{t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}\neq 0}|t^{(1)}_{i_1}|\cdots|t^{(m)}_{i_m}|=1,\end{aligned}$$

which implies that $b_{i_1\cdots i_m}=\text{sign}\big(t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}\big)$ for $t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}\neq 0$.

Thus $S\in\mathcal{F}$. Claim 2 holds. This completes the proof. □

Let $T((x_1, y_1),\ (x_2, y_2)) = a_{11}x_1x_2 + a_{22}y_1y_2 + a_{12}x_1y_2 + a_{21}y_1x_2 \in \mathcal{L}(^2\ell_1^2)$ for some $a_{ij} \in \mathbb{R}$ for $i, j = 1, 2$. For simplicity, we denote $T = (a_{11}, a_{22}, a_{12}, a_{21})$.

Kim [7] classified $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2))$ for given $(x_1, y_1), (x_2, y_2) \in S_{\ell_1^2}$.

We have the results from [7] as a corollary.

Corollary 1. *Let $(x_1, y_1), (x_2, y_2) \in S_{l_1^2}$. Then the following statements hold:*

Case 1. $x_jy_j \neq 0$ *for all* $j = 1, 2$.

If $x_jy_j > 0$ for all $j = 1, 2$,
then $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, 1, 1, 1)\}$.
If $x_jy_j < 0$ for all $j = 1, 2$,
then $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, 1, -1, -1)\}$.
If $x_1y_1 > 0$ and $x_2y_2 < 0$,
then $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, -1, -1, 1)\}$.
If $x_1y_1 < 0$ and $x_2y_2 > 0$,
then $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, -1, 1, -1)\}$.

Case 2. $x_1y_1 = 0$ *and* $x_2y_2 \neq 0$

If $x_1 = 0$ and $x_2y_2 > 0$,
then $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, c, 1) : |a| \leq 1, |c| \leq 1\}$.
If $x_1 = 0$ and $x_2y_2 < 0$,
then $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, c, -1) : |a| \leq 1, |c| \leq 1\}$.
If $y_1 = 0$ and $x_2y_2 > 0$,
then $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, 1, d) : |b| \leq 1, |d| \leq 1\}$.
If $y_1 = 0$ and $x_2y_2 < 0$,
then $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, -1, d) : |b| \leq 1, |d| \leq 1\}$.

Case 3. $x_2y_2 = 0$ *and* $x_1y_1 \neq 0$

If $x_2 = 0$ and $x_1y_1 > 0$,
then $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, 1, d) : |a| \leq 1, |d| \leq 1\}$.
If $x_2 = 0$ and $x_1y_1 < 0$,
then $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, -1, d) : |a| \leq 1, |d| \leq 1\}$.
If $y_2 = 0$ and $x_1y_1 > 0$,
then $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, c, 1) : |b| \leq 1, |c| \leq 1\}$.
If $y_2 = 0$ and $x_1y_1 < 0$,
then $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, c, -1) : |b| \leq 1, |c| \leq 1\}$.

Case 4. $x_1y_1 = x_2y_2 = 0$

If $x_1 = x_2 = 0$,
then $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, c, d) : |a| \leq 1, |c| \leq 1, |d| \leq 1\}$.
If $x_1 = y_2 = 0$,
then $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, b, c, 1) : |a| \leq 1, |b| \leq 1, |c| \leq 1\}$.
If $x_2 = y_1 = 0$,
then $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, b, 1, d) : |a| \leq 1, |b| \leq 1, |d| \leq 1\}$.

If $y_1 = y_2 = 0$,
then $NA(\mathcal{L}(^2\ell_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, c, d) : |b| \leq 1, |c| \leq 1, |d| \leq 1\}$.

Let $T((x_1, y_1, z_1),\ (x_2, y_2, z_2)) = a_{11}x_1x_2 + a_{22}y_1y_2 + a_{33}z_1z_2 + a_{12}x_1y_2 + a_{21}y_1x_2 + a_{13}x_1z_2 + a_{31}z_1x_2 + a_{23}y_1z_2 + a_{32}z_1y_2 \in \mathcal{L}(^2\ell_1^3)$ for some $a_{ij} \in \mathbb{R}$ $(j = 1, 2, 3)$. For simplicity, we denote $T = (a_{11}, a_{22}, a_{33}, a_{12}, a_{21}, a_{13}, a_{31}, a_{23}, a_{32})$.

By Theorem 2, we completely describe the elements of the set

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2)$$

for any given unit vectors $W_1 = (x_1, y_1, z_1), W_2 = (x_2, y_2, z_2) \in S_{\ell_1^3}$.

Since

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = NA(\mathcal{L}(^2\ell_1^3))(-W_1, -W_2),$$

we may assume that $x_j \geq 0$ for $j = 1, 2$.

Corollary 2. *Let* $W_1 = \big(t_1^{(1)}, t_2^{(1)}, t_3^{(1)}\big), W_2 = \big(t_1^{(2)}, t_2^{(2)}, t_3^{(2)}\big) \in S_{\ell_1^3}$ *with* $t_1^{(j)} \geq 0$ *for all* $j = 1, 2$. *Then the following statements hold:*

Case 1. $t_1^{(j)}t_2^{(j)}t_3^{(j)} \neq 0$ *for all* $j = 1, 2$.

Suppose that $t_2^{(j)}t_3^{(j)} > 0$ *for all* $j = 1, 2$.

If $t_2^{(1)} > 0, t_3^{(1)} > 0, t_2^{(2)} > 0, t_3^{(2)} > 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \big\{\pm(1, 1, 1, 1, 1, 1, 1, 1, 1)\big\}.$$

If $t_2^{(1)} > 0, t_3^{(1)} > 0, t_2^{(2)} < 0, t_3^{(2)} < 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \big\{\pm(1, -1, -1, -1, 1, -1, 1 - 1, -1)\big\}.$$

If $t_2^{(1)} < 0, t_3^{(1)} < 0, t_2^{(2)} > 0, t_3^{(2)} > 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \big\{\pm(1, -1, -1, 1, -1, 1, -1, -1, -1)\big\}.$$

If $t_2^{(1)} < 0, t_3^{(1)} < 0, t_2^{(2)} < 0, t_3^{(2)} < 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \big\{\pm(1, 1, 1, -1, -1, -1, -1, 1, 1)\big\}.$$

Suppose that $t_2^{(1)}t_3^{(1)} > 0$ *and* $t_2^{(2)}t_3^{(2)} < 0$.

If $t_2^{(1)} > 0, t_3^{(1)} > 0, t_2^{(2)} > 0, t_3^{(2)} < 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \big\{\pm(1, 1, -1, 1, 1, -1, 1, -1, 1)\big\}.$$

If $t_2^{(1)} > 0, t_3^{(1)} > 0, t_2^{(2)} < 0, t_3^{(2)} > 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \big\{ \pm (1, -1, 1, -1, 1, 1, 1, 1, -1) \big\}.$$

If $t_2^{(1)} < 0, t_3^{(1)} < 0, t_2^{(2)} > 0, t_3^{(2)} < 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \big\{ \pm (1, -1, 1, 1, -1, -1, -1, 1, -1) \big\}.$$

If $t_2^{(1)} > 0, t_3^{(1)} > 0, t_2^{(2)} < 0, t_3^{(2)} > 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \big\{ \pm (1, 1, -1, -1, -1, 1, -1, -1, 1) \big\}.$$

Suppose that $t_2^{(1)} t_3^{(1)} < 0$ *and* $t_2^{(2)} t_3^{(2)} > 0.$

If $t_2^{(1)} > 0, t_3^{(1)} < 0, t_2^{(2)} > 0, t_3^{(2)} > 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \big\{ \pm (1, 1, -1, 1, 1, 1, -1, 1, -1) \big\}.$$

If $t_2^{(1)} > 0, t_3^{(1)} > 0, t_2^{(2)} < 0, t_3^{(2)} < 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \big\{ \pm (1, -1, 1, -1, 1, -1, -1, -1, 1) \big\}.$$

If $t_2^{(1)} < 0, t_3^{(1)} > 0, t_2^{(2)} > 0, t_3^{(2)} > 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \big\{ \pm (1, -1, 1, 1, -1, 1, 1, -1, 1) \big\}.$$

If $t_2^{(1)} < 0, t_3^{(1)} > 0, t_2^{(2)} < 0, t_3^{(2)} < 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \big\{ \pm (1, 1, -1, -1, -1, -1, 1, 1, -1) \big\}.$$

Suppose that $t_2^{(j)} t_3^{(j)} < 0$ *for all* $j = 1, 2.$

If $t_2^{(1)} > 0, t_3^{(1)} < 0, t_2^{(2)} > 0, t_3^{(2)} < 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \big\{ \pm (1, 1, 1, 1, 1, -1, -1, -1, -1) \big\}.$$

If $t_2^{(1)} > 0, t_3^{(1)} < 0, t_2^{(2)} < 0, t_3^{(2)} > 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \big\{ \pm (1, -1, -1, -1, 1, 1, -1, 1, 1) \big\}.$$

If $t_2^{(1)} < 0, t_3^{(1)} > 0, t_2^{(2)} > 0, t_3^{(2)} < 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \big\{ \pm (1, -1, -1, 1, -1, -1, 1, 1, 1) \big\}.$$

If $t_2^{(1)} < 0, t_3^{(1)} > 0, t_2^{(2)} < 0, t_3^{(2)} > 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \big\{ \pm (1, 1, 1, -1, -1, 1, 1, -1, -1) \big\}.$$

Case 2. $t_1^{(j)}t_2^{(j)}t_3^{(j)} = 0$ *for some* $j = 1, 2.$

2.1. $t_1^{(1)}t_2^{(1)}t_3^{(1)} = 0$ *and* $t_1^{(2)}t_2^{(2)}t_3^{(2)} \neq 0$

If $t_1^{(1)} = 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(a_{11}, \mathrm{sign}(t_2^{(1)}t_2^{(2)}), \mathrm{sign}(t_3^{(1)}t_3^{(2)}), a_{12}, \mathrm{sign}(t_2^{(1)}t_1^{(2)}), a_{13}, \mathrm{sign}(t_3^{(1)}t_1^{(2)}), \mathrm{sign}(t_2^{(1)}t_3^{(2)}), \mathrm{sign}(t_3^{(1)}t_2^{(2)})\big) : |a_{11}| \le 1, |a_{12}| \le 1, |a_{13}| \le 1\Big\}.$$

If $t_2^{(1)} = 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm (\mathrm{sign}(t_1^{(1)}t_1^{(2)}), a_{22}, \mathrm{sign}(t_3^{(1)}t_3^{(2)}), \mathrm{sign}(t_1^{(1)}t_2^{(2)}), a_{21}, \mathrm{sign}(t_1^{(1)}t_3^{(2)}), \mathrm{sign}(t_3^{(1)}t_1^{(2)}), a_{23}, \mathrm{sign}(t_3^{(1)}t_2^{(2)})) : |a_{22}| \le 1, |a_{21}| \le 1, |a_{23}| \le 1\Big\}.$$

If $t_3^{(1)} = 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(\mathrm{sign}(t_1^{(1)}t_1^{(2)}), \mathrm{sign}(t_2^{(1)}t_2^{(2)}), a_{33}, \mathrm{sign}(t_1^{(1)}t_2^{(2)}), \mathrm{sign}(t_2^{(1)}t_1^{(2)}), \mathrm{sign}(t_1^{(1)}t_3^{(2)}), a_{31}, \mathrm{sign}(t_2^{(1)}t_3^{(2)}), a_{32}\big) : |a_{33}| \le 1, |a_{31}| \le 1, |a_{32}| \le 1\Big\}.$$

2.2. $t_1^{(2)}t_2^{(2)}t_3^{(2)} = 0$ *and* $t_1^{(1)}t_2^{(1)}t_3^{(1)} \neq 0$

If $t_1^{(2)} = 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(a_{11}, \mathrm{sign}(t_2^{(1)}t_2^{(2)}), \mathrm{sign}(t_3^{(1)}t_3^{(2)}), \mathrm{sign}(t_1^{(1)}t_2^{(2)}), a_{21}, \mathrm{sign}(t_1^{(1)}t_3^{(2)}), a_{31}, \mathrm{sign}(t_2^{(1)}t_3^{(2)}), \mathrm{sign}(t_3^{(1)}t_2^{(2)})\big) : |a_{11}| \le 1, |a_{21}| \le 1, |a_{31}| \le 1\Big\}.$$

If $t_2^{(2)} = 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(\mathrm{sign}(t_1^{(1)}t_1^{(2)}), a_{22}, \mathrm{sign}(t_3^{(1)}t_3^{(2)}), a_{12}, \mathrm{sign}(t_2^{(1)}t_1^{(2)}), \mathrm{sign}(t_1^{(1)}t_3^{(2)}), \mathrm{sign}(t_3^{(1)}t_1^{(2)}), \mathrm{sign}(t_2^{(1)}t_3^{(2)}), a_{32}\big) : |a_{22}| \le 1, |a_{12}| \le 1, |a_{32}| \le 1\Big\}.$$

If $t_3^{(2)} = 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(\mathrm{sign}(t_1^{(1)}t_1^{(2)}), \mathrm{sign}(t_2^{(1)}t_2^{(2)}), a_{33}, \mathrm{sign}(t_1^{(1)}t_2^{(2)}), \mathrm{sign}(t_2^{(1)}t_1^{(2)}), a_{13}, \mathrm{sign}((t_3^{(1)}t_1^{(2)}), a_{23}, \mathrm{sign}(t_3^{(1)}t_2^{(2)})\big) : |a_{33}| \le 1, |a_{13}| \le 1, |a_{23}| \le 1\Big\}.$$

2.3. $t_1^{(j)}t_2^{(j)}t_3^{(j)} = 0$ *for* $j = 1, 2.$

Suppose that $t_1^{(1)} = 0$.

If $t_1^{(2)} = 0, t_2^{(2)} t_3^{(2)} \neq 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(a_{11}, \text{sign}(t_2^{(1)} t_2^{(2)}), \text{sign}(t_3^{(1)} t_3^{(2)}), a_{12}, a_{21}, a_{13}, a_{31}, \text{sign}(t_2^{(1)} t_3^{(2)}), \text{sign}(t_3^{(1)} t_2^{(2)})\big) : |a_{11}| \leq 1, |a_{12}| \leq 1, |a_{21}| \leq 1, |a_{13}| \leq 1, |a_{31}| \leq 1 \Big\}.$$

If $t_1^{(2)} = t_2^{(2)} = 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(a_{11}, a_{22}, \text{sign}(t_3^{(1)} t_3^{(2)}), a_{12}, a_{21}, a_{13}, a_{31}, \text{sign}(t_2^{(1)} t_3^{(2)}), a_{32}\big) : |a_{11}| \leq 1, |a_{22}| \leq 1, |a_{12}| \leq 1, |a_{21}| \leq 1, |a_{13}| \leq 1, |a_{31}| \leq 1, |a_{32}| \leq 1 \Big\}.$$

If $t_1^{(2)} = t_3^{(2)} = 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(a_{11}, \text{sign}(t_2^{(1)} t_2^{(2)}), a_{33}, a_{12}, a_{21}, a_{13}, a_{31}, a_{23}, \text{sign}(t_3^{(1)} t_2^{(2)})\big) : |a_{11}| \leq 1, |a_{33}| \leq 1, |a_{12}| \leq 1, |a_{21}| \leq 1, |a_{13}| \leq 1, |a_{31}| \leq 1, |a_{23}| \leq 1 \Big\}.$$

If $t_2^{(2)} = 0, t_1^{(2)} t_3^{(2)} \neq 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(a_{11}, a_{22}, \text{sign}(t_3^{(1)} t_3^{(2)}), a_{12}, \text{sign}(t_2^{(1)} t_1^{(2)}), a_{13}, \text{sign}(t_3^{(1)} t_1^{(2)}), \text{sign}(t_2^{(1)} t_3^{(2)}), a_{32}\big) : |a_{11}| \leq 1, |a_{22}| \leq 1, |a_{12}| \leq 1, |a_{13}| \leq 1, |a_{32}| \leq 1 \Big\}.$$

If $t_2^{(2)} = t_3^{(2)} = 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(a_{11}, a_{22}, a_{33}, a_{12}, \text{sign}(t_2^{(1)} t_1^{(2)}), a_{13}, \text{sign}(t_3^{(1)} t_1^{(2)}), a_{23}, a_{32}\big) : |a_{11}| \leq 1, |a_{22}| \leq 1, |a_{33}| \leq 1, |a_{12}| \leq 1, |a_{13}| \leq 1, |a_{23}| \leq 1, |a_{32}| \leq 1 \Big\}.$$

If $t_3^{(2)} = 0, t_1^{(2)} t_2^{(2)} \neq 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(a_{11}, \text{sign}(t_2^{(1)} t_2^{(2)}), a_{33}, a_{12}, \text{sign}(t_2^{(1)} t_1^{(2)}), a_{13}, \text{sign}(t_3^{(1)} t_1^{(2)}), a_{23}, \text{sign}(t_3^{(1)} t_2^{(2)})\big) : |a_{11}| \leq 1, |a_{33}| \leq 1, |a_{12}| \leq 1, |a_{13}| \leq 1, |a_{23}| \leq 1 \Big\}.$$

Suppose that $t_2^{(1)} = 0$.

If $t_1^{(2)} = 0, t_2^{(2)} t_3^{(2)} \neq 0,$ *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(a_{11}, a_{22}, \text{sign}(t_3^{(1)} t_3^{(2)}), \text{sign}(t_1^{(1)} t_2^{(2)}), a_{21}, \text{sign}(t_1^{(1)} t_3^{(2)}), a_{31}, a_{23}, \text{sign}(t_3^{(1)} t_2^{(2)})\big) : |a_{11}| \leq 1, |a_{22}| \leq 1, |a_{21}| \leq 1, |a_{31}| \leq 1, |a_{23}| \leq 1 \Big\}.$$

If $t_1^{(2)} = t_2^{(2)} = 0$, *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(a_{11}, a_{22}, \mathrm{sign}(t_3^{(1)}t_3^{(2)}), a_{12}, a_{21}, \mathrm{sign}(t_1^{(1)}t_3^{(2)}), a_{31}, a_{23}, a_{32}\big) : |a| \leq 1, |b| \leq 1, |a_{12}| \leq 1, |a_{21}| \leq 1, |a_{31}| \leq 1, |a_{23}| \leq 1, |a_{32}| \leq 1\Big\}.$$

If $t_1^{(2)} = t_3^{(2)} = 0$, *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(a_{11}, a_{22}, a_{33}, \mathrm{sign}(t_1^{(1)}t_2^{(2)}), a_{21}, a_{13}, a_{31}, a_{23}, \mathrm{sign}(t_3^{(1)}t_2^{(2)})\big) : |a_{11}| \leq 1, |a_{22}| \leq 1, |a_{33}| \leq 1, |a_{21}| \leq 1, |a_{13}| \leq 1, |a_{31}| \leq 1, |a_{23}| \leq 1\Big\}.$$

If $t_2^{(2)} = 0, t_1^{(2)}t_3^{(2)} \neq 0$, *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(\mathrm{sign}(t_1^{(1)}t_1^{(2)}), a_{22}, \mathrm{sign}(t_3^{(1)}t_3^{(2)}), a_{12}, a_{21}, \mathrm{sign}(t_1^{(1)}t_3^{(2)}), \mathrm{sign}(t_3^{(1)}t_1^{(2)}), a_{23}, a_{32}\big) : |a_{22}| \leq 1, |a_{12}| \leq 1, |a_{21}| \leq 1, |a_{23}| \leq 1, |a_{32}| \leq 1\Big\}.$$

If $t_2^{(2)} = t_3^{(2)} = 0$, *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(\mathrm{sign}(t_1^{(1)}t_1^{(2)}), a_{22}, a_{33}, a_{12}, a_{21}, a_{13}, \mathrm{sign}(t_3^{(1)}t_1^{(2)}), a_{23}, a_{32}\big) : |a_{22}| \leq 1, |a_{33}| \leq 1, |a_{12}| \leq 1, |a_{21}| \leq 1, |a_{13}| \leq 1, |a_{23}| \leq 1, |a_{32}| \leq 1\Big\}.$$

If $t_3^{(2)} = 0, t_1^{(2)}t_2^{(2)} \neq 0$, *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(\mathrm{sign}(t_1^{(1)}t_1^{(2)}), a_{22}, a_{33}, \mathrm{sign}(t_1^{(1)}t_2^{(2)}), a_{21}, a_{13}, \mathrm{sign}(t_3^{(1)}t_1^{(2)}), a_{23}, \mathrm{sign}(t_3^{(1)}t_2^{(2)})\big) : |a_{22}| \leq 1, |a_{33}| \leq 1, |a_{21}| \leq 1, |a_{13}| \leq 1, |a_{23}| \leq 1\Big\}.$$

Suppose that $t_3^{(1)} = 0$.

If $t_1^{(2)} = 0, t_2^{(2)}t_3^{(2)} \neq 0$, *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(a_{11}, \mathrm{sign}(t_2^{(1)}t_2^{(2)}), a_{33}, \mathrm{sign}(t_1^{(1)}t_2^{(2)}), a_{21}, \mathrm{sign}(t_1^{(1)}t_3^{(2)}), a_{31}, \mathrm{sign}(t_2^{(1)}t_3^{(2)}), a_{32}) : |a_{11}| \leq 1, |a_{33}| \leq 1, |a_{21}| \leq 1, |a_{31}| \leq 1, |a_{32}| \leq 1\Big\}.$$

If $t_1^{(2)} = t_2^{(2)} = 0$, *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(a_{11}, a_{22}, a_{33}, a_{12}, a_{21}, \mathrm{sign}(t_1^{(1)}t_3^{(2)}), a_{31}, \mathrm{sign}(t_2^{(1)}t_3^{(2)}), a_{32}\big) : |a_{11}| \leq 1, |a_{22}| \leq 1, |a_{33}| \leq 1, |a_{12}| \leq 1, |a_{21}| \leq 1, |a_{31}| \leq 1, |a_{32}| \leq 1\Big\}.$$

If $t_1^{(2)} = t_3^{(2)} = 0$, *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(a_{11}, \mathrm{sign}(t_2^{(1)}t_2^{(2)}), a_{33}, \mathrm{sign}(t_1^{(1)}t_2^{(2)}), a_{21}, a_{13}, a_{31}, a_{23}, a_{32}\big) : |a_{11}| \leq 1, |a_{33}| \leq 1, |a_{21}| \leq 1, |a_{13}| \leq 1, |a_{31}| \leq 1, |a_{23}| \leq 1, |a_{32}| \leq 1\Big\}.$$

If $t_2^{(2)} = 0, t_1^{(2)}t_3^{(2)} \neq 0$, *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(\mathrm{sign}(t_1^{(1)}t_1^{(2)}), a_{22}, a_{33}, a_{12}, \mathrm{sign}(t_2^{(1)}t_1^{(2)}), \mathrm{sign}(t_1^{(1)}t_3^{(2)}), a_{31}, \mathrm{sign}(t_2^{(1)}t_3^{(2)}), a_{32}\big) : |a_{22}| \leq 1, |a_{33}| \leq 1, |a_{12}| \leq 1, |a_{31}| \leq 1, |a_{32}| \leq 1\Big\}.$$

If $t_2^{(2)} = t_3^{(2)} = 0$, *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(\mathrm{sign}(t_1^{(1)}t_1^{(2)}), a_{22}, a_{33}, a_{12}, \mathrm{sign}(t_2^{(1)}t_1^{(2)}), a_{13}, a_{31}, a_{23}, a_{32}\big) : |a_{22}| \leq 1, |a_{33}| \leq 1, |a_{12}| \leq 1, |a_{13}| \leq 1, |a_{31}| \leq 1, |a_{23}| \leq 1, |a_{32}| \leq 1\Big\}.$$

If $t_3^{(2)} = 0, t_1^{(2)}t_2^{(2)} \neq 0$, *then*

$$NA(\mathcal{L}(^2\ell_1^3))(W_1, W_2) = \Big\{ \pm \big(\mathrm{sign}(t_1^{(1)}t_1^{(2)}), \mathrm{sign}(t_2^{(1)}t_2^{(2)}), a_{33}, \mathrm{sign}(t_1^{(1)}t_2^{(2)}), \mathrm{sign}(t_2^{(1)}t_1^{(2)}), a_{13}, a_{31}, a_{23}, a_{32}\big) : |a_{33}| \leq 1, |a_{13}| \leq 1, |a_{31}| \leq 1, |a_{23}| \leq 1, |a_{32}| \leq 1\Big\}.$$

Proof. It is an immediate particular case of Theorem 2. □

Remark 1. *Notice that Theorem 2 is an algorithm to find a characterization of the elements in the set* $NA(\mathcal{L}(^m\ell_1^n))(W_1, \cdots, W_m)$ *for any given unit vectors* $W_1, \ldots, W_m \in S_{\ell_1^n}$*: for given* $n, m \in \mathbb{N}$ *with* $n, m \geq 2$*, we could explicitly describe the set* $NA(\mathcal{L}(^m\ell_1^n))(W_1, \cdots, W_m)$ *by computing all possible finitely many cases as like those of Corollary 2.*

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