Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 4(66), No. 2 - 2024, 159-172
https://doi.org/10.31926/but.mif.2024.4.66.2.9

PERMANENT SOLUTIONS FOR SOME MHD MOTIONS OF GENERALIZED BURGERS FLUIDS THROUGH A POROUS MEDIUM IN CYLINDRICAL DOMAINS

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Dedicated to Professor Marin Marin on the occasion of his 70th anniversary

Abstract

Some isothermal motions of the incompressible generalized Burgers fluids in cylindrical domains are investigated when the magnetic and porous effects are taken into consideration. Analytical expressions are established for the dimensionless steady state velocity fields corresponding to motions between two infinite horizontal coaxial circular cylinders. The respective motions are generated by oscillatory or constant pressure gradients which act along the common axis of cylinders. Similar velocities for motions through an infinite circular cylinder are obtained as limiting cases of previous solutions. All results can be easily particularized to give similar solutions for the incompressible Burgers, Oldroyd-B, Maxwell and Newtonian fluids. Finally, some numerical results are graphical represented and discussed. It was found that the fluid velocity diminishes for increasing values of the magnetic and porous parameters. It means the fluid moves slower in the presence of a magnetic field or porous medium.

2020 Mathematics Subject Classification: MSC: 76A05.
Key words: permanent solutions, MHD axial motions, porous medium, generalized Burgers fluids, pressure gradient.

1 Introduction

It well-known the fact that exact solutions of the different boundary value problems can characterize the behavior of a moving fluid or a solid in deformation. In addition, these solutions can be used as tests to verify numerical schemes that are developed for the study of more complex problems of motion of fluids or deformation of solids. On the other hand, the steady state (long time or permanent) solutions for unsteady motions of fluids which become steady in time

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are necessary and sufficient to determine the required time to reach the steady or permanent state. From mathematical point of view, this is the time after which the diagrams of the starting solutions overlap with those of steady state solutions. In practice, this time is very important for the experimental researchers who want to know the transition moment to the steady or permanent state. Of course, the presence of a magnetic field or porous medium during a fluid motion can increase or decrease this time and the interplay between a moving electrically conducting fluid and a magnetic field has significant effects in domains of physics, chemistry and engineering. On the other hand, the fluid motions through porous media have multiple applications in various domains such as oil reservoir technology, petroleum industry, agricultural engineering and geophysical and astrophysical investigations.

The main purpose of this work is to provide analytical expressions for the steady state solutions corresponding to some isothermal magnetohydrodynamic (MHD) motions of the largest class of rate type fluids through a porous medium induced by an oscillatory or constant pressure gradient in cylindrical domains. Among the first exact solutions for unsteady motions of non-Newtonian fluids in such domains we remember those of Rajagopal [8], Ramkisson [10], Bandelli and Rajagopal [1] and Rajagopal and Bhatnagar [9]. Interesting extensions of the previous results to a large class of rate type fluids, names incompressible generalized Burgers fluids (IGBFs), have been realized by Fetecau et al. [2], Tong and Hu [12], Corina Fetecau et al. [3], Fetecau et al. [4] and Khan et al. [7]. Some exact solutions for MHD motions of incompressible Oldroyd-B fluids through a porous medium in a circular cylinder have been recently developed by Hamza [6]. The fluid motion is induced by different time dependent pressure gradients. Extension of these results to incompressible Burgers fluids has been developed by Rabia Safdar et al. [11].

In the following, we shall provide close-form expressions for the steady state solutions of some isothermal MHD unidirectional motions of IGBFs through a porous medium between two infinite coaxial circular cylinders. The constitutive equations of these fluids, as they have been defined in [5], are given by the following relations

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S},\ \ \mathbf{S} + \lambda_1 \frac{\delta \mathbf{S}}{\delta t} + \lambda_2 \frac{\delta^2 \mathbf{S}}{\delta t^2} = \mu \left(\mathbf{A} + \lambda_3 \frac{\delta \mathbf{A}}{\delta t} + \lambda_4 \frac{\delta^2 \mathbf{A}}{\delta t^2} \right), \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $-p\mathbf{I}$ is the indeterminate spherical stress, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{A}$ is the first Rivlin-Ericksen tensor, μ is the fluid viscosity, λ_1 and λ_3 ($< \lambda_1$) are relaxation and retardation times, λ_2 and λ_4 are other material constants whose dimension is the square of time and $\delta/\delta t$ denotes the time upper-convected derivative. When $\lambda_4 = 0$, $\lambda_2 = \lambda_4 = 0$ or $\lambda_2 = \lambda_3 = \lambda_4 = 0$, the governing equations (1) define incompressible Burgers, Oldroyd-B and Maxwell fluids, respectively. The constitutive equations of incompressible Newtonian fluids are recovered when $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$.

2 Problem formulation and governing equations

Let us consider an electrical conducting IGBF at rest in a porous medium between two infinite horizontal coaxial circular cylinders of radii R_0 and R_1 ($> R_0$). In the following one considers motions of IGBFs whose velocity vector $\mathbf{w}$ is given by the relation

$$\mathbf{w} = \mathbf{w}(r,t) = w(r,t)\mathbf{e}_z;\ R_0 < r < R_1,\ t > 0, \tag{2}$$

reported to a suitable cylindrical coordinate system r, θ and z in which $\mathbf{e}_z$ is the unit vector in the z-direction. Such a motion, for which the continuity equation is identically satisfied, can be generated by an oscillating pressure gradient

$$\frac{\partial p}{\partial z} = -P\cos(\omega t) \quad \text{or} \quad \frac{\partial p}{\partial z} = -P\sin(\omega t), \tag{3}$$

which after the moment $t = 0^+$ acts on the inner fluid along the common axis of cylinders. Here, the dimensional constant P is the amplitude and ω is the frequency of oscillations.

Assuming that the extra-stress tensor $\mathbf{S}$, as well as the velocity vector $\mathbf{w}$, is a function of r and t only and using the fact that the fluid has been at rest up to the initial moment $t = 0$, the equalities $(1)_2$ and (2) allow us to show that the non-trivial shear stress $\tau(r,t) = S_{rz}(r,t)$ satisfies the partial differential equation [3, 4]

$$\left(1 + \lambda_1\frac{\partial}{\partial t} + \lambda_2\frac{\partial^2}{\partial t^2}\right)\tau(r,t) = \mu\left(1 + \lambda_3\frac{\partial}{\partial t} + \lambda_4\frac{\partial^2}{\partial t^2}\right)\frac{\partial w(r,t)}{\partial r}; \qquad R_0 < r < R_1,\ t > 0. \tag{4}$$

The balance of the linear momentum, in the presence of a circular magnetic field $\mathbf{B} = (0, B, 0)$ that acts on the fluid after the initial moment $t = 0$, reduces to the relevant partial differential equation (see for instance [11, Equation (3.21)] for Burgers fluids)

$$\rho\frac{\partial w(r,t)}{\partial t} = -\frac{\partial p}{\partial z} + \frac{\partial \tau(r,t)}{\partial r} + \frac{1}{r}\tau(r,t) - \sigma B^2 w(r,t) + \mathcal{R}(r,t); \qquad R_0 < r < R_1,\ t > 0, \tag{5}$$

in which ρ is the fluid density, σ is its electrical conductivity and the Darcy's resistance $\mathcal{R}(r,t)$ has to satisfy the partial differential equation

$$\left(1 + \lambda_1\frac{\partial}{\partial t} + \lambda_2\frac{\partial^2}{\partial t^2}\right)\mathcal{R}(r,t) = -\mu\frac{\varphi}{k}\left(1 + \lambda_3\frac{\partial}{\partial t} + \lambda_4\frac{\partial^2}{\partial t^2}\right)w(r,t); \qquad R_0 < r < R_1,\ t > 0. \tag{6}$$

In the above relation φ and k are the porosity and the permeability of the porous medium.

The initial and boundary conditions corresponding to these motion problems are

$$w(r,0) = \left.\frac{\partial w(r,t)}{\partial t}\right|_{t=0} = \left.\frac{\partial^2 w(r,t)}{\partial t^2}\right|_{t=0} = 0;\ R_0 < r < R_1, \tag{7}$$

$$\tau(r,0) = \left.\frac{\partial \tau(r,t)}{\partial t}\right|_{t=0} = \left.\frac{\partial^2 \tau(r,t)}{\partial t^2}\right|_{t=0} = 0;\quad R_0 < r < R_1, \tag{8}$$

$$\mathcal{R}(r,t) = \left.\frac{\partial \mathcal{R}(r,t)}{\partial t}\right|_{t=0} = 0;\quad R_0 < r < R_1, \tag{9}$$

respectively,

$$w(R_0,t) = w(R_1,t) = 0;\ t > 0. \tag{10}$$

Introducing the next non-dimensional variables, functions and parameter

$$\begin{aligned} r^* &= \frac{1}{R_1}\,r, \quad z^* = \frac{1}{R_1}\,z, \quad t^* = \frac{v}{R_1^2}\,t, \quad w^* = \frac{\mu}{PR_1^2}\,w, \\ \tau^* &= \frac{1}{PR_1}\,\tau, \quad \mathcal{R}^* = \frac{1}{P}\,\mathcal{R}, \quad p^* = \frac{1}{PR_1}\,p, \quad \omega^* = \frac{R_1^2}{v}\,\omega, \end{aligned} \tag{11}$$

in Equations (4), (5) and (6) and dropping out the star notation, one obtains the dimensionless forms

$$\left(1 + a\,\frac{\partial}{\partial t} + b\,\frac{\partial^2}{\partial t^2}\right)\tau(r,t) = \left(1 + c\,\frac{\partial}{\partial t} + d\,\frac{\partial^2}{\partial t^2}\right)\frac{\partial w(r,t)}{\partial r}; \quad \alpha < r < 1,\ t > 0, \tag{12}$$

$$\frac{\partial w(r,t)}{\partial t} = -\frac{\partial p}{\partial z} + \frac{\partial \tau(r,t)}{\partial r} + \frac{1}{r}\,\tau(r,t) - Mw(r,t) + \mathcal{R}(r,t); \quad \alpha < r < 1,\ t > 0, \tag{13}$$

$$\left(1 + a\,\frac{\partial}{\partial t} + b\,\frac{\partial^2}{\partial t^2}\right)\mathcal{R}(r,t) = -K\left(1 + c\,\frac{\partial}{\partial t} + d\,\frac{\partial^2}{\partial t^2}\right)w(r,t); \quad \alpha < r < 1,\ t > 0. \tag{14}$$

for the governing equations of the two motions. The initial and boundary conditions (7)–(10) remain unchanged as form while the equalities (3) become

$$\frac{\partial p}{\partial z} = -\cos(\omega t) \quad \text{or} \quad \frac{\partial p}{\partial z} = -\sin(\omega t).$$

In the above relations, $v = \mu/\rho$ is the kinematic viscosity of the fluid, $\alpha = R_0/R_1$ and the non-dimensional constants a, b, c, d are given by the relations

$$a = \frac{v}{R_1^2}\,\lambda_1,\ b = \frac{v^2}{R_1^4}\,\lambda_2,\ c = \frac{v}{R_1^2}\,\lambda_3,\ d = \frac{v^2}{R_1^4}\,\lambda_4, \tag{15}$$

while the magnetic and porous parameters M and K, respectively, have the following expressions

$$M = \frac{\sigma B^2}{\rho}\,\frac{R_1^2}{v} = \frac{R_1^2}{\mu}\,\sigma B^2, \quad K = \frac{\varphi}{k}\,R_1^2. \tag{16}$$

In the following, in order to avoid a possible confusion, we denote by $w_c(r,t)$, $\tau_c(r,t)$, $\mathcal{R}_c(r,t)$, and $w_s(r,t)$, $\tau_s(r,t)$, $\mathcal{R}_s(r,t)$ the dimensionless starting solutions of the two motion problems which become steady in time. They are induced by the non-dimensional oscillatory pressure gradients $-\cos(\omega t)$ and $-\sin(\omega t)$, respectively, and have to satisfy the system of partial differential equations (12)–(14), the initial conditions (7)–(9) and the boundary conditions (10). These solutions can describe the fluid motion some time after its initiation. After this time, which is the time to reach the steady or permanent state, the fluid behavior is described by the dimensionless steady state (permanent or long time) solutions $w_{cp}(r,t)$, $\tau_{cp}(r,t)$, $\mathcal{R}_{cp}(r,t)$ or $w_{sp}(r,t)$, $\tau_{sp}(r,t)$, $\mathcal{R}_{sp}(r,t)$.

In practice, this time is very important for the experimental researchers who want to know the transition moment to the steady state. From mathematical point of view, it is the time after which the diagrams of starting solutions superpose with those of the steady state solutions. In order to determine this time for a given motion, it is necessary and sufficient to know the steady state solutions. This is the reason that in the next section we shall provide analytical expressions for the steady state solutions only. These expressions, which satisfy the governing equations and the corresponding boundary conditions, are independent of the initial conditions. They can be used to find the need time to touch the steady state by means of graphical representations by comparing with the starting solutions (numerical solutions).

3 Analytical expressions for dimensionless steady state solutions

As we already mentioned, the steady state solutions are independent of the initial conditions. Consequently, in order to determine them only the governing equations (12)–(14) and the boundary conditions (10) are necessary. Eliminating the shear stress $\tau(r,t)$ and the Darcy's resistance $\mathcal{R}(r,t)$ between the equations (12)–(14) one finds the governing equation

$$\begin{aligned}\left(1 + a\,\frac{\partial}{\partial t} + b\,\frac{\partial^2}{\partial t^2}\right)\frac{\partial w(r,t)}{\partial t} &= \left(1 + c\,\frac{\partial}{\partial t} + d\,\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r}\right)w(r,t)\\ &-M\left(1 + a\,\frac{\partial}{\partial t} + b\,\frac{\partial^2}{\partial t^2}\right)w(r,t) - L\left(1 + c\,\frac{\partial}{\partial t} + d\,\frac{\partial^2}{\partial t^2}\right)w(r,t)\\ &-\left(1 + a\,\frac{\partial}{\partial t} + b\,\frac{\partial^2}{\partial t^2}\right)\frac{\partial p}{\partial z};\ \alpha < r < 1,\ t > 0,\end{aligned} \tag{17}$$

for the dimensionless velocity field $w(r,t)$. In order to determine the steady state velocities $w_{cp}(r,t)$ and $w_{sp}(r,t)$ in a very simple way, we use the complex velocity

$$w_p(r,t) = w_{cp}(r,t) + iw_{sp}(r,t);\ \alpha < r < 1,\ t \in \mathbb{R}, \tag{18}$$

where i is the imaginary unit.

This complex velocity $w_p(r,t)$ has to satisfy the governing equation

$$\begin{aligned}\left(1 + a\,\frac{\partial}{\partial t} + b\,\frac{\partial^2}{\partial t^2}\right)\frac{\partial w_p(r,t)}{\partial t} &= \left(1 + c\,\frac{\partial}{\partial t} + d\,\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r}\right) w_p(r,t)\\ &-M\left(1 + a\,\frac{\partial}{\partial t} + b\,\frac{\partial^2}{\partial t^2}\right) w_p(r,t) - K\left(1 + c\,\frac{\partial}{\partial t} + d\,\frac{\partial^2}{\partial t^2}\right) w_p(r,t)\\ &+(1 - b\omega^2 + i\omega a)e^{i\omega t};\ \alpha < r < 1,\ t \in \mathbb{R},\end{aligned} \tag{19}$$

and the boundary conditions

$$w_p(\alpha,t) = w_p(1,t) = 0;\ t \in \mathbb{R}. \tag{20}$$

Bearing in mind the linearity of the governing equation (19) and the form of its free term, we are looking for a solution of the form

$$w_p(r,t) = W(r)e^{i\omega t};\ \alpha < r < 1, \tag{21}$$

in which $W(\cdot)$ is an unknown complex function. Introducing $w_p(r,t)$ from Equation (21) in (19) one finds that the function $W(\cdot)$ has to satisfy the ordinary differential equation

$$\frac{d^2W(r)}{dr^2} + \frac{1}{r}\,\frac{dW(r)}{dr} - \beta W r) + \gamma = 0;\ \alpha < r < 1, \tag{22}$$

where

$$\beta = \frac{(M + i\omega)(1 - b\omega^2 + i\omega a) + K(1 - d\omega^2 + i\omega c)}{1 - d\omega^2 + i\omega c},\ \gamma = \frac{1 - b\omega^2 + i\omega a}{1 - d\omega^2 + i\omega c}. \tag{23}$$

The solution of the equation (22) is the sum of the particular solution γ/β of the non-homogeneous equation and the general solution of the homogeneous equation

$$\frac{d^2W(r)}{dr^2} + \frac{1}{r}\,\frac{dW(r)}{dr} - \beta W(r) = 0;\ \alpha < r < 1. \tag{24}$$

Making the change of variable $s = r\sqrt{\beta}$, Equation (24) takes the form of a Bessel equation, namely

$$s^2\,\frac{d^2W(s)}{ds^2} + s\,\frac{dW(s)}{ds} - s^2W(s) = 0. \tag{25}$$

Consequently, the general solution of the equation (22) is

$$W(r) = C_1 I_0(r\sqrt{\beta}) + C_2 K_0(r\sqrt{\beta}) + \gamma/\beta;\ \alpha < r < 1, \tag{26}$$

in which $I_0(\cdot)$ and $K_0(\cdot)$ are the modified Bessel functions of the first and second kind, respectively, of zero order. The complex constants C_1 and C_2 are obtained from the boundary conditions (20) and the complex velocity $w_p(r,t)$ is given by the next relation

$$w_p(r,t) = \frac{\gamma}{\beta}\left[1 + AI_0(r\sqrt{\beta}) + BK_0(r\sqrt{\beta})\right]e^{i\omega t};\ \alpha < r < 1,\ t \in \mathbb{R}. \tag{27}$$

In the above relation, the complex constants A and B have the expressions

$$\begin{aligned} A &= \frac{K_0(\alpha\sqrt{\beta}) - K_0(\sqrt{\beta})}{I_0(\alpha\sqrt{\beta})K_0(\sqrt{\beta}) - I_0(\sqrt{\beta})K_0(\alpha\sqrt{\beta})}, \\ B &= \frac{I_0(\sqrt{\beta}) - I_0(\alpha\sqrt{\beta})}{I_0(\alpha\sqrt{\beta})K_0(\sqrt{\beta}) - I_0(\sqrt{\beta})K_0(\alpha\sqrt{\beta})}. \end{aligned} \tag{28}$$

It results that $w_{cp}(r,t)$ and $w_{sp}(r,t)$ are given by the following relations

$$w_{cp}(r,t) = \text{Re}\left\{\left[1 + AI_0(r\sqrt{\beta}) + BK_0(r\sqrt{\beta})\right]\frac{\gamma e^{i\omega t}}{\beta}\right\};\ \alpha < r < 1,\ t \in \mathbb{R}, \tag{29}$$

$$w_{sp}(r,t) = \text{Im}\left\{\left[1 + AI_0(r\sqrt{\beta}) + BK_0(r\sqrt{\beta})\right]\frac{\gamma e^{i\omega t}}{\beta}\right\};\ \alpha < r < 1,\ t \in \mathbb{R}. \tag{30}$$

These solutions can be also written in simpler forms, namely

$$w_{cp}(r,t) = Q\cos(\omega t) - \omega R\sin(\omega t) + \text{Re}\left\{\left[AI_0(r\sqrt{\beta}) + BK_0(r\sqrt{\beta})\right]\frac{\gamma e^{i\omega t}}{\beta}\right\};\ \alpha < r < 1,\ t \in \mathbb{R}, \tag{31}$$

$$w_{sp}(r,t) = Q\sin(\omega t) + \omega R\cos(\omega t) + \text{Im}\left\{\left[AI_0(r\sqrt{\beta}) + BK_0(r\sqrt{\beta})\right]\frac{\gamma e^{i\omega t}}{\beta}\right\};\ \alpha < r < 1,\ t \in \mathbb{R}, \tag{32}$$

where

$$Q = \frac{q(1-b\omega^2) + ar\omega^2}{q^2 + (\omega r)^2},\quad R = \frac{aq - r(1-b\omega^2)}{q^2 + (\omega r)^2}, \tag{33}$$

$$q = (1-b\omega^2)M + (1-d\omega^2)K - a\omega^2,\quad r = aM + cK + 1 - b\omega^2. \tag{34}$$

The expressions of $\tau_{cp}(r,t)$, $\mathcal{R}_{cp}(r,t)$ and $\tau_{sp}(r,t)$, $\mathcal{R}_{sp}(r,t)$ can be immediately obtained following the same way as before and using relations (12) and (14). The velocity fields corresponding to same motions of IGBFs, but in absence of magnetic field or porous medium, can be obtained taking $M = 0$ or $K = 0$, respectively, in Equations (29), (30) or (31), (32). If both parameters M and K are zero, the dimensional forms of the relations (29) and (30) coincide with those obtained by Fetecau et al. [4, Equations (18) and (19)]. In addition, the similar solutions for incompressible Burgers, Oldroyd-B, Maxwell and Newtonian fluids performing the same motions are obtained making $d = 0$, $d = b = 0$,

$d = c = b = 0$, $d = c = b = a = 0$, respectively, in the general solutions (29) and (30).

The dimensionless steady state velocity fields $w_{Ncp}(r,t)$ and $w_{Nsp}(r,t)$ corresponding to incompressible Newtonian fluids, for instance, have the forms

$$\begin{aligned} w_{Ncp}(r,t) &= \frac{K_{eff}\cos(\omega t) + \omega\sin(\omega t)}{\omega^2 + K_{eff}^2} \\ &+ \mathrm{Re}\left\{\frac{CI_0(r\sqrt{i\omega + K_{eff}}\,) + DK_0(r\sqrt{i\omega + K_{eff}}\,)}{i\omega + K_{eff}} e^{i\omega t}\right\}; \\ &\alpha < r < 1, t \in \mathbb{R}, \end{aligned} \tag{35}$$

$$\begin{aligned} w_{Nsp}(r,t) &= \frac{K_{eff}\sin(\omega t) - \omega\cos(\omega t)}{\omega^2 + K_{eff}^2} \\ &+ \mathrm{Im}\left\{\frac{CI_0(r\sqrt{i\omega + K_{eff}}\,) + DK_0(r\sqrt{i\omega + K_{eff}}\,)}{i\omega + K_{eff}} e^{i\omega t}\right\}; \\ &\alpha < r < 1,\ t \in \mathbb{R}, \end{aligned} \tag{36}$$

in which

$$C = \frac{K_0(\alpha\sqrt{i\omega + K_{eff}}) - K_0(\sqrt{i\omega + K_{eff}})}{I_0(\alpha\sqrt{i\omega + K_{eff}})K_0(\sqrt{i\omega + K_{eff}}) - I_0(\sqrt{i\omega + K_{eff}})K_0(\alpha\sqrt{i\omega + K_{eff}}\,)}, \tag{37}$$

$$D = \frac{I_0(\sqrt{i\omega + K_{eff}}) - I_0(\alpha\sqrt{i\omega + K_{eff}})}{I_0(\alpha\sqrt{i\omega + K_{eff}})K_0(\sqrt{i\omega + K_{eff}}) - I_0(\sqrt{i\omega + K_{eff}})K_0(\alpha\sqrt{i\omega + K_{eff}}\,)}, \tag{38}$$

In the above relations, $K_{eff} = M + K$ is called the effective permeability [5]. Consequently, the steady state velocities fields $w_{Ncp}(r,t)$ and $w_{Nsp}(r,t)$ corresponding to such motions of the incompressible Newtonian fluids do not depend of parameters M and K independently but by a combination of them and a two parameter approach, as in many papers from the existing literature, is superfluous.

3.1 Limiting case $\alpha \to 0$ (MHD motions through a porous medium in a circular cylinder)

Making $\alpha \to 0$ in Equations (29), (30) and (31), (32) one obtains the dimensionless steady state velocities

$$w_{cp} = \mathrm{Re}\left\{\left[1 - \frac{I_0(r\sqrt{\beta})}{I_0(\sqrt{\beta})}\right]\frac{\gamma \mathbf{e}^{i\omega t}}{\beta}\right\};\ 0 < r < 1,\ t \in \mathbb{R}, \tag{39}$$

$$w_{sp} = \mathrm{Im}\left\{\left[1 - \frac{I_0(r\sqrt{\beta})}{I_0(\sqrt{\beta})}\right]\frac{\gamma \mathbf{e}^{i\omega t}}{\beta}\right\};\ 0 < r < 1,\ t \in \mathbb{R}, \tag{40}$$

or, equivalently,

$$w_{cp}(r,t) = Q\cos(\omega t) - \omega R\sin(\omega t) - \mathrm{Re}\left\{\frac{I_0(r\sqrt{\beta})}{I_0(\sqrt{\beta})}\ \frac{\gamma \mathbf{e}^{i\omega t}}{\beta}\right\}; \quad 0 < r < 1,\ t \in \mathbb{R}, \tag{41}$$

$$w_{sp}(r,t) = Q\sin(\omega t) + \omega R\cos(\omega t) - \mathrm{Im}\left\{\frac{I_0(r\sqrt{\beta})}{I_0(\sqrt{\beta})}\ \frac{\gamma \mathbf{e}^{i\omega t}}{\beta}\right\}; \quad 0 < r < 1,\ t \in \mathbb{R}, \tag{42}$$

corresponding to MHD motions of IGBFs through a porous medium in an infinite circular cylinder of radius R_1. It is easily to observe that taking $d = 0$ or $b = d = 0$ in Equation (39), as it was to be expected, $w_{cp}(r,t)$ become identical to the velocity fields obtained by Rabia Safdar [11, Equation (5.51)] or Hamza [6, Equation (47)], respectively, in which $P_0 = 1$.

The Newtonian velocity fields corresponding to same motions through an infinite circular cylinder can be written is simple forms

$$\begin{aligned} w_{Ncp}(r,t) &= \frac{K_{eff}\cos(\omega t) + \omega\sin(\omega t)}{\omega^2 + K_{eff}^2} \\ &- \mathrm{Re}\left\{\frac{I_0(r\sqrt{i\omega + K_{eff}})}{I_0(\sqrt{i\omega + K_{eff}})}\ \frac{\mathbf{e}^{i\omega t}}{i\omega + K_{eff}}\right\};\ 0 < r < 1,\ t \in \mathbb{R}, \end{aligned} \tag{43}$$

$$\begin{aligned} w_{Nsp}(r,t) &= \frac{K_{eff}\sin(\omega t) - \omega\cos(\omega t)}{\omega^2 + K_{eff}^2} \\ &- \mathrm{Im}\left\{\frac{I_0(r\sqrt{i\omega + K_{eff}})}{I_0(\sqrt{i\omega + K_{eff}})}\ \frac{\mathbf{e}^{i\omega t}}{i\omega + K_{eff}}\right\};\ 0 < r < 1,\ t \in \mathbb{R}. \end{aligned} \tag{44}$$

3.2 Limiting case $\omega \to 0$ (fluid motion due to a constant pressure gradient $\partial p/\partial z = -P$)

Taking the limit of the equalities (29) and (41) when the oscillations' frequency $\omega \to 0$ one finds the dimensionless steady velocities

$$w_{2p}(r) = \frac{1}{K_{eff}} + \frac{1}{K_{eff}}\ \mathrm{Re}\left[E I_0(r\sqrt{K_{eff}}) + F K_0(r\sqrt{K_{eff}})\right]; \quad \alpha < r < 1,\ \ t \in \mathbb{R}, \tag{45}$$

$$w_{1p}(r) = \frac{1}{K_{eff}} - \frac{1}{K_{eff}}\ \mathrm{Re}\left[\frac{I_0(r\sqrt{K_{eff}})}{I_0(\sqrt{K_{eff}})}\right];\ 0 < r < 1,\ \ t \in \mathbb{R}, \tag{46}$$

corresponding to the two MHD motions of IGBFs induced by a constant pressure gradient $\partial p/\partial z = -P$ through a porous medium between two infinite coaxial circular cylinders or through an infinite cylinder, respectively. In Equation (45), the two constants E and F are given by the relations

$$E = \frac{K_0(\alpha\sqrt{K_{eff}}) - K_0(\sqrt{K_{eff}})}{I_0(\alpha\sqrt{K_{eff}})K_0(\sqrt{K_{eff}}) - I_0(\sqrt{K_{eff}})K_0(\alpha\sqrt{K_{eff}})}, \tag{47}$$

$$F = \frac{I_0(\sqrt{K_{eff}}) - I_0(\alpha\sqrt{K_{eff}})}{I_0(\alpha\sqrt{K_{eff}})K_0(\sqrt{K_{eff}}) - I_0(\sqrt{K_{eff}})K_0(\alpha\sqrt{K_{eff}})}. \tag{48}$$

The steady velocities $w_{2p}(r)$ and $w_{1p}(r)$, which can be also obtained making $\omega \to 0$ in Equations (35) and (39), are identical to the similar solutions of incompressible Newtonian fluids performing the same motions. This is not a surprise because the governing equations and the boundary conditions corresponding to steady motions of incompressible Newtonian and non-Newtonian fluids are identical. Finally, taking the limits of the equalities (45) and (46) when $K_{eff} \to 0$ one obtains the steady velocity fields

$$w_{2p}(r)) = \frac{1-r^2}{4} + \frac{\alpha^2-1}{4}\frac{\ln r}{\ln\alpha};\ \alpha < r < 1, \tag{49}$$

$$w_{1p}(r) = \frac{1-r^2}{4};\ 0 < r < 1, \tag{50}$$

corresponding to the same motions of IGBFs or incompressible Newtonian fluids but in the absence of magnetic and porous effects. Convergence of expressions of $w_{2p}(r)$ and $w_{1p}(r)$ given by Equations (45) and (46) to those from the relations (49) and (50) for $K_{eff} \to 0$ is also graphically proved by Figures 1 when $R_0 = 0.2$ and $R_1 = 0.8$ ($\alpha = 0.25$). In addition, from these figures it clearly results that the fluid velocity decreases for increasing values of the effective permeability K_{eff}. It means that the fluid moves slower in the presence of a magnetic field or porous medium.

Simple computations show that these velocities satisfy the governing equation and the corresponding boundary conditions. In addition, as it was to be expected taking the limit of $w_{2p}(r)$ from Equations (45) and (49) when $\alpha \to 0$ one finds the expressions of $w_{1p}(r)$ given by Equations (46) and (50), respectively. The dimensionless steady shear stresses corresponding to the steady motion due to a constant pressure gradient between two infinite coaxial cylinders or through an infinite circular cylinder are given by the following two relations

$$\tau_{2p}(r) = \frac{\alpha^2-1}{4}\frac{1}{r\ln\alpha} - \frac{r}{2},\ \alpha < r < 1;\quad \tau_{1p}(r) = -\frac{r}{2},\ 0 < r < 1. \tag{51}$$

4 Some numerical results and discussion

In this work there have been investigated isothermal MHD motions of IGBFs through a porous medium in cylindrical domains. The fluid motion is generated

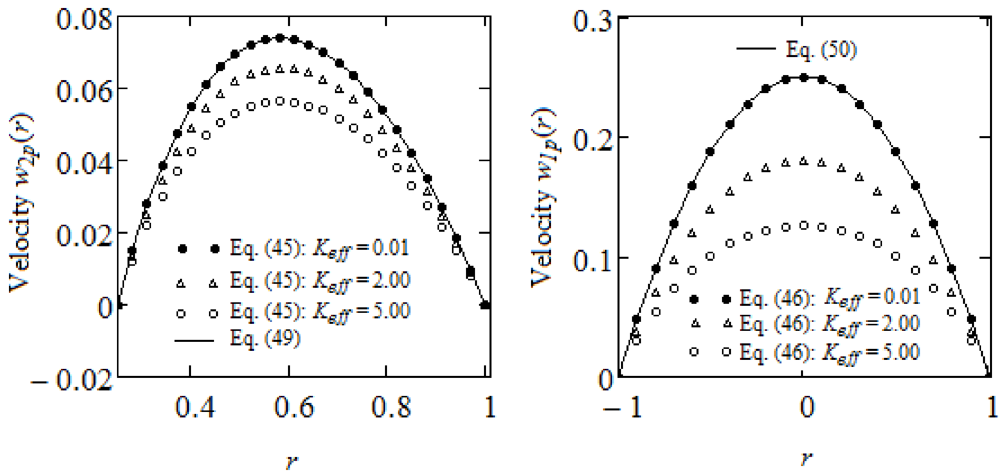

Figure 1: Convergence of steady velocities $w_{2p}(r)$ and $w_{1p}(r)$ given by Equations (45) and (46), respectively, to $w_{2p}(r)$ and $w_{1p}(r)$ given by Equations (49) and (50) when $K_{eff} \to 0$.

by an oscillatory or constant pressure gradient. Closed-form expressions have been established for dimensionless steady state velocity fields corresponding to motions between two infinite horizontal coaxial circular cylinders and through an infinite horizontal circular cylinder. All solutions are presented in simple forms in terms of the modified Bessel functions $I_0(\cdot)$ and $K_0(\cdot)$. They can be immediately particularized to give similar solutions for incompressible Burgers, Oldroyd-B, Maxwell and Newtonian fluids performing the same motions. The Newtonian solutions do not depend of the magnetic and porous parameters independently but by a combination of them that is called the effective permeability.

In order to bring to light some characteristics of the fluid behavior in such motions of IGBFs, in Figures 2 and 3 have been presented, for comparison, the variations in time of the velocities $w_{cp}(r,t)$ and $w_{sp}(r,t)$ given by Equations (39) and (40) along the axis of cylinder (i.e., when the spatial variable $r = 1/2$) for $\omega = \pi/6$, fixed values of the material constants $a = 1$, $b = 0.8$, $c = 0.6$, $d = 0.5$. and increasing values of the magnetic and porous parameters M and K, respectively. In all cases the oscillatory behavior of the two motions, as well as the phase difference between them, is easily observable. Furthermore, at same values of physical parameters, the amplitudes of the oscillations are identical for motions due to cosine or sine oscillations of the pressure gradient. In addition, both velocities are decreasing functions with respect to the two parameters M and K. As expected, this result is in accord with that resulting from Figures 1.

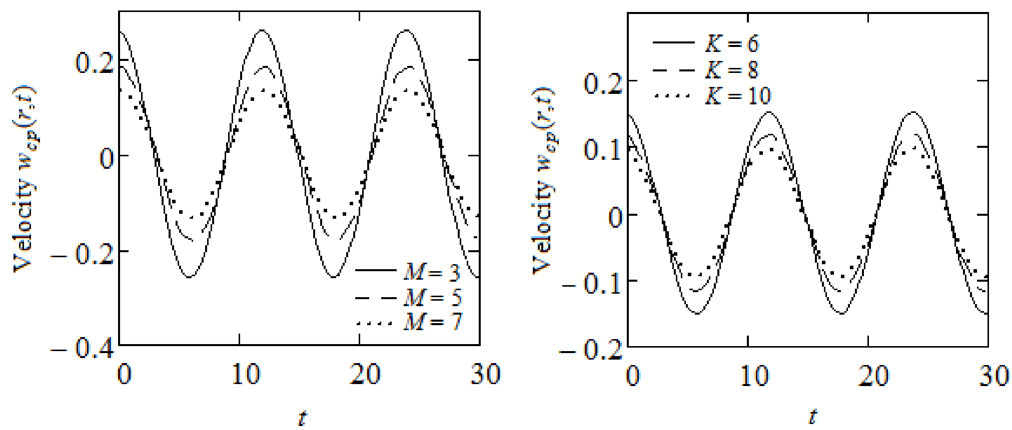

Figure 2: Time variation of the velocity $w_{cp}(r,t)$ given by Equation (39) along the cylinder axis for $K = 0.5$ with three values of M, and $M = 0.8$ with three values of K.

5 Conclusions

In this paper, we presented inaugural closed-form expressions delineating the steady-state velocities pertinent to MHD motions of IGBFs within the interstice of two infinitely extended coaxial circular cylinders across a porous medium. Additionally, we derive analogous solutions for MHD motions of same fluids traversing through an infinite circular cylinder embedded within a porous medium, which are discerned as limiting scenarios stemming from prior findings. In both instances, the fluidic dynamics are engendered by oscillatory or constant pressure gradients. Solutions furnished herein readily lend themselves to adaptation, yielding commensurate outcomes for incompressible fluids of various rheological characteristics, encompassing Burgers, Oldroyd-B, Maxwell, and Newtonian fluids, executing analogous motions. Furthermore, it merits highlighting that the steady solutions corresponding to the motions of incompressible Newtonian and non-Newtonian fluids, instigated by a constant pressure gradient, are congruent. This congruency arises from the identical nature of the equations governing the respective fluidic motions.

Conclusively, to elucidate certain facets of the fluidic behavior, graphical depictions have been included in Figures 1 through 3. Figure 1 clearly shows that the fluid velocity declines for increasing values of the effective permeability K_{eff}. It means that the fluid flows slower in the presence of a magnetic field or porous medium. Oscillatory behavior of the two motions and the phase difference between them are easily observed from Figures 2 and 3.

Finally, we end with two very important observations, namely:

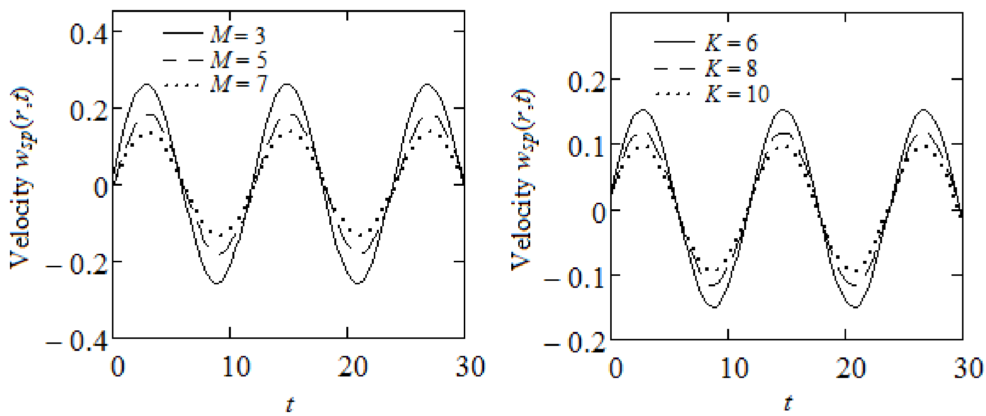

Figure 3: Time variation of the velocity $w_{sp}(r,t)$ given by Equation (40) along the cylinder axis for $K = 0.5$ with three values of M, and $M = 0.8$ with three values of K.

1. Present solutions can be used to determine the required time to reach the steady state.

2. In absence of the porous medium, eliminating the velocity $w(r,t)$ between relations (12) and (13) one obtains a governing equation for the shear stress $\tau(r,t)$, namely:

$$\begin{aligned}\left(1 + a\frac{\partial}{\partial t} + b\frac{\partial^2}{\partial t^2}\right)\frac{\partial \tau(r,t)}{\partial t} \\ = \left(1 + c\frac{\partial}{\partial t} + d\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\tau(r,t) \\ -M\left(1 + a\frac{\partial}{\partial t} + b\frac{\partial^2}{\partial t^2}\right)\tau(r,t)\ \alpha < r < 1,\ t > 0.\end{aligned} \tag{52}$$

Consequently, in the future, isothermal MHD motion problems of IGBFs in cylindrical domains can be solved when the shear stress is prescribed on the boundary.

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