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ANALYSIS OF A DYNAMIC ELECTRO-VISCOELASTIC CONTACT PROBLEM

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Dedicated to Professor Marin Marin on the occasion of his 70th anniversary

Abstract

In this work, we analyze a mathematical problem for dynamic contact between two electro-viscoelastic bodies with adhesion, normal compliance, and damage. An inclusion of the parabolic type describes the evolution of damage. A first order differential equation explains the development of the bonding field. We create a variational formulation for the model and demonstrate the existence and uniqueness of the weak solution. Parabolic inequalities, variational inequalities, and the Banach fixed point theorem form the foundation for the proof.

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Key words: dynamic process, piezoelectric materials, normal compliance, fixed point, damage field, adhesion field.

1 Introduction

The study of different techniques to analyse contact problems is developing rapidly in recent years, Therefore, the engineering literature on this subject is rather abundant (see, e.g., [7, 13] and the references therein).

Piezoelectricity is the capacity of certain crystals, ceramics, DNA, and various proteins to produce a voltage when they are subjected to mechanical stress. General models for piezoelectric effects can be found in [2, 9].

The constitutive laws which utilize internal variables to characterize the changing state of a material during a deformation process have been proposed by several investigators. The damage is one of these internal state variables, it is an

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extremely important topic in design engineering. There exists a very large engineering literature on it. General models of mechanical damage, were introduced in [6], and considered by many authors, we can see [5, 13]. The normal compliance contact condition allows the interpenetration of the body's surface into the obstacle and it was used in various references, see e.g. [10, 12]. Adhesion processes are important in many industrial settings, we can found The results of the mathematical analysis of various adhesive contact problems in [4, 13] and references therein.

The aim of this paper is to study the coupling of an electro-viscoelastic problem with normal compliance, and a dynamic contact problem with adhesion and damage. The contact is modelled with adhesion and normal compliance. We model the material's behavior with an electro-viscoelastic constitutive law with damage. We derive a variational formulation of the problem and prove the existence of a unique weak solution.

The document is structured as shown below. The physical environment and the mechanical issue are both described in Section 2. We introduce a few notations, describe the presumptions based on the problem data, and construct the variational formulation of the model in Section 3. Our primary existence and uniqueness result, Theorem 1, is stated in Section 4. Nonlinear evolution equations using monotone operators, a conventional existence and uniqueness conclusion based on parabolic inequalities, and fixed-point arguments are used in the theorem's proof.

2 Problem statement

This section studies the physical setting as it is depicted below. Two electro-viscoelastic bodies, occupy two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d(d = 2, 3)$. The boundary Γ^ω is assumed to be Lipschitz continuous, and is decomposed into two measurable parts Γ^ω_a and Γ^ω_b, on one hand, and on three separate measurable parts Γ^ω_1, Γ^ω_2 and Γ^ω_3, on the other hand, such that $meas(\Gamma^\omega_1) > 0$, $meas(\Gamma^\omega_a) > 0$. Let's indicate by $[0, T]$, the time period of importance, where $T > 0$. The Ω^ω bodies are exposed to volume electric charges of density q^ω_0 and $\boldsymbol{f}^\omega_0$ forces. The displacement field disappears at $\Gamma^\omega_1 \times (0, T)$, due to fixation of the two bodies there. Furthermore, we suppose that the electrical potential disappears on $\Gamma^\omega_a \times (0, T)$. A surface electric charge of density q^ω_2 is required on $\Gamma^\omega_b \times (0, T)$ and the surface tractions $\boldsymbol{f}^\omega_2$ act on $\Gamma^\omega_2 \times (0, T)$. The two bodies are in adhesive contact along the common part $\Gamma^1_3 = \Gamma^2_3 = \Gamma_3$.

With these assumptions, the following is the classical formulation of the dynamic problem for adhesive contact between two electro-viscoelastic substances with damage and normal compliance.

Problem P. For $\omega = 1, 2$, find a displacement field $\boldsymbol{u}^\omega : \Omega^\omega \times (0, T) \longrightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\omega : \Omega^\omega \times (0, T) \longrightarrow \mathbb{S}^d$, an electric potential field $\zeta^\omega : \Omega^\omega \times (0, T) \longrightarrow \mathbb{R}$, a damage field $\xi^\omega : \Omega^\omega \times (0, T) \longrightarrow \mathbb{R}$, a bonding field $\varsigma : \Gamma_3 \times (0, T) \longrightarrow \mathbb{R}$ and an electric displacement field $\boldsymbol{D}^\omega : \Omega^\omega \times (0,T) \longrightarrow \mathbb{R}^d$ such that

$$\boldsymbol{\sigma}^\omega(t) = \mathcal{A}^\omega(\varepsilon(\dot{\boldsymbol{u}}^\omega(t))) + \mathcal{B}^\omega(\varepsilon(\boldsymbol{u}^\omega(t)), \xi^\omega) - (\mathcal{E}^\omega)^* E^\omega(\zeta^\omega) \text{ in } \Omega^\omega \times (0,T), \tag{1}$$

$$\boldsymbol{D}^\omega = \mathcal{E}^\omega \varepsilon(\boldsymbol{u}^\omega) - B^\omega \nabla \zeta^\omega \quad \text{in } \Omega^\omega \times (0,T), \tag{2}$$

$$\dot{\xi}^\omega - \kappa^\omega \Delta \xi^\omega + \partial\varphi_{K^\omega}(\xi^\omega) \ni \Psi^\omega(\varepsilon(\boldsymbol{u}^\omega), \xi^\omega) \quad \text{in } \Omega^\omega \times (0,T), \tag{3}$$

$$\rho^\omega \ddot{\boldsymbol{u}}^\omega = \operatorname{Div} \boldsymbol{\sigma}^\omega + \boldsymbol{f}_0^\omega \quad \text{in } \Omega^\omega \times (0,T), \tag{4}$$

$$\operatorname{div} \boldsymbol{D}^\omega - q_0^\omega = 0 \quad \text{in } \Omega^\omega \times (0,T), \tag{5}$$

$$\boldsymbol{u}^\omega = 0 \quad \text{on } \Gamma_1^\omega \times (0,T), \tag{6}$$

$$\boldsymbol{\sigma}^\omega \boldsymbol{\nu}^\omega = \boldsymbol{f}_2^\omega \quad \text{on } \Gamma_2^\omega \times (0,T), \tag{7}$$

$$-\sigma_\nu = p_\nu(u_\nu^1 + u_\nu^2) - \gamma_\nu \varsigma^2 R_\nu(u_\nu^1 + u_\nu^2) \text{ on } \Gamma_3 \times (0,T), \tag{8}$$

$$-\sigma_\tau = p_\tau(\varsigma) R_\tau(|\boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2|) \text{ on } \Gamma_3 \times (0,T), \tag{9}$$

$$\dot{\varsigma} = H_{ad}(\varsigma, \hat{\varsigma}, R_\nu(u_\nu^1 + u_\nu^2), R_\tau(|\boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2|)) \text{ on } \Gamma_3 \times (0,T), \tag{10}$$

$$\frac{\partial \xi^\omega}{\partial \nu^\omega} = 0 \quad \text{on } \Gamma^\omega \times (0,T), \tag{11}$$

$$\zeta^\omega = 0 \quad \text{on } \Gamma_a^\omega \times (0,T), \tag{12}$$

$$\boldsymbol{D}^\omega . \boldsymbol{\nu}^\omega = q_2^\omega \quad \text{on } \Gamma_b^\omega \times (0,T), \tag{13}$$

$$\boldsymbol{u}^\omega(0) = \boldsymbol{u}_0^\omega,\ \dot{\boldsymbol{u}}^\omega(0) = \boldsymbol{v}_0^\omega,\ \xi^\omega(0) = \xi_0^\omega \quad \text{in } \Omega^\omega, \tag{14}$$

$$\varsigma(0) = \varsigma_0 \text{ on } \Gamma_3. \tag{15}$$

First, The electro-viscoelastic constitutive law with damage, is shown in equations (1) and (2). The relation (3) indicates the damage field's evolution. The equilibrium equations for the stress and electric displacement fields are expressed by formulas (4) and (5), respectively. The displacement and traction boundary conditions are given by equations (6) and (7) , respectively. Equations (8) and (9) discribe the normal compliance condition with adhesion, where R_ν, R_τ are the truncation operators, γ_ν is the adhesion coefficient and p_ν, p_τ are given functions. The evolution of the bonding field is described by the ordinary differential equation (10). The homogeneous Neumann boundary condition is described by quation (11), where $\frac{\partial \xi^\omega}{\partial \nu^\omega}$ is the normal derivative of ξ^ω. The electric boundary conditions are shown in (12) and (13). Finally, the initial data are given in (14) and (15).

3 Variational formulation and preliminaries

For a weak formulation of the problem, we introduce some notation and preliminary material to be used in the rest of the paper. Further details can be found in [13, 1, 5]. In the sequel, $\mathbb{S}^d$ represent the space of second-order symmetric tensors on $\mathbb{R}^d$, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Let us introduce the following function spaces:

$$H^\omega = \{\boldsymbol{v}^\omega = (v_i^\omega);\ v_i^\omega \in L^2(\Omega^\omega)\}, \quad H_1^\omega = \{\boldsymbol{v}^\omega = (v_i^\omega);\ v_i^\omega \in H^1(\Omega^\omega)\},$$

$$\mathcal{H}^\omega = \{\boldsymbol{\tau}^\omega = (_{ij});\ _{ij} =_{ji} \in L^2(\Omega^\omega)\}, \quad \mathcal{H}_1^\omega = \{\boldsymbol{\tau}^\omega = (_{ij}) \in \mathcal{H}^\omega;\ \operatorname{div} \boldsymbol{\tau}^\omega \in H^\omega\}.$$

The spaces H^ω, H_1^ω, $\mathcal{H}^\omega$ and $\mathcal{H}_1^\omega$ are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{aligned}
(\boldsymbol{u}^\omega, \boldsymbol{v}^\omega)_{H^\omega} &= \int_{\Omega^\omega} \boldsymbol{u}^\omega . \boldsymbol{v}^\omega dx, \\
(\boldsymbol{u}^\omega, \boldsymbol{v}^\omega)_{H_1^\omega} &= \int_{\Omega^\omega} \boldsymbol{u}^\omega . \boldsymbol{v}^\omega dx + \int_{\Omega^\omega} \nabla \boldsymbol{u}^\omega . \nabla \boldsymbol{v}^\omega dx, \\
(\boldsymbol{\sigma}^\omega, \boldsymbol{\tau}^\omega)_{\mathcal{H}^\omega} &= \int_{\Omega^\omega} \boldsymbol{\sigma}^\omega . \boldsymbol{\tau}^\omega dx, \\
(\boldsymbol{\sigma}^\omega, \boldsymbol{\tau}^\omega)_{\mathcal{H}_1^\omega} &= \int_{\Omega^\omega} \boldsymbol{\sigma}^\omega . \boldsymbol{\tau}^\omega dx + \int_{\Omega^\omega} \operatorname{div} \boldsymbol{\sigma}^\omega . \operatorname{Div} \boldsymbol{\tau}^\omega dx.
\end{aligned}$$

and the associated norms $\|.\|_{H^\omega}$, $\|.\|_{H_1^\omega}$, $\|.\|_{\mathcal{H}^\omega}$, and $\|.\|_{\mathcal{H}_1^\omega}$ respectively. Here and below we use the notation

$$\nabla \boldsymbol{u}^\omega = (u^\omega_{i,j}),\ \varepsilon(\boldsymbol{u}^\omega) = (\varepsilon_{ij}(\boldsymbol{u}^\omega)), \quad \varepsilon_{ij}(\boldsymbol{u}^\omega) = \frac{1}{2}(u^\omega_{i,j} + u^\omega_{j,i}), \quad \forall \boldsymbol{u}^\omega \in H_1^\omega,$$
$$\operatorname{Div} \boldsymbol{\sigma}^\omega = (\sigma^\omega_{ij,j}), \quad \forall \boldsymbol{\sigma}^\omega \in \mathcal{H}_1^\omega.$$

Also, we introduce the sets $\mathcal{Z}, V^\omega$ for the bonding and displacement fields, respectively.

$$\mathcal{Z} = \left\{ \theta \in L^\infty\big(0,T; L^2(\Gamma_3)\big);\ 0 \le \theta(t) \le 1 \ \ \forall t \in [0,T], \text{ a.e. on } \Gamma_3 \right\},$$
$$V^\omega = \left\{ \boldsymbol{v}^\omega \in H_1^\omega;\ \boldsymbol{v}^\omega = 0 \text{ on } \Gamma_1^\omega \right\}.$$

Since $meas(\Gamma_1^\omega) > 0$, V^ω is a real Hilbert space (see [11, p.79]), with the inner product and the associated norm given by

$$(\boldsymbol{u}^\omega, \boldsymbol{v}^\omega)_{V^\omega} = (\varepsilon(\boldsymbol{u}^\omega), \varepsilon(\boldsymbol{v}^\omega))_{\mathcal{H}^\omega}, \quad \|\boldsymbol{u}^\omega\|_{V^\omega} = \|\varepsilon(u^\omega)\|_{\mathcal{H}^\omega}. \tag{16}$$

Moreover, by the Sobolev trace theorem, the Korn's inequality and (16), there exists a constant $c_0 > 0$, such that

$$\|\boldsymbol{v}^\omega\|_{L^2(\Gamma_3)^d} \le c_0 \|\boldsymbol{v}^\omega\|_{V^\omega} \quad \forall \boldsymbol{v}^\omega \in V^\omega. \tag{17}$$

Since $meas(\Gamma_a^\omega) > 0$, W^ω is a real Hilbert space, with the inner product given by

$$(\zeta^\omega, \Psi^\omega)_{W^\omega} = \int_{\Omega^\omega} \nabla \zeta^\omega . \nabla \Psi^\omega dx, \tag{18}$$

and let $||.||_{W^\omega}$ be the associated norm. By the Sobolev trace theorem, the Friedrichs-Poincaré's inequality and (18), there exists a constant $c_1 > 0$, such that

$$\|\zeta^\omega\|_{L^2(\Omega)^\omega} \le c_1 \|\zeta^\omega\|_{W^\omega} \quad \forall \zeta^\omega \in W^\omega. \tag{19}$$

In addition, we introduce the spaces

$$\mathbb{L}_0^\omega = L^2(\Omega^\omega),\ \mathbb{L}_1^\omega = H^1(\Omega^\omega),\ W^\omega = \left\{ \Psi^\omega \in \mathbb{L}_1^\omega;\ \Psi^\omega = 0 \text{ on } \Gamma_a^\omega \right\},$$
$$\mathcal{W}^\omega = \left\{ \boldsymbol{D}^\omega = (D_i^\omega);\ D_i^\omega \in L^2(\Omega^\omega),\ \operatorname{div} \boldsymbol{D}^\omega \in L^2(\Omega^\omega) \right\}.$$

On the space $\mathcal{W}^\omega$, we use the inner product

$$(\boldsymbol{D}^\omega, \boldsymbol{\Psi}^\omega)_{\mathcal{W}^\omega} = \int_{\Omega^\omega} \boldsymbol{D}^\omega.\boldsymbol{\Psi}^\omega dx + \int_{\Omega^\omega} \operatorname{div} \boldsymbol{D}^\omega. \operatorname{div} \boldsymbol{\Psi}^\omega dx,$$

and the associated norm $\||.||_{\mathcal{W}^\omega}$. In order to simplify the notations, we define the spaces

$$\begin{gathered} \boldsymbol{V} = V^1 \times V^2, \quad H = H^1 \times H^2, \quad H_1 = H_1^1 \times H_1^2, \\ \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2, \quad \mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2, \quad \mathbb{L}_0 = \mathbb{L}_0^1 \times \mathbb{L}_0^2, \\ \mathbb{L}_1 = \mathbb{L}_1^1 \times \mathbb{L}_1^2, \quad W = W^1 \times W^2, \quad \mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2. \end{gathered}$$

The spaces $\boldsymbol{V}$, $\mathbb{L}_1$, W and $\mathcal{W}$ are real Hilbert spaces endowed with the canonical inner products denoted by $(.,.)_{\boldsymbol{V}}$, $(.,.)_{\mathbb{L}_1}$, $(.,.)_W$, and $(.,.)_{\mathcal{W}}$.

The associate norms will be denoted by $\|.\|_{\boldsymbol{V}}$, $\|.\|_{\mathbb{L}_1}$, $\|.\|_W$ and $\|.\|_{\mathcal{W}}$, respectively.

In the study of the Problem **P**, we consider the following assumptions:

The viscosity operator $\mathcal{A}^\omega : \Omega^\omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\left\{ \begin{array}{l} \text{(a) There exists } C^1_{\mathcal{A}^\omega}, C^2_{\mathcal{A}^\omega} > 0 \text{ such that,} \\ \qquad |\mathcal{A}^\omega(\boldsymbol{x},\boldsymbol{\epsilon})| \le C^1_{\mathcal{A}^\omega}|\boldsymbol{\epsilon}| + C^2_{\mathcal{A}^\omega} \quad \forall \boldsymbol{\epsilon} \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\omega. \\ \text{(b) There exists } m_{\mathcal{A}^\omega} > 0 \text{ such that} \\ \qquad (\mathcal{A}^\omega(\boldsymbol{x},\boldsymbol{\epsilon}_1) - \mathcal{A}^\omega(\boldsymbol{x},\boldsymbol{\epsilon}_2)) \cdot (\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2) \ge m_{\mathcal{A}^\omega}|\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2|^2 \\ \qquad \forall \boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\omega. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\omega(\boldsymbol{x},\boldsymbol{\epsilon}) \text{ is Lebesgue measurable on } \Omega^\omega, \\ \qquad \text{for any } \boldsymbol{\epsilon} \in \mathbb{S}^d. \\ \text{(d) The mapping } \boldsymbol{\epsilon} \mapsto \mathcal{A}^\omega(\boldsymbol{x},\boldsymbol{\epsilon}) \text{ is continuous on } \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\omega. \end{array} \right. \tag{20}$$

The elasticity operator $\mathcal{B}^\omega : \Omega^\omega \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{S}^d$ satisfies

$$\left\{ \begin{array}{l} \text{(a) There exists a constant } M_{\mathcal{B}^\omega} > 0 \text{ such that} \\ \qquad |\mathcal{B}^\omega(\boldsymbol{x},\boldsymbol{\epsilon}_1,r_1,d_1) - \mathcal{B}^\omega(\boldsymbol{x},\boldsymbol{\epsilon}_2,r_2,d_2)| \\ \qquad \le M_{\mathcal{B}^\omega}(|\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2| + |r_1 - r_2| + |d_1 - d_2|) \\ \qquad \forall \boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2 \in \mathbb{S}^d, \forall r_1, r_2, d_1, d_2 \in \mathbb{R}, \quad \text{a.e. } \boldsymbol{x} \in \Omega^\omega. \\ \text{(b) The mapping } \boldsymbol{x} \to \mathcal{B}^\omega(\boldsymbol{x},\boldsymbol{\epsilon},r,d) \text{ is Lebesgue} \\ \qquad \text{measurable on } \Omega^\omega, \text{ for any } \boldsymbol{\epsilon} \in \mathbb{S}^d \text{ and } r,d \in \mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{B}^\omega(\boldsymbol{x},\boldsymbol{0},0,0) \text{ belongs to } \mathcal{H}^\omega. \end{array} \right. \tag{21}$$

The electric permittivity operator $B^\omega = \left(B^\omega_{ij}\right) : \Omega^\omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\left\{ \begin{array}{l} \text{(a) } B^\omega_{ij} = B^\omega_{ji} \in L^\infty(\Omega^\omega), \ 1 \le i,j \le d. \\ \text{(b) There exists a constant } M_{B^\omega} > 0 \text{ such that} \\ \qquad B^\omega \pi.\pi \ge M_{B^\omega}|\pi|^2 \quad \forall \pi = (\pi_i) \in \mathbb{R}^d \text{ a.e. } \ x \in \Omega^\omega. \end{array} \right. \tag{22}$$

The piezoelectric operator $\mathcal{E}^\omega = (e^\omega_{ijk}) : \Omega^\omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies

$$e^\omega_{ijk} = e^\omega_{ikj} \in L^\infty(\Omega^\omega), \ 1 \le i,j,k \le d. \tag{23}$$

The damage source function $\Psi^\omega : \Omega^\omega \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\left\{\begin{array}{l} \text{(a) There exists a constant } M_{\Psi^\omega} > 0 \text{ such that} \\ \quad |\Psi^\omega(\boldsymbol{x}, \boldsymbol{\epsilon}_1, r_1, d_1) - \Psi^\omega(\boldsymbol{x}, \boldsymbol{\epsilon}_2, r_2, d_2)| \\ \quad \le M_{\Psi^\omega}(|\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2| + |r_1 - r_2| + |d_1 - d_2|) \\ \quad \forall \boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2 \in \mathbb{S}^d, \quad \forall r_1, r_2, d_1, d_2 \in \mathbb{R}, \text{ a.e.} \boldsymbol{x} \in \Omega^\omega. \\ \text{(b) } \forall \boldsymbol{\epsilon} \in \mathbb{S}^d \text{ et } r, d \in \mathbb{R}, \quad \Psi^\omega(., \boldsymbol{\epsilon}, r, d) \text{ is Lebesgue measurable} \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \Psi^\omega(\boldsymbol{x}, \boldsymbol{0}, 0, 0) \text{ belongs to } L^2(\Omega^\omega). \end{array}\right. \tag{24}$$

The normal compliance function $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\left\{\begin{array}{l} \text{(a) There exists a constant } L_\nu > 0 \text{ such that} \\ \quad |p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \le L_\nu |r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}, \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) } p_\nu(., r) \quad \text{is Lebesgue measurable on } \Gamma_3, \quad \text{for any } r \in \mathbb{R}. \\ \text{(c) } p_\nu(\boldsymbol{x}, r) = 0 \quad \forall r \le 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{array}\right. \tag{25}$$

The tangential contact function $p_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\left\{\begin{array}{l} \text{(a) There exists a constant } L_\tau > 0 \text{ such that} \\ \quad |p_\tau(\boldsymbol{x}, r_1) - p_\tau(\boldsymbol{x}, r_2)| \le L_\tau |r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}\ , \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) There exists a constant } M_\tau > 0 \text{ such that} \\ \qquad |p_\tau(\boldsymbol{x}, r)| \le M_\tau \ \forall r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) } p_\tau(., r) \text{ is Lebesgue measurable on } \Gamma_3, \ \forall r \in \mathbb{R}. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto p_\tau(\boldsymbol{x}, 0) \text{ belongs to } L^2(\Gamma_3). \end{array}\right. \tag{26}$$

The adhesion rate function $H_{ad} : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}$ satisfies:

$$\left\{\begin{array}{l} \text{(a) There exists } L_{ad} > 0 \text{ such that :} \\ \quad |H_{ad}(\boldsymbol{x}, r_1, d_1, s_1, \lambda_1) - H_{ad}(\boldsymbol{x}, r_2, d_2, s_2, \lambda_2)| \\ \quad \le L_\nu |r_1 - r_2| + |d_1 - d_2| + |s_1 - s_2| + |\lambda_1 - \lambda_2|, \\ \quad \forall r_1, r_2, d_1, d_2, s_1, s_2 \in \mathbb{R}, \ \lambda_1, \lambda_2 \in \mathbb{R}^{d-1}, \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) The map } \boldsymbol{x} \mapsto H_{ad}(\boldsymbol{x}, r, d, s, \lambda) \text{ is measurable on } \Gamma_3, \\ \quad \text{for any } r, d, s \in \mathbb{R}, \ \lambda \in \mathbb{R}^{d-1}. \\ \text{(c) The map } (r, d, s, \lambda) \mapsto H_{ad}(\boldsymbol{x}, r, d, s, \lambda) \\ \quad \text{is continuous on } \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(d) } H_{ad}(\boldsymbol{x}, 0, d, s, \lambda) = 0, \quad \forall d, s \in \mathbb{R}, \ \lambda \in \mathbb{R}^{d-1}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \\ \text{(e) } H_{ad}(\boldsymbol{x}, r, d, s, \lambda) \ge 0, \quad \forall r \le 0, d, s \in \mathbb{R}, \ \lambda \in \mathbb{R}^{d-1}, \\ \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3, \text{ and } H_{ad}(\boldsymbol{x}, r, d, s, \lambda) \le 0, \\ \quad \forall r \ge 1, d, s \in \mathbb{R}, \ \lambda \in \mathbb{R}^{d-1}, \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{array}\right. \tag{27}$$

The mass density satisfies

$$\rho^\omega \in L^\infty(\Omega^\omega), \text{ there exists } \rho^* > 0 \text{ such that } \quad \rho^\omega \ge \rho^* \text{ a.e. } \boldsymbol{x} \in \Omega^\omega. \tag{28}$$

We also suppose that The forces, tractions, volume and surface free charge densities have the regularity

$$\boldsymbol{f}_0^\omega \in L^2(0, T; L^2(\Gamma_2^\omega)^d), \quad \boldsymbol{f}_2^\omega \in L^2(0, T; L^2(\Gamma_2^\omega)^d), \tag{29}$$

$$q_0^\omega \in C(0,T;L^2(\Omega^\omega)), \quad q_2^\omega \in C(0,T;L^2(\Gamma_b^\omega)), \tag{30}$$

$$q_2^\omega(t) = 0 \quad \text{on } \Gamma_3 \ \ \forall t \in [0,T]. \tag{31}$$

The adhesion coefficient γ_ν satisfies: $\gamma_\nu \in L^\infty(\Gamma_3)$, $\gamma_\nu \geq 0$ a.e. on Γ_3.
The microcrak difussion k^ω coefficient verify: $k^\omega > 0$.
The initial data satisfy

$$\boldsymbol{u}_0^\omega \in \boldsymbol{V}^\omega, \quad \boldsymbol{v}_0^\omega \in H^\omega, \quad \xi_0^\omega \in K^\omega \ \ \text{a.e.} x \in \Omega^\omega, \tag{32}$$

$$\varsigma_0 \in L^2(\Gamma_3), \quad 0 \leq \varsigma_0 \leq 1, \ \ \text{a.e.} x \in \Gamma_3. \tag{33}$$

We will use a modified inner product on the Hilbert space, given by

$$((\boldsymbol{u},\boldsymbol{v}))_H = \sum_{\omega=1}^{2} (\rho^\omega \boldsymbol{u}^\omega, \boldsymbol{v}^\omega)_{H^\omega} \quad \forall\ \boldsymbol{u},\boldsymbol{v} \in H. \tag{34}$$

and we let $|||.|||_H$ be the associated norm given by

$$|||\boldsymbol{v}|||_H = ((\boldsymbol{v},\boldsymbol{v}))_H^{\frac{1}{2}} \quad \forall \boldsymbol{v} \in H. \tag{35}$$

We can prove that $|||.|||_H$ and $||.||_H$ are equivalent norms on H, and also the inclusion mapping of $(\boldsymbol{V}, ||.||_{\boldsymbol{V}})$ into $(H, ||.||_H)$ is continuous and dense. We denote by $\boldsymbol{V}'$ the dual space of $\boldsymbol{V}$. The Gelfand triple $\boldsymbol{V} \subset H \subset \boldsymbol{V}'$ can be written by matching H with its own dual. To indicate the duality pairing between $\boldsymbol{V}'$ and $\boldsymbol{V}$, we use the notation $(.,.)_{\boldsymbol{V}'\times\boldsymbol{V}}$, defined as follows

$$(\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = ((\boldsymbol{u},\boldsymbol{v}))_H \ \ \forall \boldsymbol{u} \in H, \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{36}$$

Next, we define the following mappings $a:\ \mathbb{L}_1 \times \mathbb{L}_1 \to \mathbb{R}$, $f:\ [0,T] \to \boldsymbol{V}'$, $q:\ [0,T] \to W$, $j_{ad}:\ L^\infty(\Gamma_3) \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ and $j_{vc}:\ L^\infty(\Gamma_3) \times \boldsymbol{V} \times V \to \mathbb{R}$ as follows:

$$a(\xi,\zeta) = \sum_{\omega=1}^{2} k^\omega \int_{\Omega^\omega} \nabla\xi^\omega . \nabla\zeta^\omega \ dx, \tag{37}$$

$$(\boldsymbol{f}(t),\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\omega=1}^{2} \int_{\Omega^\omega} \boldsymbol{f}_0^\omega(t).\boldsymbol{v}^\omega dx + \sum_{\omega=1}^{2} \int_{\Gamma_2^\omega} \boldsymbol{f}_2^\omega(t).v^\omega da \ \ \forall\ \boldsymbol{v} \in \boldsymbol{V}, \tag{38}$$

$$(q(t),\Psi)_W = \sum_{\omega=1}^{2} \int_{\Omega^\omega} q_0^\omega(t)\Psi^\omega dx - \sum_{\omega=1}^{2} \int_{\Gamma_b^\omega} q_2^\omega(t)\Psi^\omega da \ \ \forall \Psi \in W, \tag{39}$$

$$j_{ad}(\varsigma,u,v) = \int_{\Gamma_3} (-\gamma_\nu \varsigma^2 R_\nu(u_\nu^1 + u_\nu^2)v_\nu + p_\tau(\varsigma)R_\tau(|\boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2|)).v_\tau)da, \tag{40}$$

$$j_{vc}(u,v) = \int_{\Gamma_3} p_\nu(u_\nu^1 + u_\nu^2)v_\nu da. \tag{41}$$

By a standard procedure based on Green's formula, we derive the following variational formulation of the electro-mechanical problem (1)-(15).

Problem PV. Find a displacement field $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2) : [0,T] \to V$, a stress field $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) : [0,T] \to \mathcal{H}$, an electric potential field $\zeta = (\zeta^1, \zeta^2) : [0,T] \to W$, a damage field $\xi = (\xi^1, \xi^2) : [0,T] \to \mathbb{L}_1$, and an adhesion field $\varsigma : [0,T] \to L^\infty(\Gamma_3)$ such that

$$\boldsymbol{\sigma}^\omega(t) = \mathcal{A}^\omega(\varepsilon(\dot{\boldsymbol{u}}^\omega) + \mathcal{B}^\omega(\varepsilon(\boldsymbol{u}^\omega(t)), \xi^\omega) + (\mathcal{E}^\omega)^*\nabla\zeta^\omega \quad \text{in } \Omega^\omega \times (0,T), \tag{42}$$

$$\begin{gathered}(\ddot{\boldsymbol{u}}(t), \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\omega=1}^{2}(\mathcal{A}^\omega\varepsilon(\dot{\boldsymbol{u}}^\omega(t)), \varepsilon(\boldsymbol{v}^\omega))_{\mathcal{H}^\omega} \\ + \sum_{\omega=1}^{2}(\mathcal{B}^\omega(\varepsilon(u^\omega(t)\xi^\omega), \varepsilon(v^\omega))_{\mathcal{H}^\omega} + \sum_{\omega=1}^{2}\big((\mathcal{E}^\omega)^*\nabla\zeta^\omega(t), \varepsilon(v^\omega)\big)_{\mathcal{H}^\omega} \\ + j_{ad}(\varsigma(t), \boldsymbol{u}(t), \boldsymbol{v}) + j_{vc}(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \quad \forall v \in V, t \in (0,T),\end{gathered} \tag{43}$$

$$\begin{gathered}\xi(t) \in K \quad \forall t \in [0,T], \quad \sum_{\omega=1}^{2}(\dot{\xi^\omega}(t), \delta^\omega - \xi^\omega(t))_{L_0^\omega} + a(\xi(t), \delta - \xi(t)) \\ \geq \sum_{\omega=1}^{2}(\Psi^\omega(\varepsilon(u^\omega(t)), (t), \xi^\omega(t)), \delta^\omega - \xi^\omega(t))_{L_0^\omega} \quad \forall \delta \in K,\end{gathered} \tag{44}$$

$$\begin{gathered}\sum_{\omega=1}^{2}(B^\omega\nabla\varphi^\omega(t), \nabla\Psi^\omega)_{H^\omega} - \sum_{\omega=1}^{2}(\mathcal{E}^\omega\varepsilon(\boldsymbol{u}^\omega(t)), \nabla\Psi^\omega)_{H^\omega} = \\ (q(t), \Psi)_W \quad \forall \Psi \in W,\ t \in (0,T),\end{gathered} \tag{45}$$

$$\dot{\varsigma} = H_{ad}(\varsigma, \hat{\varsigma}, R_\nu(u_\nu^1 + u_\nu^2), R_\tau(|\boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2|)) \quad \text{on } \Gamma_3 \times (0,T) \tag{46}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0,\ \xi(0) = \xi_0,\ \varsigma(0) = \varsigma_0. \tag{47}$$

4 Existence and uniqueness result

Now, we propose our existence and uniqueness result.

Theorem 1. *Assume that* (20)-(33) *hold. Then there exists a unique solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}, \zeta, \xi, \varsigma, \boldsymbol{D}\}$ *to problem PV. Moreover, the solution satisfies*

$$\boldsymbol{u} \in W^{1,2}(0,T;V) \cap C^1(0,T;H),\ \ddot{\boldsymbol{u}} \in L^2(0,T;V'), \tag{48}$$

$$\zeta \in C(0,T;W), \tag{49}$$

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \quad (\mathrm{Div}\boldsymbol{\sigma}^1, \mathrm{Div}\boldsymbol{\sigma}^2) \in L^2(0,T;V'), \tag{50}$$

$$\xi \in H^1(0,T;\mathbb{L}_0) \cap L^2(0,T;\mathbb{L}_1), \tag{51}$$

$$\varsigma \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}. \tag{52}$$

$$\boldsymbol{D} \in C(0,T;\mathcal{W}). \tag{53}$$

The functions $\boldsymbol{u}, \boldsymbol{\sigma}, \zeta, \boldsymbol{D}, \xi, \varsigma$ which satisfy (42)-(47) and (48)-(53) are called a weak solution of the contact problem P. We conclude that, under the assumptions (20)-(39), the mechanical problem (1)-(15) has a unique weak solution satisfying (48)-(53). Let us now move on to the proof of theorem 1 which is carried out in several steps.

Let $\eta \in L^2(0,T;V')$ be given, in the first step we consider the following variational problem.

Problem $\mathbf{PV}_\eta^{\boldsymbol{u}}$. Find $\boldsymbol{u}_\eta = (\boldsymbol{u}_\eta^1, \boldsymbol{u}_\eta^2) : [0,T] \to \boldsymbol{V}$ such that

$$(\ddot{\boldsymbol{u}}_\eta(t), \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\omega=1}^{2} (\mathcal{A}^\omega \varepsilon(\dot{\boldsymbol{u}}^\omega(t)),\ \varepsilon(\boldsymbol{v}^\omega))_{\mathcal{H}^\omega} + (\eta(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} = (\boldsymbol{f}(t), \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{V}, a.e.\ t \in (0,T), \tag{54}$$

$$\boldsymbol{u}^\omega(0) = \boldsymbol{u}_0^\omega, \quad \dot{\boldsymbol{u}}^\omega(0) = \boldsymbol{v}_0^\omega \quad \text{in } \Omega^\omega. \tag{55}$$

To solve Problem $\mathbf{PV}_\eta^u$, we apply the following abstract existence and uniqueness result, which can be found in[13, p.48].

Theorem 2. *Let $\boldsymbol{V}$, H be as above, and let $A : \boldsymbol{V} \to \boldsymbol{V}'$ be a hemicontinuous and monotone operator which satisfies*

$$(A\boldsymbol{v}, \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \geq w\|\boldsymbol{v}\|_{\boldsymbol{V}}^2 + \lambda \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{56}$$

$$\|A\boldsymbol{v}\|_{\boldsymbol{V}'} \leq C(\|\boldsymbol{v}\|_{\boldsymbol{V}} + 1) \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{57}$$

for some constants $w > 0$, $C > 0$ and $\lambda \in \mathbb{R}$. Then, given $\boldsymbol{v}_0 \in H$ and $f \in L^2(0,T;\boldsymbol{V}')$, there exists a unique function $\boldsymbol{v}$ which satisfies

$$\boldsymbol{v} \in L^2(0,T;\boldsymbol{V}) \cap C(0,T;H), \quad \dot{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{V}'),$$
$$\dot{\boldsymbol{v}}(t) + A\boldsymbol{v}(t) = \boldsymbol{f}(t) \text{ a.e. } t \in (0,T),$$
$$\boldsymbol{v}(0) = \boldsymbol{v}_0.$$

We have the following result for the problem.

Lemma 1. *There exists a unique solution to Problem $\boldsymbol{PV}_\eta^u$ and it has its regularity expressed in* (48)*.*

Proof. We define the operator $A : \boldsymbol{V} \to \boldsymbol{V}'$ by

$$(A\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\omega=1}^{2} (\mathcal{A}^\omega \varepsilon(\boldsymbol{u}^\omega),\ \varepsilon(\boldsymbol{v}^\omega))_{\mathcal{H}^\omega} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}. \tag{58}$$

Using (16), (20) and (58) it follows that

$$\|A\boldsymbol{u} - A\boldsymbol{v}\|_{\boldsymbol{V}'}^2 \leq \sum_{\omega=1}^{2} \|\mathcal{A}^\omega \varepsilon(\boldsymbol{u}^\omega) - \mathcal{A}^\omega \varepsilon(\boldsymbol{v}^\omega)\|_{\mathcal{H}^\omega}^2 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V},$$

and keeping in mind the Krasnoselski Theorem (see [8, p.60]), we deduce that $A : \boldsymbol{V} \to \boldsymbol{V}'$ is a continuous operator and so is hemicontinuous. Now, by (16), (20) and (58), we find

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \geq \min\{m_{\mathcal{A}^1}, m_{\mathcal{A}^2}\}\|\boldsymbol{u} - \boldsymbol{v}\|_{\boldsymbol{V}}^2 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}, \tag{59}$$

i.e., that A is a monotone operator. Choosing $\boldsymbol{v}=0$ in (59) we obtain

$$\begin{aligned}(A\boldsymbol{u},\boldsymbol{u})_{\boldsymbol{V}'\times\boldsymbol{V}} &\geq m\|\boldsymbol{u}\|^2_{\boldsymbol{V}} - \|Ao\|^2_{\boldsymbol{V}'}\|\boldsymbol{u}\|_{\boldsymbol{V}}\\ &\geq \frac{1}{2}m\|\boldsymbol{u}\|^2_{\boldsymbol{V}} - \frac{1}{2m}\|Ao\|^2_{\boldsymbol{V}'} \quad \forall \boldsymbol{u}\in\boldsymbol{V}. \end{aligned} \tag{60}$$

Moreover, by (20) and (58) we find

$$\|A\boldsymbol{u}\|_{\boldsymbol{V}'} \leq C^1\|u\|_{\boldsymbol{V}} + C^2 \quad \forall \boldsymbol{u}\in\boldsymbol{V}. \tag{61}$$

where $C^1=\max\{C^1_{\mathcal{A}^1}, C^1_{\mathcal{A}^2}\}$ and $C^2=\max\{C^2_{\mathcal{A}^1}, C^2_{\mathcal{A}^2}\}$.

It follows now from Theorem 2 that there exists a unique function $\boldsymbol{v}_\eta$ which satisfies

$$\boldsymbol{v}_\eta \in L^2(0,T;\boldsymbol{V})\cap C(0,T;H), \quad \dot{\boldsymbol{v}}_\eta\in L^2(0,T;\boldsymbol{V}'), \tag{62}$$

$$\dot{\boldsymbol{v}}_\eta(t) + A\boldsymbol{v}_\eta(t) + \eta(t) = \boldsymbol{f}(t), \quad a.e.\ t\in[0,T] \tag{63}$$

$$\boldsymbol{v}_\eta(0)=\boldsymbol{v}_0. \tag{64}$$

Let $\boldsymbol{u}_\eta : [0,T]\to\boldsymbol{V}$ be the function defined by

$$\boldsymbol{u}_\eta(t) = \int_0^t \boldsymbol{v}_\eta(s)ds + \boldsymbol{u}_0 \quad \forall t\in[0,T]. \tag{65}$$

It follows from (58) and (62)–(65) that $\boldsymbol{u}_\eta$ is a unique solution of the variational problem $\mathbf{PV}^u_\eta$ and it satisfies the regularity expressed in (48). □

In the second step, let $\eta\in L^2(0,T;\boldsymbol{V}')$, we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 1 and we consider the following variational problem.

Problem $\mathbf{PV}^\zeta_\eta$. Find $\zeta_\eta=(\zeta^1_\eta,\zeta^2_\eta):[0,T]\to W$ such that

$$\begin{aligned}\sum_{\omega=1}^{2}(B^\omega\nabla\zeta^\omega_\eta(t),\nabla\phi^\omega)_{H^\omega} - \sum_{\omega=1}^{2}(\mathcal{E}^\omega\varepsilon(\boldsymbol{u}^\omega_\eta(t)),\nabla\phi^\omega)_{H^\omega}\\ = (q(t),\phi)_W \quad \forall\phi\in W, t\in(0,T).\end{aligned} \tag{66}$$

Lemma 2. *Problem PV^ζ_η has a unique solution ζ_η which satisfies the regularity* (51)*. Moreover, if ζ_i represents the solution of Problem PV^ζ_η corresponding to u_i, $i=1,2$, then there exists $C>0$ such that*

$$\|\zeta_1(t)-\zeta_2(t)\|_W \leq C\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V \qquad \forall t\in[0,T]. \tag{67}$$

Proof. We define a bilinear form $b: W\times W\to\mathbb{R}$ such that

$$b(\zeta,\phi) = \sum_{\omega=1}^{2}(B^\omega\nabla\zeta^\omega,\nabla\phi^\omega)_{H^\omega} \quad \forall\zeta,\phi\in W. \tag{68}$$

We use (19) and (22) to show that the bilinear form b is continuous, symmetric and coercive on W. Moreover using Riesz Representation Theorem we may define an element $L_\eta : [0,T] \to W$ such that

$$(L_\eta(t), \phi)_W = (q(t), \phi)_W + \sum_{\omega=1}^{2} (\mathcal{E}^\omega \varepsilon(\boldsymbol{u}_\eta^\omega(t)), \nabla \phi^\omega)_{H^\omega} \quad \forall \phi \in W, t \in (0,T).$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\zeta_\eta(t) \in W$ such that

$$b(\zeta_\eta(t), \phi) = (L_\eta(t), \phi)_W \quad \forall \phi \in W. \tag{69}$$

We conclude that $\zeta_\eta(t)$ is a solution of PV_η^ζ. Let $t_1, t_2 \in [0,T]$, it follows from (19),(22), (23) and (66) that

$$\|\zeta_\eta(t_1) - \zeta_\eta(t_2)\|_W \le C(\|\boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2)\|_V + \|q(t_1) - q(t_2)\|_W). \tag{70}$$

Since $\boldsymbol{u} \in C(0,T;V)$, and $q \in C(0,T;W)$ we deduce from (70), $\zeta_\eta \in C(0,T;W)$. Finally, inequality (67) is obtained by arguments similar to those used in the proof of the previous inequality, which concludes the proof. □

In the third step, we let $\mu \in L^2(0,T;\mathbb{L}_0)$, be given and consider the following variational problem for the damage field.

Problem $\mathbf{PV}_\mu^\xi$. Find $\xi_\mu = (\xi_\mu^1, \xi_\mu^2) : [0,T] \to \mathbb{L}_0$ such that

$$\begin{aligned} \xi_\mu(t) \in K, \quad & \sum_{\omega=1}^{2} (\dot{\xi}_\mu^\omega(t), \delta^\omega - \xi_\mu^\omega)_{\mathbb{L}_0^\omega} + a(\xi_\mu(t), \delta - \xi_\mu(t)) \\ & \ge \sum_{\omega=1}^{2} (\mu^\omega(t), \delta^\omega - \xi_\mu^\omega(t))_{\mathbb{L}_0^\omega} \quad \forall \delta \in K, \text{a.e.} t \in (0,T), \end{aligned} \tag{71}$$

$$\xi_\mu(0) = \xi_0. \tag{72}$$

Where $K = K^1 \times K^2$. The following abstract result for parabolic variational inequalities.

Lemma 3. *For all $\mu \in L^2(0,T;\mathbb{L}_0)$, there exists a unique solution ξ_μ to the auxiliary problem PV_μ^ξ satisfying* (51).

Proof. Using classical arguments of functional analysis concerning parabolic inequalities [3] with some algebraic computations, we find that that (71) has a unique solution $\tilde{\xi}_\mu$ having the regularity (51)

□

In the fifth step, we use the displacement field u_η obtained in Lemma 1 and we consider the following initial-value problem.

Problem $\mathbf{PV}_\eta^\varsigma$. Find $\varsigma_\eta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\varsigma_\eta} = H_{ad}(\varsigma_\eta, \hat{\varsigma}_\eta, R_\nu(u_{\eta\nu}^1 + u_{\eta\nu}^2), R_\tau(|\boldsymbol{u}_{\eta\tau}^1 - \boldsymbol{u}_{\eta\tau}^2|)) \quad \text{on } \Gamma_3 \times (0,T), \tag{73}$$

$$\varsigma_\eta(0) = \varsigma_0 \text{ in } \Omega^\omega. \tag{74}$$

Lemma 4. *There exists a unique solution $\varsigma_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))\bigcap \mathcal{Z}$ to problem PV^{ς}_{η}. Moreover, if $\boldsymbol{u}_i$ represents the solution of Problem PV^u_η for $\eta_i \in L^2(0,T;V')$, $i=1,2$, then there exists $C>0$ such that*

$$||\varsigma_1(t)-\varsigma_2(t)||_{L^2(\Gamma_3)} \le C\int_0^t ||\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)||_{\boldsymbol{V}}\,ds \quad \forall t\in[0,T]. \tag{75}$$

Proof. For the sake of simplicity we suppress the dependence of various functions on Γ_3, and note that the equalities and inequalities below are valid a.e. on Γ_3. Consider the mapping $G_\eta:[0,T]\times L^2(\Gamma_3)\to L^2(\Gamma_3)$ defined by

$$G_\eta(t,\varsigma) = H_{ad}(\varsigma_\eta,\hat{\varsigma}_\eta,R_\nu(u^1_{\eta\nu}+u^2_{\eta\nu}),R_\tau(|\boldsymbol{u}^1_{\eta\tau}-\boldsymbol{u}^2_{\eta\tau}|)). \tag{76}$$

It follows from the properties of the truncation operator R_ν and R_τ, that G_η is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\varsigma\in L^2(\Gamma_3)$, the mapping $t\to G_\eta(t,\varsigma)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. We deduce that there exists a unique function $\varsigma_\eta\in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution to problem $\mathrm{PV}^{\varsigma}_{\eta}$, such that $0\le\varsigma_\eta(t)\le 1$ for all $t\in[0,T]$, a.e. on Γ_3. Therefore, from the definition of the set $\mathcal{Z}$, we find that $\varsigma_\eta\in\mathcal{Z}$. From the Cauchy problem (73)-(74) we can write

$$\varsigma_i(t)=\varsigma_0-\int_0^t H_{ad}(\varsigma_i(s),\hat{\varsigma}_i(s),R_\nu(u^1_{i\nu}+u^2_{i\nu})(s),R_\tau(|\boldsymbol{u}^1_{i\tau}-\boldsymbol{u}^2_{i\tau}|)(s))ds$$

and then

$$\begin{aligned}
&||\varsigma_1(t)-\varsigma_2(t)||_{L^2(\Gamma_3)} \le C\int_0^t ||\varsigma_1(s)-\varsigma_2(s)||_{L^2(\Gamma_3)}ds\\
&+C\int_0^t ||R_\nu(u^1_{1\nu}(s)+u^2_{1\nu}(s))-R_\nu(u^1_{2\nu}(s)+u^2_{2\nu}(s))||_{L^2(\Gamma_3)}ds\\
&+C\int_0^t ||R_\tau(|\boldsymbol{u}^1_{1\tau}(s)-\boldsymbol{u}^2_{1\tau}(s)|)-R_\tau(|\boldsymbol{u}^1_{2\tau}(s)-\boldsymbol{u}^2_{2\tau}(s)|)||_{L^2(\Gamma_3)}ds.
\end{aligned}$$

Using the definition of R_ν and R_τ and writing $\varsigma_1=\varsigma_1-\varsigma_2+\varsigma_2$, we get

$$||\varsigma_1(t)-\varsigma_2(t)||_{L^2(\Gamma_3)} \le C\bigg(\int_0^t ||\varsigma_1(s)-\varsigma_2(s)||_{L^2(\Gamma_3)}ds+\int_0^t ||\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)||_{L^2(\Gamma_3)^d}ds\bigg).$$

Next, we apply Gronwall's inequality to deduce

$$||\varsigma_1(t)-\varsigma_2(t)||_{L^2(\Gamma_3)} \le C\int_0^t ||\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)||_{L^2(\Gamma_3)^d}ds,$$

and from the relation (17) we obtain

$$\|\varsigma_1(t)-\varsigma_2(t)\|_{L^2(\Gamma_3)} \le C\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V ds,$$

which concludes the proof of lemma 4. □

Finally as a consequence of these results and using the properties of the operator $\mathcal{B}^\omega$, the operator $\mathcal{E}^\omega$, the functional j and the functions Ψ^ω for $t \in [0,T]$, we consider the operator

$$\Pi(\eta,\mu)(t) = \Big(\Pi^1(\eta,\mu)(t), \Pi^2(\eta,\mu)(t)\Big) \in V' \times \mathbb{L}_0 \tag{77}$$

defined by the equations

$$\Big(\Pi^1(\eta,\mu)(t), \boldsymbol{v}\Big)_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\omega=1}^{2}\Big(\mathcal{B}^\omega(\varepsilon(\boldsymbol{u}_\eta^\omega(t), \xi_\mu^\omega(t)), \varepsilon(\boldsymbol{v}^\omega)\Big)_{\mathcal{H}^\omega} \tag{78}$$
$$+\sum_{\omega=1}^{2}\Big((\mathcal{E}^\omega)^*\nabla\varphi_\eta^\omega(t), \varepsilon(\boldsymbol{v}^\omega)\Big)_{\mathcal{H}^\omega} + j_{ad}(\varsigma_\eta(t), \boldsymbol{u}_\eta(t), \boldsymbol{v}) + j_{vc}(\boldsymbol{u}_\eta, \boldsymbol{v}) \quad \forall v \in \boldsymbol{V},$$

$$\Pi^2(\eta,\mu)(t) = \Big(\Psi^1(\varepsilon(\boldsymbol{u}_\eta^1), \xi_\mu^1), \Psi^2(\varepsilon(\boldsymbol{u}_\eta^2), \xi_\mu^2)\Big). \tag{79}$$

Here we use $\boldsymbol{u}_\eta, \zeta_\eta, \xi_\mu$ and ς_η obtained in lemmas 1, 2, 3 and 4.

Lemma 5. *The operator* Π *has a fixed point*

$$(\eta^*, \mu^*) \in L^2(0,T; V' \times \mathbb{L}_0).$$

Proof. Let $t \in (0,T)$ and $(\eta_1,\mu_1), (\eta_2,\mu_2), (\eta_3,\mu_3) \in L^2(0,T; \boldsymbol{V}' \times \mathbb{L}_0)$. We use the notation $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{\eta_i} = \dot{\boldsymbol{u}}_i$, $\ddot{\boldsymbol{u}}_{\eta_i} = \ddot{\boldsymbol{u}}_i$, $\varsigma_{\eta_i} = \varsigma_i$, $\zeta_{\eta_i} = \zeta_i$ and $\xi_{\mu_i} = \xi_i$, for $i = 1,2$. Let us start by using the hypotheses (21), (23), (25) and (26) and the definition of R_ν, R_τ, we can rewrite

$$\begin{aligned}
&||\Pi^1(\eta_1(t),\mu_1(t)) - \Pi^1(\eta_2(t),\mu_2(t))||_{\boldsymbol{V}'\times\mathbb{L}_0\times\mathbb{L}_0} \\
&\le \sum_{\omega=1}^{2} ||\mathcal{B}^\omega(\varepsilon(\boldsymbol{u}_1^\omega(t)), \tau_1^\omega(t), \xi_1^\omega(t)) - \mathcal{B}^\omega(\varepsilon(\boldsymbol{u}_2^\omega(t)), \tau_2^\omega(t), \xi_2^\omega(t))||_{\mathcal{H}^\omega} \\
&\quad + \sum_{\omega=1}^{2} ||(\mathcal{E}^\omega)^*\nabla\zeta_1^\omega(t) - (\mathcal{E}^\omega)^*\nabla\zeta_2^\omega(t)||_{\mathcal{H}^\omega} \\
&\quad + C_1\Big(||\zeta_1(t) - \zeta_2(t)||_W\Big) + C_2\Big(||p_\nu(u_{1\eta\nu}^1 + u_{1\eta\nu}^2) - p_\nu(u_{2\eta\nu}^1 + u_{2\eta\nu}^2)||_{L^2(\Gamma_3)} \\
&\quad + ||\varsigma_1^2(t)R_\nu(u_{1\eta\nu}^1 + u_{1\eta\nu}^2) - \varsigma_2^2(t)R_\nu(u_{2\eta\nu}^1 + u_{2\eta\nu}^2)||_{L^2(\Gamma_3)} \\
&\quad + ||p_\tau(\varsigma_1(t))R_\tau(|\boldsymbol{u}_{1\eta\tau}^1 - \boldsymbol{u}_{1\eta\tau}^2|) - p_\tau(\varsigma_2(t))R_\tau(|\boldsymbol{u}_{2\eta\tau}^1 - \boldsymbol{u}_{2\eta\tau}^2|)||_{L^2(\Gamma_3)}\Big).
\end{aligned}$$

Therefore

$$\begin{aligned}
&||\Pi^1(\eta_1(t),\mu_1(t)) - \Pi^1(\eta_2(t),\mu_2(t))||_{\boldsymbol{V}'\times\mathbb{L}_0\times\mathbb{L}_0} \\
&\le C\Big(||\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)||_{\boldsymbol{V}} + ||\tau_1(t) - \tau_2(t)||_{\mathbb{L}_0} \\
&\qquad + ||\xi_1(s) - \xi_2(s)||_{\mathbb{L}_0} + ||\zeta_1(t) - \zeta_2(t)||_W \\
&\qquad + ||\varsigma_1(t) - \varsigma_2(t)||_{L^2(\Gamma_3)}\Big) \quad \text{a.e. } t \in (0,T).
\end{aligned} \tag{80}$$

On the other hand, since $\boldsymbol{u}_i(t) = \boldsymbol{u}_0 + \int_0^t \dot{\boldsymbol{u}}_i(s)ds$, we know that for a.e. $t \in (0,T)$,

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\boldsymbol{V}} \leq \int_0^t \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|_{\boldsymbol{V}} ds. \tag{81}$$

We use (22), (23) and (66) to obtain

$$||\zeta_1(t) - \zeta_2(t)||_W^2 \leq C||\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)||_{\boldsymbol{V}}^2. \tag{82}$$

Applying Young's inequalitie, (80) becomes, via (75) , (81) and (82)

$$\begin{aligned} &||\Pi^1(\eta_1(t), \mu_1(t)) - \Pi^1(\eta_2(t), \mu_2(t))||_{\boldsymbol{V}'}^2 \\ &\leq C\Big(||\xi_1(t) - \xi_2(t)||_{\mathbb{L}_0}^2 + ||\tau_1(t) - \tau_2(t)||_{\mathbb{L}_0}^2 + ||\varsigma_1(t) - \varsigma_2(t)||_{L^2(\Gamma_3)} \\ &\quad + \int_0^t ||\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)||_{\boldsymbol{V}}^2 + ||\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)||_{\boldsymbol{V}}^2\, ds\Big) \ a.e.t \in (0,T). \end{aligned} \tag{83}$$

Furthermore, we find by taking the substitution $\eta = \eta_1$, $\eta = \eta_2$ in (58) and choosing $\boldsymbol{v} = \dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2$ as test function

$$\begin{aligned} &(\ddot{\boldsymbol{u}}_1 - \ddot{\boldsymbol{u}}_2, \dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2)_{\boldsymbol{V}' \times \boldsymbol{V}} + \sum_{\omega=1}^{2} (A^\omega \varepsilon(\dot{\boldsymbol{u}}_1^\omega) - A^\omega \varepsilon(\dot{\boldsymbol{u}}_2^\omega), \varepsilon(\dot{\boldsymbol{u}}_1^\omega - \dot{\boldsymbol{u}}_2^\omega))_{\mathcal{H}^\omega} \\ &\quad + (\eta_1 - \eta_2, \dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2)_{\boldsymbol{V}' \times \boldsymbol{V}} = 0 \quad \text{a.e. } t \in (0,T). \end{aligned}$$

By virtue of (20) and (28), using (34)−(36) this equation becomes

$$\begin{aligned} &\frac{(\rho^*)^2}{2} \frac{d}{dt} \|\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)\|_H^2 + \min(m_{\mathcal{A}^1}, m_{\mathcal{A}^2}) \|\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)||_{\boldsymbol{V}}^2 \\ &\quad \leq ||\eta_2(t) - \eta_1(t)|_{\boldsymbol{V}'} ||\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)||_{\boldsymbol{V}}. \end{aligned}$$

Integrating this inequality over the interval time variable $(0,t)$, Young inequality leads to

$$\begin{aligned} &(\rho^*)^2 \|\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)\|_H^2 + \min(m_{\mathcal{A}^1}, m_{\mathcal{A}^2}) \int_0^t ||\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)||_{\boldsymbol{V}}^2 ds \\ &\leq \frac{2}{\min(m_{\mathcal{A}^1}, m_{\mathcal{A}^2})} \int_0^t ||\eta_1(s) - \eta_2(s)||_{\boldsymbol{V}'}^2 ds. \end{aligned}$$

Consequently,

$$\int_0^t ||\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)||_{\boldsymbol{V}}^2 ds \leq C \int_0^t ||\eta_1(s) - \eta_2(s)||_{\boldsymbol{V}'}^2 ds \quad \text{a.e. } t \in (0,T). \tag{84}$$

which also implies, using a variant of (81), that

$$||\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)||_{\boldsymbol{V}}^2 \leq C \int_0^t ||\eta_1(s) - \ \eta_2(s)||_{\boldsymbol{V}'}^2 ds \quad \text{a.e.} t \in (0,T), \tag{85}$$

From the relation (71) we deduce that

$$(\dot{\xi}_1 - \dot{\xi}_2, \xi_1 - \xi_2)_{\mathbb{L}_0} + a(\xi_1 - \xi_2, \xi_1 - \xi_2) \leq (\mu_1 - \mu_2, \xi_1 - \xi_2)_{\mathbb{L}_0} \text{ a.e.} t \in [0, T].$$

Integrating the previous inequality with respect to time, using the initial conditions $\xi_1(0) = \xi_0$ and $\xi_2(0) = \xi_0$ and the inequality $a(\xi_1 - \xi_2, \xi_1 - \xi_2) \geq 0$ to find

$$\frac{1}{2} ||\xi_1(t) - \xi_2(t)||^2_{\mathbb{L}_0} \leq \int_0^t (\mu_1(s) - \mu_2(s), \xi_1(s) - \xi_2(s))_{\mathbb{L}_0} ds,$$

which implies that

$$||\xi_1(t) - \xi_2(t)||^2_{\mathbb{L}_0} \leq \int_0^t \|\mu_1(s) - \mu_2(s)\|^2_{\mathbb{L}_0} ds + \int_0^t \|\xi_1(s) - \xi_2(s)\|^2_{\mathbb{L}_0} ds.$$

This inequality, combined with Gronwall's inequality, leads to

$$\|\xi_1(t) - \xi_2(t)\|^2_{\mathbb{L}_0} \leq C \int_0^t \|\mu_1(s) - \mu_2(s)\|^2_{\mathbb{L}_0} ds. \tag{86}$$

We can infer, using (83)–(86), that

$$\begin{aligned} &\|\Pi^1(\eta_1(t), \mu_1(t)) - \Pi^1(\eta_2(t), \mu_2(t))\|^2_{\boldsymbol{V}' \times \mathbb{L}_0 \times \mathbb{L}_0} \\ &\quad \leq C\big(\|\eta_1(t) - \eta_2(t)\|^2_{V'}\|^2_{\mathbb{L}_0} + \|\mu_1(t) - \mu_2(t)\|^2_{\mathbb{L}_0}\big). \end{aligned} \tag{87}$$

Similarly, using (81)–(84), (86) and (87), we obtain the following estimate for Π^2

$$\begin{aligned} &\|\Pi^2(\eta_1(t), \mu_1(t)) - \Pi^2(\eta_2(t), \mu_2(t))\|^2_{\boldsymbol{V}' \times \mathbb{L}_0 \times \mathbb{L}_0} \\ &\quad = \|\Psi\big(\varepsilon(\boldsymbol{u}_1(t)), \xi_1(t)\big) - \Psi\big(\varepsilon(\boldsymbol{u}_2(t)), \xi_2(t)\big)\|^2_{\boldsymbol{V}' \times \mathbb{L}_0 \times \mathbb{L}_0} \\ &\quad \leq C\big(\|\eta_1(t) - \eta_2(t)\|^2_{\boldsymbol{V}'} + \|\mu_1(t) - \mu_2(t)\|^2_{\mathbb{L}_0}\big). \end{aligned} \tag{88}$$

From (87) and (88), we conclude that there exists a positive constant $C > 0$ verifying

$$\begin{aligned} &\|\Pi(\eta_1(t), \mu_1(t)) - \Pi(\eta_2(t), \mu_2(t))\|^2_{\boldsymbol{V}' \times \mathbb{L}_0 \times \mathbb{L}_0} \\ &\quad \leq C\|(\eta_1(t) - \eta_2(t), \mu_1(t) - \mu_2(t))\|^2_{\boldsymbol{V}' \times \mathbb{L}_0}. \end{aligned} \tag{89}$$

We generalize this procedure by recurrence on m. Then we obtain the formula

$$\begin{aligned} &\|\Pi^m(\eta_1, \mu_1) - \Pi^m(\eta_2, \mu_2)\|^2_{L^2(0,T;\boldsymbol{V}' \times \mathbb{L}_0)} \\ &\quad \leq \frac{C^m T^m}{m!} \|(\eta_1 - \eta_2, \mu_1 - \mu_2)\|^2_{L^2(0,T;\boldsymbol{V}' \times \mathbb{L}_0)}. \end{aligned} \tag{90}$$

Thus, for m sufficiently large, Π^m is a contraction on $L^2(0, T; \boldsymbol{V}' \times \mathbb{L}_0)$. Hence, Banach's fixed point theorem shows that Π admits a unique fixed point $(\eta^*, \mu^*) \in L^2(0, T; \boldsymbol{V}' \times \mathbb{L}_0)$. □

Now, in the latest step, we have all the ingredients to prove Theorem 1.

Proof. Existence. Let $(\eta^*, \mu^*) \in L^2(0,T; \boldsymbol{V}' \times \mathbb{L}_0)$ be the fixed point of Π defined by (77)-(79) and denote by

$$\boldsymbol{u}_* = \boldsymbol{u}_{\eta^*}, \quad \xi_* = \xi_{\eta^*}, \quad \zeta_* = \zeta_{\eta^*}, \quad \varsigma_* = \varsigma_{\mu^*}. \tag{91}$$

Let $\boldsymbol{\sigma}_* = (\boldsymbol{\sigma}^1_*, \boldsymbol{\sigma}^2_*) : [0,T] \to \mathcal{H}$ and $\boldsymbol{D}_* = (\boldsymbol{D}^1_*, \boldsymbol{D}^2_*) : [0,T] \to H$ the functions defined by

$$\sigma^\omega_* = \mathcal{A}^\omega(\varepsilon(\dot{\boldsymbol{u}}^\omega_*(t))) + \mathcal{B}^\omega(\varepsilon(\boldsymbol{u}^\omega_*(t)), \xi_*) - (\mathcal{E}^\omega)^* E^\omega(\zeta^\omega_*), \quad \omega = 1, 2, \tag{92}$$

$$\boldsymbol{D}^\omega_* = \mathcal{E}^\omega \boldsymbol{\varepsilon}(\boldsymbol{u}^\omega_*) - B^\omega \nabla \zeta^\omega_*, \quad \omega = 1, 2. \tag{93}$$

It is easy to verify that the $\{\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \zeta_*, \xi_*, \varsigma_*\}$ is the unique solution to problem PV possessing regularities (48)-(53).

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Π defined by (77)-(79) and the unique solvability of problems PV^μ_η, PV^ζ_η, PV^ξ_μ and $\mathrm{PV}^\varsigma_\eta$.

□

References

[1] Barboteu, M., Fernández, J.R. and Raffat, T., *Numerical analysis of a dynamic piezoelectric contact problem arising in viscoelasticity.* Comput. Methods Appl. Mech. Engrg., **197** (2008), 3724–3732.

[2] Batra, R.C and Yang, J.S., *Saint-Venant's principle in linear piezoelectricity.* J. Elasticity, **38** (1995), 209–218.

[3] Brézis, H., *Equations et inéquations non linéaires dans les espaces en dualité,* Ann. Inst. Fourier, **18** (1968), 115–175.

[4] Chau, O., Shillor, M. and Sofonea, M., *Dynamic frictionless contact with adhesion.* Z. Angew. Math. Phys., **55** (2004), 32–47.

[5] Ferhat, M.S., Aissaoui, A. and Rimi, K., *Variational analysis of an elastic-thermo-viscoplastic contact problem with normal damped response*, Adv. Math., Sci. J., **11** (3) (2022), 131-156.

[6] Frémond, M. and Nedjar, B., *Damage, gradient of damage and principle of virtual work*, Internat. J. Solids Structures, **33** (8) (1996), 1083–1103.

[7] Ikeda, T., *Fundamentals of piezoelectricity*, Oxford University Press , Oxford, 1990.

[8] Kavian, O., *Introduction à la thèorie des points critiques et applications aux problémes elliptiques,* Springer-Verlage, 1993.

[9] Mindlin, R.D., *Elasticity, piezoelasticity and crystal lattice dynamics.* J. Elasticity, **4**, (1972), 217–280.

[10] Montreanu, D. and Sofonea, M., *Quasivariational inequalities and applications in frictional contact problems with normal compliance.* Adv. Math. Sci. Appl., **10** (2000), 103–118.

[11] Nečas, J. and Hlaváček, I., *Mathematical theory of elastic and elastico-plastic bodies: an introduction,* Elsevier Scientific Publishing Company, Amsterdam, Oxford, New York, 1981.

[12] Rochdi, M., Shillor, M. and Sofonea, M., *A quasistatic viscoelastic contact problem with normal compliance and friction.* J. Elasticity, **51** (1998), 105–126.

[13] Sofonea, M., Han, W. and Shillor, M., *Analysis and approximation of contact problems with adhesion or damage,* Pure Appl. Math. **276**, New york, Chapman-Hall/CRC Press, 2006.