Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 4(66), No. 2 - 2024, 109-140
https://doi.org/10.31926/but.mif.2024.4.66.2.7

ON A KURAMOTO-VELARDE TYPE EQUATION

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Dedicated to Professor Marin Marin on the occasion of his 70th anniversary

Abstract

Kuramoto-Velarde type equations describe the evolution of the spinodal decomposition of phase separating systems in an external field, or, the spatio-temporal evolution of the morphology of steps on crystal surfaces. Under appropriate assumptions on the initial data, on the time T, and on the coefficients of such equation, we prove the well-posedness of the classical solutions for the Cauchy problem, associated with this equation.

2020 Mathematics Subject Classification: 35G25, 35K55.
Key words: existence, uniqueness, stability, Kuramoto-Velarde-type equation, Cauchy problem.

1 Introduction

In this paper, we investigate the well-posedness of the following Cauchy problem:

$$\begin{cases} \partial_t u + \partial_x f(u) + \alpha\partial_x^3 u + \beta^2\partial_x^4 u \\ \qquad\qquad +\delta(\partial_x u)^2 + \kappa u\partial_x^2 u + \gamma\partial_x^2 u = 0, & 0 < t < T, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

with

$$\alpha,\ \beta, \delta,\ \kappa,\ \gamma \in \mathbb{R}, \qquad \beta \neq 0. \tag{2}$$

On the flux $f(u)$, we assume

$$f \in C^1(\mathbb{R}), \quad |f'(u)| \le C_0\big(1 + |u|^3\big), \quad u \in \mathbb{R}, \tag{3}$$

for some positive constant C_0.

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On the initial datum and on the coefficients, we assume

$$u_0 \in H^2(\mathbb{R}), \quad u_0 \not\equiv 0 \tag{4}$$

and one of the following

$$\left(2\gamma^2+3\right)\beta^2 - (\delta-2\kappa)^4 \left\|u_0\right\|^4_{L^2(\mathbb{R})}\left(e^{\frac{2\left(2\gamma^2+3\right)T}{\beta^2}}-1\right) > 0, \tag{5}$$

$$|\delta-2\kappa|^4 < \frac{\beta^2\left(2\gamma^2+3\right)}{\|u_0\|^4_{L^2(\mathbb{R})}\left(e^{\frac{2\left(2\gamma^2+3\right)T}{\beta^2}}-1\right)}, \tag{6}$$

$$\begin{cases} \delta \neq 2\kappa, \\ \beta^2 \geq \sup_{A>0} \frac{T\left(2\gamma^2+3\right)+\sqrt{T^2(2\gamma+3)^2+2TA\log(A+1)(\delta-2\kappa)^4\|u_0\|^4_{L^2(\mathbb{R})}}}{\log(A+1)}, \end{cases} \tag{7}$$

$$\begin{cases} 0 \neq (\delta-2\kappa)^4 = B\beta^6, \text{ for some } B \in \left(0,\, \inf_{A>0}\frac{16A\|u_0\|^4_{L^2(\mathbb{R})}\left(2\gamma^2+3\right)T^2}{\log^2(A+1)}\right), \\ \left|2BA\left\|u_0\right\|^4_{L^2(\mathbb{R})}T\beta^2 - \log\left(A+1\right)\right|^2 \\ \qquad \leq \log^2\left(A+1\right) - 16AB\left\|u_0\right\|^4_{L^2(\mathbb{R})}\left(3\gamma^2+3\right)T^2, \end{cases} \tag{8}$$

$$\begin{cases} 2\gamma+3 = F\left(\delta-2\kappa\right)^8, \text{ for some, } F>0, \delta\neq 2\kappa, \\ (\delta-2\kappa)^4 \leq \inf_{A>0}\frac{-TA\|u_0\|^4_{L^2(\mathbb{R})}+\sqrt{T^2A^2\|u_0\|^8_{L^2(\mathbb{R})}+2\beta^6T\log(A+1)F}}{2TF\beta^2}, \end{cases} \tag{9}$$

$$\begin{cases} f(u) = au^2+bu^3, \quad b\neq 0, \quad \alpha \neq 0, \\ \delta = 4\kappa, \quad \gamma = -h^2 \neq 0, \quad \frac{b}{\alpha}<0, \quad \ell_3^2 = -\frac{b}{\alpha}, \quad (a,\,\kappa)\neq(0,\,0), \\ \|\partial_x u_0\|_{L^2(\mathbb{R})} < \frac{-\ell_3^2\|u_0\|^3_{L^2(\mathbb{R})}+\sqrt{\ell_3^4\|u_0\|^6_{L^2(\mathbb{R})}+2A_1^2}}{2}, \\ A_1 := \frac{4\ell_3^3\beta^3}{\sqrt{2a^2+\kappa^2}\sqrt{e^{\frac{16\ell_3^4\beta^4T}{h^2}}-1}}. \end{cases} \tag{10}$$

Observe that, if $\beta^2 = T$, Condition (5) reads

$$T > \frac{(\delta-2\kappa)^4\|u_0\|^4_{L^2(\mathbb{R})}}{2\gamma^2+3}\left(e^{2\left(2\gamma^2+3\right)}-1\right). \tag{11}$$

Taking

$$f(u) = au^2+bu^3+cu^4, \tag{12}$$

Equation (1) reads

$$\begin{aligned} &\partial_t u + \partial_x\left(au^2+bu^3+cu^4\right)+\alpha\partial_x^3 u \\ &\quad + \beta^2\partial_x^4 u + \delta(\partial_x u)^2 + \kappa u\partial_x^2 u + \gamma\partial_x^2 u = 0 \end{aligned} \tag{13}$$

and models the spinodal decomposition of phase separating systems in an external field [21, 43, 62], the spatiotemporal evolution of the morphology of steps on crystal surfaces [26, 36, 54] and the growth of thermodynamically unstable crystal surfaces with strongly anisotropic surface tension [27, 28, 29]. In the case of a growing crystal surface with strongly anisotropic surface tension, the function u represents the surface slope, while the constants a, b and c are the growth driving forces proportional to the difference between the bulk chemical potentials of the solid and fluid phases. Equation (13) is also deduced as a small-slope approximation of the crystal growth model obtained in [20].

Taking $b = c = 0$ in (13), we have

$$\partial_t u + a\partial_x u^2 + \alpha\partial_x^3 u + \beta^2\partial_x^4 u + \delta(\partial_x u)^2 + \kappa u\partial_x^2 u + \gamma\partial_x^2 u = 0. \tag{14}$$

It is known as the Kuramoto-Velarde equation and describes slow space-time variations of disturbances at interfaces, diffusion-reaction fronts and plasma instability fronts [8, 24, 23]. It also describes Benard-Marangoni cells that occur when there is large surface tension on the interface [32, 60, 63] in a microgravity environment. This situation arises in crystal growth experiments aboard an orbiting space station, although the free interface is metastable with respect to small perturbations. The nonlinearities, caused by $\delta(\partial_x u)^2$ and $\kappa u\partial_x^2 u$, model pressure destabilization effects striving to rupture the interface. (14) is deduced in [59] to describe the long waves on a viscous fluid owing down an inclined plane, and in [19], as particular case of (13), to model the drift waves in a plasma. From a mathematical point of view, in [34], the exact solutions for (14) are studied, while in [53], the initial boundary problem is analyzed. In [8, 7], the authors prove the existence of the solitons for (14). Instead, in [49], the existence of traveling wave solutions for (14) is studied. In [33], the author analyzes the existence of the periodic solution for (14), under appropriate assumptions on a, α, β, δ, κ and γ. The well-posedness of the Cauchy problem for (14) is proven in [52], using the energy space technique and taking $a = 0$ and, in [12], through a priori estimates together with an application of the Cauchy-Kovalevskaya Theorem and under assumptions (3) and (4). Finally, in [17], the well-posedness of the classical solutions of (14) is proven under appropriate assumption on the initial data, of the time T, and the coefficient β.

Observe that, in [52], under assumption $a = 0$, the author gives some suitable conditions on α, β, δ, κ and γ and prove the local well-posedness of (14). Instead, in [12], assuming (4), the well-posedness of the Cauchy problem for (14) is proven, for each choice of β, T and u_0, while, in [17], the well-posedness of classical solutions is proven, under appropriate assumptions on β, T and H^1- norm of the initial datum. Hence, in this paper, we prove that it also possible to prove the well-posedness of classical solutions of (14), under appropriate assumption on β, δ, κ, T and L^2- norm of the initial datum.

Taking $\delta = \kappa = 0$ in (14), we have

$$\partial_t u + a\partial_x u^2 + \alpha\partial_x^3 u + \beta^2\partial_x^4 u + \gamma\partial_x^2 u = 0, \tag{15}$$

that was also independently deduced by Kuramoto [38, 39, 40] to describe the phase turbulence in reaction-diffusion systems, and by Sivashinsky [56] to describe plane flame propagation, taking into account the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front.

Equation (15) can be used to study incipient instabilities in several physical and chemical systems [5, 30, 41]. Moreover, (15), which is also known as the Benney-Lin equation [2, 47], was derived by Kuramoto in the study of phase turbulence in Belousov-Zhabotinsky reactions [44].

From a mathematical point of view, the dynamical properties and the existence of exact solutions for (15) have been investigated in [22, 35, 37, 50, 51, 61]. The control problem for (15) are studied in [1, 4, 25], respectively. In [6], the problem of global exponential stabilization of (15) with periodic boundary conditions is analyzed. In [31], it is proposed a generalization of optimal control theory for (15), while in [48] the problem of global boundary control of (15) is considered. In [54], the existence of solitonic solutions for (15) is proven. In [3, 57, 12, 18], the well-posedness of the Cauchy problem for (15) is proven, using the energy space technique, the fixed point method, a priori estimates together with an application of the Cauchy-Kovalevskaya Theorem and a priori estimates together with an application of the Aubin-Lions Lemma, respectively. Instead, in [16, 45, 46], the initial-boundary value problem for (15) is studied, using a priori estimates together with an application of the Cauchy-Kovalevskaya Theorem, and the energy space technique, respectively. Finally, following [9, 42, 55], in [10], the convergence of the solution of (15) to the unique entropy one of the Burgers equation is proven.

The main result of this paper is the following theorem.

Theorem 1. *Assuming that*

- (2), (3), (4) *and one within* (5), (6) (7), (8), (9) *hold*

or

- (2), (4), *and* (10) *hold*

there exists a solution u *of* (1), *such that*

$$u \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^2(\mathbb{R})) \cap L^4(0,T;W^{2,4}(\mathbb{R})). \tag{16}$$

Moreover, if $f \in C^2(\mathbb{R})$, *the solution is unique and if* u_1 *and* u_2 *are two solutions of* (1), *in correspondence of the initial data* $u_{1,0}$ *and* $u_{2,0}$, *we have that*

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R})} \le e^{Ct} \|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{R})}, \tag{17}$$

for some suitable $C > 0$ *and every* $0 \le t \le T$.

Theorem 1 improves the results of [17] and [52], because it gives some conditions on L^2- norm of u_0, β, δ, κ and T, to guarantee the existence of classical solutions for (1), under Assumption (4). Moreover, it shows that, if

$$\delta = 2\kappa \tag{18}$$

and, in particular, if

$$\delta = \kappa = 0,$$

the solutions of (1) verifies (16) and (17), for every β and T (see Lemma 1). Finally, the argument of Theorem 1 relies on deriving suitable a priori estimates together with the existence result in [12, 14, 11]. Unfortunately, our arguments do not provide any blow-up results but only local-in-time existence criteria.

The paper is organized as follows. In Section 2, we prove Theorem 1, under assumptions (5), (6), (7), (8), (9) and (18), while, in Section 3, we prove Theorem 1, under assumptions (10).

2 A priori estimates

In this section, we prove Theorem 1 assuming (5), (6), (7), (8), (9), and (18).

We prove some a priori estimates on u. In what follows we denote with C and c the positive constants independent on the parameters.

Lemma 1. *We have that*

$$\frac{e^{\frac{2\left(2\gamma^2+3\right)t}{\beta^2}}}{\|u(t,\cdot)\|^4_{L^2(\mathbb{R})}} + \frac{(\delta-2\kappa)^4}{\beta^2\left(\gamma^2+3\right)}\left(e^{\frac{2\left(2\gamma^2+3\right)t}{\beta^2}}-1\right) \geq \frac{1}{\|u_0\|^4_{L^2(\mathbb{R})}}, \tag{19}$$

for every $0 \leq t \leq T$. *Moreover, if* (5) *holds,*

$$\|u(t,\cdot)\|^4_{L^2(\mathbb{R})} \leq C, \tag{20}$$

$$\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C, \tag{21}$$

$$\int_0^t \|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \leq C, \tag{22}$$

for every $0 \leq t \leq T$.

The proof of the previous lemma is based on the following result.

Lemma 2. *For every* $t \geq 0$, *we have that*

$$\begin{aligned}\int_{\mathbb{R}} &|u(t,x)|(\partial_x u(t,x))^2 dx \\ &\leq \sqrt{2}\,\|u(t,\cdot)\|^{\frac{3}{2}}_{L^2(\mathbb{R})}\,\|\partial_x u(t,\cdot)\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}.\end{aligned} \tag{23}$$

Proof. We begin by observing that

$$\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \partial_x u \partial_x u dx = -\int_{\mathbb{R}} u \partial_x^2 u dx.$$

Therefore, by the Hölder inequality,

$$\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq \int_{\mathbb{R}} |u||\partial_x^2 u| dx \leq \|u(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}. \tag{24}$$

Again by the the Hölder inequality,

$$u^2(t,x) = 2\int_{-\infty}^{x} u\partial_x u dy \le 2\int_{\mathbb{R}} |u||\partial_x u| dx \le 2\left\|u(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x u(t,\cdot)\right\|_{L^2(\mathbb{R})}.$$

Hence,

$$\left\|u(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})} \le 2\left\|u(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x u(t,\cdot)\right\|_{L^2(\mathbb{R})},$$

which gives

$$\left\|u(t,\cdot)\right\|_{L^\infty(\mathbb{R})} \le \sqrt{2}\left\|u(t,\cdot)\right\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\left\|\partial_x u(t,\cdot)\right\|^{\frac{1}{2}}_{L^2(\mathbb{R})}. \tag{25}$$

From (24) and (25)

$$\begin{aligned}\int_{\mathbb{R}} |u|(\partial_x u)^2 dx \le& \left\|u(t,\cdot)\right\|_{L^\infty(\mathbb{R})}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \le& \left\|u(t,\cdot)\right\|_{L^\infty(\mathbb{R})}\left\|u(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}\\ \le&\sqrt{2}\left\|u(t,\cdot)\right\|^{\frac{3}{2}}_{L^2(\mathbb{R})}\left\|\partial_x u(t,\cdot)\right\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})},\end{aligned}$$

this is (23). □

Proof of Lemma 1. Let $0 \le t \le T$. Multiplying (1) by $2u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}&\frac{d}{dt}\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = 2\int_{\mathbb{R}} u\partial_t u dx\\ =&\underbrace{-2\int_{\mathbb{R}} uf'(u)\partial_x u dx}_{=0} -2\alpha\int_{\mathbb{R}} u\partial_x^3 u dx - 2\beta^2\int_{\mathbb{R}} u\partial_x^4 u dx\\ &-2\delta\int_{\mathbb{R}} u(\partial_x u)^2 dx - 2\kappa\int_{\mathbb{R}} u^2\partial_x^2 u dx - 2\gamma\int_{\mathbb{R}} u\partial_x^2 u dx\\ =&2\alpha\int_{\mathbb{R}}\partial_x u\partial_x^2 u dx + 2\beta^2\int_{\mathbb{R}}\partial_x u\partial_x^3 u dx\\ &-2(\delta-2\kappa)\int_{\mathbb{R}} u(\partial_x u)^2 dx - 2\gamma\int_{\mathbb{R}} u\partial_x^2 u dx\\ =&-2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2(\delta-2\kappa)\int_{\mathbb{R}} u(\partial_x u)^2 dx - 2\gamma\int_{\mathbb{R}} u\partial_x^2 u dx.\end{aligned}$$

Consequently, we have that

$$\begin{aligned}&\frac{d}{dt}\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\qquad = -2(\delta-2\kappa)\int_{\mathbb{R}} u(\partial_x u)^2 dx - 2\gamma\int_{\mathbb{R}} u\partial_x^2 u dx.\end{aligned} \tag{26}$$

Due to (23) and the Young inequality,

$$2\left|\delta-2\kappa\right|\int_{\mathbb{R}} |u|(\partial_x u)^2 dx$$

$$
\begin{aligned}
&\leq 2\sqrt{2}\,|\delta-2\kappa|\,\|u(t,\cdot)\|^{\frac{3}{2}}_{L^2(\mathbb{R})}\,\|\partial_x u(t,\cdot)\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\,\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
&\leq \frac{2\,(\delta-2\kappa)^2}{\beta^2}\,\|u(t,\cdot)\|^3_{L^2(\mathbb{R})}\,\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{(\delta-2\kappa)^4}{\beta^4}\,\|u(t,\cdot)\|^6_{L^2(\mathbb{R})}+\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
2|\gamma|&\int_{\mathbb{R}}|u||\partial_x^2 u|dx\\
&=\int_{\mathbb{R}}\left|\frac{2\gamma u}{\beta}\right|\left|\beta\partial_x^2 u\right|dx\leq\frac{2\gamma^2}{\beta^2}\,\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (26) that

$$
\begin{aligned}
&\frac{d}{dt}\,\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\leq\frac{2\gamma^2}{\beta^2}\,\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{(\delta-2\kappa)^4}{\beta^4}\,\|u(t,\cdot)\|^6_{L^2(\mathbb{R})}+\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{27}
$$

Due to the Young inequality,

$$
\begin{aligned}
\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}&=\int_{\mathbb{R}}\partial_x u\partial_x u dx=-\int_{\mathbb{R}}u\partial_x^2 u dx\leq 2\int_{\mathbb{R}}\left|\frac{\sqrt{3}u}{2\beta}\right|\left|\frac{\beta\partial_x^2 u}{\sqrt{3}}\right|dx\\
&\leq\frac{3}{4\beta^2}\,\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{3}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Consequently, by (27),

$$
\begin{aligned}
&\frac{d}{dt}\,\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{6}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\leq\frac{2\gamma^2+3}{\beta^2}\,\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{(\delta-2\kappa)^4}{\beta^4}\,\|u(t,\cdot)\|^6_{L^2(\mathbb{R})}.
\end{aligned}
\tag{28}
$$

Denoting

$$
\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}:=X(t),\quad \ell_1:=\frac{2\gamma^2+3}{\beta^2},\quad \ell_2:=\frac{(\delta-2\kappa)^4}{\beta^4},
\tag{29}
$$

(28) and (29) give

$$
\frac{dX(t)}{dt}\leq\ell_1 X(t)+\ell_2 X^3(t),
\tag{30}
$$

which gives

$$
\frac{1}{X^3(t)}\frac{dX(t)}{dt}\leq\frac{\ell_1}{X^2(t)}+\ell_2
\tag{31}
$$

and

$$
\frac{1}{X^3(t)}\frac{dX(t)}{dt}=\frac{1}{2}\frac{d}{dt}\left(\frac{-1}{X^2(t)}\right).
\tag{32}
$$

Consequently, by (31) and (32),

$$\frac{d}{dt}\left(\frac{-1}{X^2(t)}\right)+\frac{-2\ell_1}{X^2(t)}\le 2\ell_2 \tag{33}$$

Multiplying (33) by $e^{2\ell_1 t}$, we have that

$$e^{2\ell_1 t}\frac{d}{dt}\left(\frac{-1}{X^2(t)}\right)+\frac{-2\ell_1 e^{2\ell_1 t}}{X^2(t)}\le 2\ell_2 e^{2\ell_1 t},$$

that is

$$\frac{d}{dt}\left(\frac{-e^{2\ell_1 t}}{X^2(t)}\right)\le 2\ell_2 e^{2\ell_1 t}.$$

Therefore,

$$\frac{d}{dt}\left(\frac{e^{2\ell_1 t}}{X^2(t)}\right)\ge -2\ell_2 e^{2\ell_1 t}.$$

Integrating on $(0, t)$, by (4) and (29), we have that

$$\frac{e^{2\ell_1 t}}{X^2(t)}-\frac{1}{X_0^2}\ge -\frac{\ell_2}{\ell_1}\left(e^{2\ell_1 t}-1\right),$$

that is,

$$\frac{e^{2\ell_1 t}}{X^2(t)}+\frac{\ell_2}{\ell_1}\left(e^{2\ell_1 t}-1\right)\ge \frac{1}{X_0^2}. \tag{34}$$

Using (29) in (34), we have (19).

Assume (5) and we prove (19). By (19) and (34), we have that

$$\frac{\ell_1 e^{2\ell_1 t}+\ell_2\left(e^{2\ell_1 t}-1\right)X^2(t)}{\ell_1 X^2(t)}\ge \frac{1}{X_0^2}.$$

Therefore,

$$\frac{\ell_1 X^2(t)}{\ell_1 e^{2\ell_1 t}+\ell_2\left(e^{2\ell_1 t}-1\right)X^2(t)}\le X_0^2. \tag{35}$$

Hence,

$$\begin{aligned}\ell_1 X^2(t)\le& X_0^2\ell_1 e^{2\ell_1 t}+X_0^2\ell_2\left(e^{2\ell_1 t}-1\right)X^2(t)\\ \le& X_0^2\ell_1 e^{2\ell_1 T}+X_0^2\ell_2\left(e^{2\ell_1 T}-1\right)X^2(t),\end{aligned}$$

which gives

$$\left(\ell_1-X_0^2\ell_2\left(e^{2\ell_1 T}-1\right)\right)X^2(t)\le X_0^2\ell_1 e^{2\ell_1 T}. \tag{36}$$

Thanks to (5) and (29), there exists a constant $C>0$, such that

$$CX^2(t)\le X_0^2\ell_1 e^{2\ell_1 T}. \tag{37}$$

Using (29) in (37), we have (20).

We prove (21). By (27), we have that

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{6}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C.$$

Integrating on $(0,t)$, by (4), we get

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{6}\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le\|u_0\|^2_{L^2(\mathbb{R})}+Ct\le C,$$

which gives (21).

Finally, we prove (22). Thanks to (20) and (24), we have that

$$\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\le C\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}.$$

Thanks to the Young inequality,

$$\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\le\frac{C}{2}+\frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

An integration on $(0,t)$ and (21) give (22). □

Corollary 1. *Fix $T>0$. If* (6) *holds, then we have* (20), (21) *and* (22).

Proof. Let $0\le t\le T$. We begin by observing that, by (5), we have that

$$(\delta-2\kappa)^4<\frac{\left(2\gamma^2+3\right)\beta^2}{\|u_0\|^4_{L^2(\mathbb{R})}\left(e^{\frac{2\left(2\gamma^2+3\right)T}{\beta^2}}-1\right)},$$

Therefore,

$$-\sqrt[4]{\frac{\left(2\gamma^2+3\right)\beta^2}{\|u_0\|^4_{L^2(\mathbb{R})}\left(e^{\frac{2\left(2\gamma^2+3\right)T}{\beta^2}}-1\right)}}<\delta-2\kappa<\sqrt[4]{\frac{\left(2\gamma^2+3\right)\beta^2}{\|u_0\|^4_{L^2(\mathbb{R})}\left(e^{\frac{2\left(2\gamma^2+3\right)T}{\beta^2}}-1\right)}},$$

which gives (6).

Finally, arguing as in Lemma 1 we obtain (20), (21) and (22). □

Corollary 2. *Fix $T>0$. If* (18) *holds, then*

$$\|u(t,\cdot)\|^4_{L^2(\mathbb{R})}\le\|u_0\|^4_{L^2(\mathbb{R})}e^{\frac{2\left(2\gamma^2+3\right)T}{\beta^2}},\tag{38}$$

for every $0\le t\le T$. Moreover, we have (21) *and* (22).

Proof. Arguing as in Lemma 1, we have (19). Thanks to (18) and (19), we obtain that

$$\frac{1}{\|u(t,\cdot)\|^4_{L^2(\mathbb{R})}}\ge\frac{1}{\|u_0\|^4_{L^2(\mathbb{R})}e^{\frac{2\left(2\gamma^2+3\right)t}{\beta^2}}},$$

Hence,

$$\|u(t,\cdot)\|^4_{L^2(\mathbb{R})} \le \|u_0\|^4_{L^2(\mathbb{R})} e^{\frac{2\left(2\gamma^2+3\right)t}{\beta^2}} \le \|u_0\|^4_{L^2(\mathbb{R})} e^{\frac{2\left(2\gamma^2+3\right)T}{\beta^2}},$$

which gives (38).

Finally, arguing as in Lemma 1, we have (21) and (22). □

Lemma 3. *We have that*

$$\frac{e^{2(\ell_1+\lambda)t}}{\|u(t,\cdot)\|^4_{L^2(\mathbb{R})}} + \frac{\ell_2}{\ell_1+\lambda}\left(e^{2(\ell_1+\lambda)t}-1\right) \ge \frac{1}{\|u_0\|^4_{L^2(\mathbb{R})}}, \tag{39}$$

for every $0 \le t \le T$, where λ is a positive constant and ℓ_1, ℓ_2 are defined in (29), *respectively. In particular, if* (7) *holds and taking*

$$\lambda = A\ell_2 \|u_0\|^4_{L^2(\mathbb{R})}, \quad A > 0, \tag{40}$$

(20) *holds. Moreover, we have* (21) *and* (22).

Proof. Let $0 \le t \le T$. Arguing as in Lemma 1 we have (30). Let λ be a positive constant. By (30), we have that

$$\frac{dX(t)}{dt} \le \ell_1 X(t) + \ell_2 X^3(t) \le (\ell_1+\lambda)\, X(t) + \ell_2 X^3(t).$$

Arguing as in Lemma 1 we obtain (39).

Assume (7) and (40) and we prove (20). We begin by observing that, by (39), we have

$$\frac{(\ell_1+\lambda)\, e^{2(\ell_1+\lambda)t} + \ell_2\left(e^{2(\ell_1+\lambda)t}-1\right)\|u(t,\cdot)\|^4_{L^2(\mathbb{R})}}{(\ell_1+\lambda)\,\|u(t,\cdot)\|^4_{L^2(\mathbb{R})}} \ge \frac{1}{\|u_0\|^4_{L^2(\mathbb{R})}}.$$

Therefore, we get

$$\begin{aligned}(\ell_1+\lambda)\,\|u(t,\cdot)\|^4_{L^2(\mathbb{R})} &\le \|u_0\|^4_{L^2(\mathbb{R})}\,(\ell_1+\lambda)\, e^{2(\ell_1+\lambda)t} \\ &\quad + \ell_2\left(e^{2(\ell_1+\lambda)t}-1\right)\|u(t,\cdot)\|^4_{L^2(\mathbb{R})} \\ &\le \|u_0\|^4_{L^2(\mathbb{R})}\,(\ell_1+\lambda)\, e^{2(\ell_1+\lambda)T} \\ &\quad + \ell_2\,\|u_0\|^4_{L^2(\mathbb{R})}\left(e^{2(\ell_1+\lambda)T}-1\right)\|u(t,\cdot)\|^4_{L^2(\mathbb{R})}.\end{aligned}$$

Consequently,

$$\begin{aligned}\left[\ell_1+\lambda-\ell_2\,\|u_0\|^4_{L^2(\mathbb{R})}\left(e^{2(\ell_1+\lambda)T}-1\right)\right]\|u(t,\cdot)\|^4_{L^2(\mathbb{R})} \\ le\,\|u_0\|^4_{L^2(\mathbb{R})}\,(\ell_1+\lambda)\, e^{2(\ell_1+\lambda)T}.\end{aligned}$$

In order to have (20), we require that

$$\lambda - \ell_2\,\|u_0\|^4_{L^2(\mathbb{R})}\left(e^{2(\ell_1+\lambda)T}-1\right) \ge 0. \tag{41}$$

By (40), we obtain that

$$A+1-e^{2\left(\ell_1+A\ell_2\|u_0\|^4_{L^2(\mathbb{R})}\right)T}\geq 0,$$

which gives

$$\ell_1+A\ell_2\left\|u_0\right\|^4_{L^2(\mathbb{R})}\leq\frac{\log\left(A+1\right)}{2T}. \tag{42}$$

Thanks to (29), (42) reads

$$\frac{2\gamma^2+3}{\beta^2}+\frac{A\left\|u_0\right\|^4_{L^2(\mathbb{R})}\left(\delta-2\kappa\right)^4}{\beta^4}\leq\frac{\log\left(A+1\right)}{2T}. \tag{43}$$

Hence, we have that

$$\frac{\log\left(A+1\right)\beta^4-2T\left(2\gamma^2+3\right)\beta^2-2TA\left\|u_0\right\|^4_{L^2(\mathbb{R})}\left(\delta-2\kappa\right)^4}{2T\beta^2}\geq 0, \tag{44}$$

which is guaranteed by (7).

Finally, arguing as in Lemma 1, we have (21) and (22). □

Corollary 3. *Fix $T>0$ and assume* (8)*. We have* (20), (21) *and* (22).

Proof. Let $0\leq t\leq T$. Arguing as in Lemma 3, we have (43). We consider

$$(\delta-2\kappa)^4=B\beta^6,\quad B>0,\quad \delta\neq 2\kappa.$$

Consequently, (43) reads

$$\frac{2\gamma^2+3}{\beta^2}+BA\left\|u_0\right\|^4_{L^2(\mathbb{R})}\beta^2\leq\frac{\log\left(A+1\right)}{2T}.$$

Hence, we have that

$$\frac{2BAT\left\|u_0\right\|^4_{L^2(\mathbb{R})}\beta^4-\log\left(A+1\right)\beta^2+2\left(\gamma^2+3\right)T}{2T\beta^2}\leq 0. \tag{45}$$

which is guaranteed by (8).

Finally, arguing as in Lemma 1, we have (21) and (22). □

Corollary 4. *Fix $T>0$ and assume* (9)*. We have* (20), (21) *and* (22).

Proof. Let $0\leq t\leq T$. Arguing as in Lemma 3, we have (44). We consider

$$2\gamma^2+3=F\left(\delta-2\kappa\right)^8,\quad F>0.$$

Therefore, (44) reads

$$\frac{\log\left(A+1\right)\beta^4-2TF\beta^2\left(\delta-2\kappa\right)^8-2TA\left\|u_0\right\|^4_{L^2(\mathbb{R})}\left(\delta-2\kappa\right)^4}{2T\beta^2}\geq 0,$$

that is

$$\frac{2TF\beta^2\left(\delta-2\kappa\right)^8+2TA\left\|u_0\right\|^4_{L^2(\mathbb{R})}\left(\delta-2\kappa\right)^4-\log\left(A+1\right)\beta^4}{2T\beta^2}\le 0.$$

Consequently,

$$\left(\delta-2\kappa\right)^4\le\frac{-TA\left\|u_0\right\|^4_{L^2(\mathbb{R})}+\sqrt{T^2A^2\left\|u_0\right\|^8_{L^2(\mathbb{R})}+2\beta^6T\log\left(A+1\right)F}}{2TF\beta^2},\tag{46}$$

which is (9).

Finally, arguing as in Lemma 1, we have (21) and (22). □

Lemma 4. *Assume one within* (5), (6), (18), (7), (8), *and* (9). *Fix* $T>0$, *there exists a constant* $C>0$, *such that*

$$\|u\|_{L^\infty((0,T)\times\mathbb{R})}\le C.\tag{47}$$

In particular, we have that

$$\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\int_0^t\left\|\partial_x^3u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C,\tag{48}$$

for every $0\le t\le T$. *Moreover,*

$$\int_0^t\|\partial_x u(s,\cdot)\|^4_{L^4(\mathbb{R})}ds\le C,\tag{49}$$

for every $0\le t\le T$.

Proof. Let $0\le t\le T$. Multiplying (1) by $-2\partial_x^2u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}=&-2\int_{\mathbb{R}}u\partial_x^2udx\\ =&2\int_{\mathbb{R}}f'(u)\partial_xu\partial_x^2udx-2\alpha\int_{\mathbb{R}}\partial_x^2u\partial_x^3udx+2\beta^2\int_{\mathbb{R}}\partial_x^2u\partial_x^4udx\\ &-2\delta\int_{\mathbb{R}}(\partial_xu)^2\partial_x^2udx+2\kappa\int_{\mathbb{R}}u(\partial_x^2u)^2dx-2\gamma\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ =&2\int_{\mathbb{R}}f'(u)\partial_xu\partial_x^2udx-2\beta^2\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &+2\kappa\int_{\mathbb{R}}u(\partial_x^2u)^2dx-2\gamma\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore, we have that

$$\begin{aligned}&\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\qquad=2\int_{\mathbb{R}}f'(u)\partial_xu\partial_x^2udx+2\kappa\int_{\mathbb{R}}u(\partial_x^2u)^2dx-2\gamma\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}\tag{50}$$

Due to (3) and the Young inequality,

$$\begin{aligned}
2\int_{\mathbb{R}} |f'(u)||\partial_x u||\partial_x^2 u|dx \leq & 2C_0 \int_{\mathbb{R}} \left(1+|u|+|u|^2+|u|^3\right)|\partial_x u||\partial_x^2 u|dx \\
\leq & C_0 \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0 \int_{\mathbb{R}} u^2(\partial_x u)^2 dx + C_0 \int_{\mathbb{R}} u^4 (\partial_x u)^2 dx \\
& + C_0 \int_{\mathbb{R}} u^6 (\partial_x u)^2 dx + 4C_0 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
\leq & C_0 \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0 \left\|u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
& + C_0 \int_{\mathbb{R}} u^4 (\partial_x u)^2 dx + C_0 \int_{\mathbb{R}} u^6 (\partial_x u)^2 dx + 4C_0 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
2\kappa \int_{\mathbb{R}} |u|(\partial_x^2 u)^2 dx \leq & 2|\kappa| \left\|u\right\|_{L^\infty((0,T)\times\mathbb{R})} \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Consequently, by (50),

$$\begin{aligned}
& \frac{d}{dt}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
& \quad \leq C_0 \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0 \left\|u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
& \qquad + \left(4C_0 + 2|\gamma| + 2|\kappa| \left\|u\right\|_{L^\infty((0,T)\times\mathbb{R})}\right) \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
& \qquad + C_0 \int_{\mathbb{R}} u^4 (\partial_x u)^2 dx + C_0 \int_{\mathbb{R}} u^6 (\partial_x u)^2 dx.
\end{aligned} \tag{51}$$

[13, Lemma 2.6] says that

$$\int_{\mathbb{R}} u^4 (\partial_x u)^2 dx \leq 4 \left\|u(t,\cdot)\right\|^4_{L^2(\mathbb{R})} \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \tag{52}$$

Hence, by (20) and (52), we have that

$$\begin{aligned}
C_0 \int_{\mathbb{R}} u^4 (\partial_x u)^2 dx \leq & C_0 \left\|u(t,\cdot)\right\|^4_{L^2(\mathbb{R})} \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq C \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
C_0 \int_{\mathbb{R}} u^6 (\partial_x u)^2 dx \leq & C_0 \left\|u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} u^4 (\partial_x u)^2 dx \\
\leq & C_0 \left\|u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \left\|u(t,\cdot)\right\|^4_{L^2(\mathbb{R})} \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
\leq & C \left\|u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned} \tag{53}$$

It follows from (51) and (53) that

$$\begin{aligned}
& \frac{d}{dt}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
& \quad \leq C_0 \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0 \left\|u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
& \qquad + C\left(1 + \left\|u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \left\|u\right\|_{L^\infty((0,T)\times\mathbb{R})}\right) \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Integrating on $(0,t)$, by (4), (21), (22) and the Young inequality,

$$\begin{aligned}
&\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2 \int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\
&\quad\le \|\partial_x u_0\|^2_{L^2(\mathbb{R})} + C_0 \left(1 + \|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\right) \int_0^t \|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \qquad (54)\\
&\qquad + C \left(1 + \|u\|^2_{L^\infty((0,T)\times\mathbb{R})} + \|u\|_{L^\infty((0,T)\times\mathbb{R})}\right) \int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\
&\quad\le C \left(1 + \|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned}$$

We prove (47). Due to (25), (20) and (54),

$$\|u\|^2_{L^\infty((0,T)\times\mathbb{R})} \le C\sqrt{1 + \|u\|^2_{L^\infty((0,T)\times\mathbb{R})}}.$$

Hence,

$$\|u\|^4_{L^\infty((0,T)\times\mathbb{R})} - C\,\|u\|^2_{L^\infty((0,T)\times\mathbb{R})} - C \le 0,$$

which gives (47).

(48) follows from (47) and (54).

Finally, we prove (49). [15, Lemma 2.3] says that

$$\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} \le 6\left(\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right) \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Therefore, by (20) and (48), we have that

$$\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} \le C \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0,t)$, by (21), we obtain (49). □

Lemma 5. *Assume one within* (5), (6), (18), (7), (8), *and* (9). *Fix* $T > 0$, *there exists a constant* $C > 0$, *such that*

$$\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2} \int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C, \qquad (55)$$

$$\|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})} \le C, \qquad (56)$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. Multiplying (1) by $2\partial_x^4 u$, integration on $\mathbb{R}$ gives

$$\begin{aligned}
\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &= 2\int_{\mathbb{R}} \partial_x^4 u \partial_t u dx \\
&= -2\int_{\mathbb{R}} f'(u)\partial_x u \partial_x^4 u dx - 2\alpha \int_{\mathbb{R}} \partial_x^3 u \partial_x^4 u dx - 2\beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad - 2\delta \int_{\mathbb{R}} (\partial_x u)^2 \partial_x^4 u dx - 2\kappa \int_{\mathbb{R}} u \partial_x^2 u \partial_x^4 u dx - 2\gamma \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx
\end{aligned}$$

$$= -2\int_{\mathbb{R}} f'(u)\partial_x u\partial_x^4 u dx - 2\beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\delta\int_{\mathbb{R}}(\partial_x u)^2\partial_x^4 u dx$$
$$- 2\kappa\int_{\mathbb{R}} u\partial_x^2 u\partial_x^4 u dx + 2\gamma\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Consequently, we have that

$$\begin{aligned}\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}&\\ = -2\int_{\mathbb{R}} f'(u)\partial_x u\partial_x^4 u dx - 2\delta\int_{\mathbb{R}}(\partial_x u)^2\partial_x^4 u dx&\\ - 2\kappa\int_{\mathbb{R}} u\partial_x^2 u\partial_x^4 u dx + 2\gamma\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.&\end{aligned} \tag{57}$$

Due to (47), (48) and the Young inequality,

$$\begin{aligned}2\int_{\mathbb{R}}|f'(u)||\partial_x u||\partial_x^4 u|dx &\le 2\left\|f'\right\|_{L^\infty(-C,C)}\int_{\mathbb{R}}|\partial_x u||\partial_x^4 u|dx\\ &\le C\int_{\mathbb{R}}|\partial_x u||\partial_x^4 u|dx = \int_{\mathbb{R}}\left|\frac{C\partial_x u}{\beta}\right|\left|\beta\partial_x^4 u\right|dx\\ &\le C\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\le C + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ 2|\delta|\int_{\mathbb{R}}(\partial_x u)^2|\partial_x^4 u|dx &\le \int_{\mathbb{R}}\left|\frac{2\delta(\partial_x u)^2}{\beta^2}\right|\left|\beta\partial_x^4 u\right|dx\\ &\le \frac{2\delta^2}{\beta^2}\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ 2|\kappa|\int_{\mathbb{R}}|u||\partial_x^2 u||\partial_x^4 u|dx &\le 2|\kappa|\left\|u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u||\partial_x^4 u|dx\\ &\le C\int_{\mathbb{R}}|\partial_x^2 u||\partial_x^4 u|dx = \int_{\mathbb{R}}\left|\frac{C\partial_x^2 u}{\beta}\right|\left|\beta\partial_x^4 u\right|dx\\ &\le C\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

It follows from (57) that

$$\begin{aligned}&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad\le C + \frac{2\delta^2}{\beta^2}\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + C\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2|\gamma|\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Integrating on $(0,t)$, by (4), (21), (48) and (49), we get

$$\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\int_0^t\left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds$$

$$\begin{aligned}&\le \left\|\partial_x^2 u_0\right\|^2_{L^2(\mathbb{R})} + Ct + \frac{2\delta^2}{\beta^2}\int_0^t \left\|\partial_x u(s,\cdot)\right\|^4_{L^4(\mathbb{R})} ds\\&\quad + C\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + 2|\gamma|\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C,\end{aligned}$$

which gives (55).

Finally, we prove (56). Thanks to (48), (55) and the Hölder inequality,

$$\begin{aligned}(\partial_x u(t,x))^2 =&2\int_{-\infty}^x |\partial_x u|\partial_x^2 u| dy \le 2\int_{\mathbb{R}} |\partial_x u||\partial_x^2 u| dx\\ \le&2\left\|\partial_x u(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})} \le C.\end{aligned}$$

Hence,

$$\left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \le C,$$

which gives (56). □

Lemma 6. *Assume one within* (5), (6), (18), (7), (8), *and* (9). *Given* $T > 0$, *we have*

$$\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^4_{L^2(\mathbb{R})} ds \le C, \tag{58}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. We begin by observing that

$$\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} = \int_{\mathbb{R}} \partial_x^2 u(\partial_x^2 u)^3 dx = -3\int_{\mathbb{R}} \partial_x u(\partial_x^2 u)^2 \partial_x^3 u dx. \tag{59}$$

Due to (56) and the Young inequality,

$$\begin{aligned}&3\int_{\mathbb{R}} |\partial_x u|(\partial_x^2 u)^2|\partial_x^3 u| dx\\&\quad\le\frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \frac{9}{2}\int_{\mathbb{R}} (\partial_x u)^2(\partial_x^3 u)^2 dx\\&\quad\le\frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \frac{9}{2}\left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&\quad\le\frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

If follows from (59) that

$$\frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} \le C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0,t)$, by (48), we have (58). □

Lemma 7. *Assume one within* (5), (6), (18), (7), (8), *and* (9). *Fix* $T > 0$, *we have*

$$\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{6}\int_0^t \left\|\partial_t u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C, \tag{60}$$

for every $0 \le t \le T$.

Proof. Let $0 \leq t \leq T$. Multiplying (1) by $2\partial_t u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}&\frac{d}{dt}\left(\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \gamma \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\ &\quad = -2\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\int_{\mathbb{R}} f'(u)\partial_x u\partial_t u dx - 2\alpha\int_{\mathbb{R}} \partial_x^3 u\partial_t u dx\\ &\qquad - 2\delta\int_{\mathbb{R}}(\partial_x u)^2\partial_t u dx - 2\kappa\int_{\mathbb{R}} u\partial_x^2 u\partial_t u dx.\end{aligned}$$

Therefore, we have that

$$\begin{aligned}&\frac{d}{dt}\left(\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \gamma \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right) + 2\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad = -2\int_{\mathbb{R}} f'(u)\partial_x u\partial_t u dx - 2\alpha\int_{\mathbb{R}} \partial_x^3 u\partial_t u dx - 2\delta\int_{\mathbb{R}}(\partial_x u)^2\partial_t u dx \qquad (61)\\ &\qquad - 2\kappa\int_{\mathbb{R}} u\partial_x^2 u\partial_t u dx.\end{aligned}$$

Thanks to (47), (48), (55), (56) and the Young inequality,

$$\begin{aligned}&2\int_{\mathbb{R}} |f'(u)||\partial_x u||\partial_t u| dx\\ &\quad \leq 2\left\|f'\right\|_{L^\infty(-C,C)}\int_{\mathbb{R}}|\partial_x u||\partial_t u|dx \leq C\int_{\mathbb{R}}|\partial_x u||\partial_t u|dx\\ &\quad \leq C\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad \leq C + \frac{1}{2}\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ &2|\alpha|\int_{\mathbb{R}}|\partial_x^3 u||\partial_t u|dx\\ &\quad \leq 2\alpha^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ &2|\delta|\int_{\mathbb{R}}(\partial_x u)^2|\partial_t u|dx\\ &\quad \leq 2|\delta|\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x u||\partial_t u|dx\\ &\quad \leq C\int_{\mathbb{R}}|\partial_x u||\partial_t u|dx\\ &\quad \leq C\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad \leq C + \frac{1}{2}\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ &2|\kappa|\int_{\mathbb{R}}|u||\partial_x^2 u||\partial_t u|dx\\ &\quad \leq 2|\kappa|\left\|u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|dx \leq C\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|dx\end{aligned}$$

$$= 2\int_{\mathbb{R}} \left|\frac{\sqrt{3}C\partial_x^2 u}{2}\right| \left|\frac{\partial_t u}{\sqrt{3}}\right| dx \le C\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{3}\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$
$$\le C + \frac{1}{3}\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

It follows from (61) that

$$\frac{d}{dt}\left(\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \gamma\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right) + \frac{1}{6}\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$
$$\le C + \alpha^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0,t)$, by (4) and (48), we have

$$\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \gamma\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{6}\int_0^t \left\|\partial_t u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds$$
$$\le \beta^2\left\|\partial_x^2 u_0\right\|^2_{L^2(\mathbb{R})} - \gamma\left\|\partial_x u_0\right\|^2_{L^2(\mathbb{R})} + Ct + 2\alpha^2\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds$$
$$\le \beta^2\left\|\partial_x^2 u_0\right\|^2_{L^2(\mathbb{R})} + |\gamma|\left\|\partial_x u_0\right\|^2_{L^2(\mathbb{R})} + Ct + 2\alpha^2\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C.$$

Therefore, by (48),

$$\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{6}\int_0^t \left\|\partial_t u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds$$
$$\le C + \gamma\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C + |\gamma|\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C,$$

which gives (60). □

Now, we prove the following result

Lemma 8. *Assuming that* (2), (4), (3) *and one within* (5), (6), (18), (7), (8), (9) *hold, there exists a solution u of* (1), *which verifies* (16).

Proof. Thanks to Lemma 1, Corollaries 1 and 2, or, Lemma 3, Corollaries 3 and 4, Lemmas 4, 5, 6, 7 and the Cauchy-Kovalevskaya Theorem [58], we have that u is solution of (1) and (16) holds. □

Lemma 9. *If $f \in C^2(\mathbb{R})$, we have* (17).

Proof. We begin by observing that Lemma 8 gives the existence of a solution u of (1), such that (17) holds.

Arguing as in [17, Theorem 1.1], we have (17). □

Proof of Theorem 1. Theorem 1 follows from Lemmas 8 and (9). □

3 Proof of Theorem 1 assuming (10)

In this section, we prove Theorem 1 assuming (10). Thanks to (10), (1) reads

$$\begin{cases} \partial_t u + 2au\partial_x u + 3bu^2\partial_x u + \alpha\partial_x^3 u + \beta^2\partial_x^4 u \\ \qquad\qquad +4\kappa(\partial_x u)^2 + \kappa u\partial_x^2 u - h^2\partial_x^2 u = 0, \quad 0 < t < T, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), \qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}. \end{cases} \tag{62}$$

We prove some a priori estimates on u.

Lemma 10. *Estimates* (21), (47), (49) *hold. Moreover, we have that*

$$\frac{e^{\frac{16\ell_3^2\beta^2 t}{h^2}}}{\left(2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \ell_3^2\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}\right)^2} + \frac{2\left(a^2+\kappa^2\right)}{2h^2\ell_3^6\beta^6}\left(e^{\frac{16\ell_3^2\beta^2 t}{h^2}} - 1\right) \geq \frac{1}{\left(2\left\|\partial_x u_0\right\|^2_{L^2(\mathbb{R})} + \ell_3^2\left\|u_0\right\|^4_{L^4(\mathbb{R})}\right)^2}, \tag{63}$$

$$2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \ell_3^2\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} \leq C, \tag{64}$$

$$\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C, \tag{65}$$

$$\int_0^t \left\|u(s,\cdot)\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C,$$

$$\int_0^t \left\|u(s,\cdot)\partial_x u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C,$$

for every $0 \leq t \leq T$.

Proof. Multiplying (62) by $-\partial_x^2 u + \ell_3^2 u^3$, we have

$$\begin{aligned} &\left(-\partial_x^2 u + \ell_3^2 u^3\right)\partial_t u + 2a\left(-\partial_x^2 u + \ell_3^2 u^3\right)u\partial_x u + 3b\left(-\partial_x^2 u + \ell_3^2 u^3\right)u^2\partial_x u \\ &\quad + \alpha\left(-\partial_x^2 u + \ell_3^2 u^3\right)\partial_x^3 u + \beta^2\left(-\partial_x^2 u + \ell_3^2 u^3\right)\partial_x^4 u \\ &\quad + 4\kappa\left(-\partial_x^2 u + \ell_3^2 u^3\right)(\partial_x u)^2 + \kappa\left(-\partial_x^2 u + \ell_3^2 u^3\right)u\partial_x^2 u \\ &\quad - h^2\left(-\partial_x^2 u + \ell_3^2 u^3\right)\partial_x^2 u = 0. \end{aligned} \tag{66}$$

Observe that

$$\begin{aligned} &\int_{\mathbb{R}}\left(-\partial_x^2 u + \ell_3^2 u^3\right)\partial_t u dx = \frac{d}{dt}\left(\frac{1}{2}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\ell_3^2}{4}\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}\right), \\ &2a\int_{\mathbb{R}}\left(-\partial_x^2 u + \ell_3^2 u^3\right)u\partial_x u dx = -2a\int_{\mathbb{R}} u\partial_x u\partial_x^2 u dx, \\ &3b\int_{\mathbb{R}}\left(-\partial_x^2 u + \ell_3^2 u^3\right)u^2\partial_x u dx = -3b\int_{\mathbb{R}} u^2\partial_x u\partial_x^2 u dx, \\ &\alpha\int_{\mathbb{R}}\left(-\partial_x^2 u + \ell_3^2 u^3\right)\partial_x^3 u dx = -3\alpha\ell_3^2\int_{\mathbb{R}} u^2\partial_x u\partial_x^2 u dx, \end{aligned} \tag{67}$$

$$\begin{aligned}
&\beta^2\int_{\mathbb{R}}\left(-\partial_x^2 u+\ell_3^2u^3\right)\partial_x^4 u dx=\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-3\ell_2^2\beta^2\int_{\mathbb{R}}u^2\partial_x u\partial_x^3 u dx\\
&\quad=6\ell_3^2\beta^2\int_{\mathbb{R}}u(\partial_x u)^2\partial_x^2 u dx+3\ell_3^2\beta^2\left\|u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad=-2\ell_3^2\beta^2\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}+3\ell_3^2\beta^2\left\|u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&4\kappa\int_{\mathbb{R}}\left(-\partial_x^2 u+\ell_3^2u^3\right)(\partial_x u)^2dx=4\kappa\ell_3^2\int_{\mathbb{R}}u^3(\partial_x u)^2dx,\\
&\kappa\int_{\mathbb{R}}\left(-\partial_x^2 u+\ell_3^2u^3\right)u\partial_x^2 u dx=-\kappa\int_{\mathbb{R}}u(\partial_x^2 u)^2dx-4\kappa\ell_3^2\int_{\mathbb{R}}u^3(\partial_x u)^2dx,\\
&-h^2\int_{\mathbb{R}}\left(-\partial_x^2 u+\ell_3^2u^3\right)\partial_x^2 u dx\\
&\quad=h^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+3h^2\ell_3^2\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Integrating (66) on $\mathbb{R}$, by (67), we get

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\ell_3^2}{4}\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}\right)+\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad+3\ell_3^2\beta^2\left\|u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+h^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad+3h^2\ell_3^2\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad=3\left(b+\alpha\ell_3^2\right)\int_{\mathbb{R}}u^2\partial_x u\partial_x^2 u dx+2a\int_{\mathbb{R}}u\partial_x u\partial_x^2 u dx\\
&\qquad+2\ell_3^2\beta^2\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}+\kappa\int_{\mathbb{R}}u(\partial_x^2 u)^2dx.
\end{aligned}$$

Thanks to (10), we have that

$$\begin{aligned}
&\frac{d}{dt}\left(2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\ell_3^2\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}\right)+4\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad+12\ell_3^2\beta^2\left\|u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+4h^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad+12h^2\ell_3^2\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad=8a\int_{\mathbb{R}}u\partial_x u\partial_x^2 u dx+8\ell_3^2\beta^2\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}+4\kappa\int_{\mathbb{R}}u(\partial_x^2 u)^2dx.
\end{aligned}\tag{68}$$

Due to the Young inequality,

$$\begin{aligned}
&8|a|\int_{\mathbb{R}}|u\partial_x u||\partial_x^2 u|dx=8\int_{\mathbb{R}}|h\ell_3 u\partial_x u|\left|\frac{a\partial_x^2 u}{h\ell_3}\right|dx\\
&\qquad\le 4h^2\ell_3^2\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{4a^2}{h^2\ell_3^2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&4|\kappa|\int_{\mathbb{R}}|u|(\partial_x^2 u)^2dx=4\int_{\mathbb{R}}\left|h\ell_3 u\partial_x^2 u\right|\left|\frac{\kappa\partial_x^2 u}{h\ell_3}\right|dx\\
&\qquad\le 2h^2\ell_3^2\left\|u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{2\kappa^2}{h^2\ell_3^2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

It follows from (68) that

$$\begin{aligned}\frac{d}{dt}&\left(2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\ell_3^2\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}\right)+4\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&+8\ell_3^2\beta^2\left\|u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+4h^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&+10h^2\ell_3^2\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&\leq 8\ell_3^2\beta^2\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}+\frac{2\left(2a^2+\kappa^2\right)}{h^2\ell_3^2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{69}$$

Due to the Young inequality,

$$\begin{aligned}\frac{2\left(2a^2+\kappa^2\right)}{h^2\ell_3^2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}&=\frac{2\left(2a^2+\kappa^2\right)}{h^2\ell_3^2}\int_{\mathbb{R}}\partial_x^3 u\partial_x^3 u dx\\&=-\frac{2\left(2a^2+\kappa^2\right)}{h^2\ell_3^2}\int_{\mathbb{R}}\partial_x u\partial_x^3 u dx\leq\frac{2\left(2a^2+\kappa^2\right)}{h^2\ell_3^2}\int_{\mathbb{R}}|\partial_x u||\partial_x^3 u|dx\\&=2\int_{\mathbb{R}}\left|\frac{\left(2a^2+\kappa^2\right)\partial_x u}{h^2\ell_3^2\beta}\right|\left|\beta\partial_x^3 u\right|dx\\&\leq\frac{\left(2a^2+\kappa^2\right)^2}{h^4\ell_3^2\beta^2}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Consequently, by (69),

$$\begin{aligned}\frac{d}{dt}&\left(2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\ell_3^2\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}\right)+3\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&+8\ell_3^2\beta^2\left\|u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+4h^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&+10h^2\ell_3^2\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&\leq 8\ell_3^2\beta^2\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}+\frac{\left(2a^2+\kappa^2\right)^2}{h^4\ell_3^2\beta^2}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{70}$$

Due to the Hölder inequality,

$$\begin{aligned}8\ell_3^2\beta^2\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}=&8\ell_3^2\beta^2\left\|\partial_x u(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\\leq&16\ell_3^2\beta^2\left\|\partial_x u(t,\cdot)\right\|^3_{L^2(\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Thanks to the Young inequality,

$$8\ell_3^2\beta^2\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}\leq\frac{64\ell_3^4\beta^4}{h^2}\left\|\partial_x u(t,\cdot)\right\|^6_{L^2(\mathbb{R})}+h^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Hence, by (70),

$$\begin{aligned}\frac{d}{dt}&\left(2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\ell_3^2\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}\right)+3\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&+8\ell_3^2\beta^2\left\|u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+3h^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\end{aligned} \tag{71}$$

$$
\begin{aligned}
&+ 10h^2\ell_3^2 \left\| u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le &\frac{64\ell_3^4\beta^4}{h^2} \left\|\partial_x u(t,\cdot)\right\|^6_{L^2(\mathbb{R})} + \frac{\left(2a^2+\kappa^2\right)^2}{h^4\ell_3^2\beta^2} \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Denoting

$$
\begin{aligned}
Y(t) :=&2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \ell_3^2\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})},\\
\ell_4 :=&\frac{8\ell_3^4\beta^4}{h^2}, \quad \ell_5 := \frac{\left(2a^2+\kappa^2\right)^2}{2h^4\ell_3^2\beta^2}.
\end{aligned}
\tag{72}
$$

and using (72),

$$
\ell_5\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le \ell_5\left(\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \ell_3^2\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}\right) = \ell_5 Y(t). \tag{73}
$$

Since

$$
2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le 2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \ell_3^2\left\|u(t,\cdot)\right\|^2_{L^4(\mathbb{R})},
$$

then, by (72),

$$
8\left\|\partial_x u(t,\cdot)\right\|^6_{L^2(\mathbb{R})} \le \left(2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \ell_3^2\left\|u(t,\cdot)\right\|^2_{L^4(\mathbb{R})}\right)^6 = Y^3(t). \tag{74}
$$

Consequently, by (71), (72), (72) and (74),

$$
\frac{dY(t)}{dt} \le \ell_4 Y + \ell_5 Y^3.
$$

Arguing as in Lemma 1, we get

$$
\frac{e^{2\ell_4 t}}{Y^2(t)} - \frac{1}{Y_0^2} \ge -\frac{\ell_5}{\ell_4}\left(e^{2\ell_4 t}-1\right),
$$

that is

$$
\frac{e^{2\ell_4 t}}{Y^2(t)} + \frac{\ell_5}{\ell_4}\left(e^{2\ell_4 t}-1\right) \ge \frac{1}{Y_0^2}. \tag{75}
$$

Using (72) in (75), we have (63).

We prove (64). We begin by observing that, by the Hölder inequality,

$$
\ell_3^2\left\|u_0\right\|^4_{L^4(\mathbb{R})} \le \ell_3^2\left\|u_0\right\|^2_{L^\infty(\mathbb{R})}\left\|u_0\right\|^2_{L^2(\mathbb{R})} \le 2\ell_3^2\left\|u_0\right\|^3_{L^2(\mathbb{R})}\left\|\partial_x u_0\right\|_{L^2(\mathbb{R})}.
$$

Therefore,

$$
2\left\|\partial_x u_0\right\|^2_{L^2(\mathbb{R})} + \ell_3^2\left\|u_0\right\|^4_{L^4(\mathbb{R})} \le 2\left\|\partial_x u_0\right\|^2_{L^2(\mathbb{R})} + 2\ell_3^2\left\|u_0\right\|^3_{L^2(\mathbb{R})}\left\|\partial_x u_0\right\|_{L^2(\mathbb{R})},
$$

which gives

$$
\begin{aligned}
&\left(2\left\|\partial_x u_0\right\|^2_{L^2(\mathbb{R})} + \ell_3^2\left\|u_0\right\|^4_{L^4(\mathbb{R})}\right)^2\\
&\quad\le \left(2\left\|\partial_x u_0\right\|^2_{L^2(\mathbb{R})} + 2\ell_3^2\left\|u_0\right\|^3_{L^2(\mathbb{R})}\left\|\partial_x u_0\right\|_{L^2(\mathbb{R})}\right)^2.
\end{aligned}
\tag{76}
$$

Denote

$$Y_{1,0}^2 := \left(2\left\|\partial_x u_0\right\|_{L^2(\mathbb{R})}^2 + 2\ell_3^2\left\|u_0\right\|_{L^2(\mathbb{R})}^3\left\|\partial_x u_0\right\|_{L^2(\mathbb{R})}\right)^2. \tag{77}$$

It follows from (72), (50) and (77) that

$$\frac{1}{X_0^2} \geq \frac{1}{Y_{1,0}^2}. \tag{78}$$

Consequently, by (63), (72) and (78),

$$\frac{e^{2\ell_4 t}}{Y^2(t)} + \frac{\ell_5}{\ell_4}\left(e^{2\ell_4 t} - 1\right) \geq \frac{1}{Y_{1,0}^2},$$

that is

$$\frac{\ell_4 e^{2\ell_4 t} + \ell_5\left(e^{2\ell_4 t} - 1\right)Y^2(t)}{\ell_4 Y^2(t)} \geq \frac{1}{Y_{1,0}^2}.$$

Therefore,

$$\begin{aligned}\ell_4 Y^2(t) \leq & Y_{1,0}^2\ell_4 e^{2\ell_4 t} + \ell_5 Y_{1,0}^2\left(e^{2\ell_4 t} - 1\right)Y^2(t)\\ \leq & Y_{1,0}^2\ell_4 e^{2\ell_4 T} + \ell_5 Y_{1,0}^2\left(e^{2\ell_4 T} - 1\right)Y^2(t).\end{aligned}$$

Hence,

$$\left[\ell_4 - \ell_5 Y_{1,0}^2\left(e^{2\ell_4 T} - 1\right)\right]Y^2(t) \leq Y_{1,0}^2\ell_4 e^{2\ell_4 T}. \tag{79}$$

We require that

$$\ell_4 - \ell_5 Y_{1,0}^2\left(e^{2\ell_4 T} - 1\right) > 0. \tag{80}$$

Thanks to (72) and (77), (80) reads

$$\begin{aligned}\frac{8\ell_3^4\beta^4}{h^2} - \frac{\left(2a^2+\kappa^2\right)^2}{2h^4\ell_3^2\beta^2}&\left(2\left\|\partial_x u_0\right\|_{L^2(\mathbb{R})}^2\right.\\ &\left.+2\ell_3^2\left\|u_0\right\|_{L^2(\mathbb{R})}^3\left\|\partial_x u_0\right\|_{L^2(\mathbb{R})}\right)^2\left(e^{\frac{16\ell_3^4\beta^4 T}{h^2}} - 1\right) > 0,\end{aligned}$$

which is verified when

$$\begin{aligned}\left(2a^2+\kappa^2\right)^2\left(e^{\frac{16\ell_3^4\beta^4 T}{h^2}} - 1\right)\left(2\left\|\partial_x u_0\right\|_{L^2(\mathbb{R})}^2 + 2\ell_3^2\left\|u_0\right\|_{L^2(\mathbb{R})}^3\left\|\partial_x u_0\right\|_{L^2(\mathbb{R})}\right)^2&\\ - 16\ell_3^6\beta^6 < 0.&\end{aligned}$$

Thanks to (10), we have that

$$2\left\|\partial_x u_0\right\|_{L^2(\mathbb{R})}^2 + 2\ell_3^2\left\|u_0\right\|_{L^2(\mathbb{R})}^3\left\|\partial_x u_0\right\|_{L^2(\mathbb{R})} - A_1^2 < 0.$$

Therefore, by (10), (72), (79) and (80), there exists a constant $C > 0$, such that

$$\frac{Y^2(t)}{C} \le Y_{1,0}^2 \ell_4 e^{2\ell_4 T},$$

that is

$$Y^2(t) \le C Y_{1,0}^2 \ell_4 e^{2\ell_4 T} \tag{81}$$

Using (72) and (77) in (81), we have (64).

We prove (21) and (65). By (64) and (71), we have that

$$\begin{aligned}\frac{d}{dt}\Big(2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \ell_3^2\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}\Big) + 3\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ + 8\ell_3^2\beta^2\left\|u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 4h^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ + 10h^2\ell_3^2\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C.\end{aligned}$$

Integrating on $(0,t)$, by (4), we get

$$\begin{aligned}2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \ell_3^2\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + 3\beta^2\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ + 8\ell_3^2\beta^2\int_0^t\left\|u(s,\cdot)\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ + 4h^2\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ + 10h^2\ell_3^2\int_0^t\left\|u(s,\cdot)\partial_x u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ \le 2\left\|\partial_x u_0\right\|^2_{L^2(\mathbb{R})} + \ell_3^2\left\|u_0\right\|^4_{L^4(\mathbb{R})} + Ct \le C,\end{aligned}$$

which gives (21) and (65).

We prove (47). Thanks to (64) and the Hölder inequality,

$$\begin{aligned}|u(t,x)|^3 =& 3\left|\int_{-\infty}^x u^2\partial_x u dy\right| \le 3\int_{\mathbb{R}} u^2|\partial_x u||dx\\ \le& 3\left\|u(t,\cdot)\right\|^2_{L^4(\mathbb{R})}\left\|\partial_x u(t,\cdot)\right\|_{L^2(\mathbb{R})} \le C.\end{aligned}$$

Hence,

$$\|u\|^3_{L^\infty((0,T)\times\mathbb{R})} \le C,$$

which gives (47).

Finally, we prove (49). [16, Lemma 2.3] says that

$$\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} \le \int_{\mathbb{R}} u^2(\partial_x^2 u)^2 dx.$$

Consequently, by (47),

$$\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} \le \|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0,t)$, by (21), we have (49). □

Lemma 11. *Assume* (10) *and fix* $T > 0$. *Then*

$$\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + h^2 \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t \left\|\partial_t u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C \tag{82}$$

for every $0 \le t \le T$. *Moreover,* (56) *and* (58) *hold.*

Proof. Let $0 \le t \le T$. Multiplying (62) by $2\partial_t u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
&\frac{d}{dt}\left(\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + h^2 \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad = 2\beta^2 \int_{\mathbb{R}} \partial_x^4 u \partial_t u dx - 2h^2 \int_{\mathbb{R}} \partial_x^2 u \partial_t u dx\\
&\quad = -2\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 4a \int_{\mathbb{R}} u\partial_x u \partial_t u dx - 6b \int_{\mathbb{R}} u^2 \partial_x u \partial_t u dx\\
&\qquad - 2\alpha \int_{\mathbb{R}} \partial_x^3 u \partial_t u dx - 8\kappa \int_{\mathbb{R}} (\partial_x u)^2 \partial_t u dx - 2\kappa \int_{\mathbb{R}} u \partial_x^2 u \partial_t u dx.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\frac{d}{dt}\left(\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + h^2 \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right) + 2\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad = -4a \int_{\mathbb{R}} u\partial_x u \partial_t u dx - 6b \int_{\mathbb{R}} u^2 \partial_x u \partial_t u dx - 2\alpha \int_{\mathbb{R}} \partial_x^3 u \partial_t u dx\\
&\qquad - 8\kappa \int_{\mathbb{R}} (\partial_x u)^2 \partial_t u dx - 2\kappa \int_{\mathbb{R}} u \partial_x^2 u \partial_t u dx.
\end{aligned} \tag{83}$$

Due to (47) and the Young inequality,

$$\begin{aligned}
4|a| \int_{\mathbb{R}} |u\partial_x u||\partial_t u| dx &= 2\int_{\mathbb{R}} \left|\frac{2au\partial_x u}{\sqrt{D_1}}\right| \left|\sqrt{D_1}\partial_t u\right| dx\\
&\le \frac{4a^2}{D_1} \left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_1 \left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
6|b| \int_{\mathbb{R}} u^2 |\partial_x u||\partial_t u| dx &\le 6|b| \left\|u\right\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |u\partial_x u||\partial_t u| dx\\
&\le 2C \int_{\mathbb{R}} |u\partial_x u||\partial_t u| dx = 2\int_{\mathbb{R}} \left|\frac{Cu\partial_x u}{\sqrt{D_1}}\right| \left|\sqrt{D_1}\partial_t u\right| dx\\
&\le \frac{C}{D_1} \left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_1 \left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\alpha| \int_{\mathbb{R}} |\partial_x^3 u||\partial_t u| dx &= 2\int_{\mathbb{R}} \left|\frac{\alpha\partial_x^3 u}{\sqrt{D_1}}\right| \left|\sqrt{D_1}\partial_t u\right| dx\\
&\le \frac{\alpha^2}{D_1} \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_1 \left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
8|\kappa| \int_{\mathbb{R}} (\partial_x u)^2 \partial_t u dx &= 2\int_{\mathbb{R}} \left|\frac{4\kappa(\partial_x u)^2}{\sqrt{D_1}}\right| \left|\sqrt{D_1}\partial_t u\right| dx\\
&\le \frac{16\kappa^2}{D_1} \left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + D_1 \left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

$$\begin{aligned}2|\kappa|\int_{\mathbb{R}}|u\partial_x^2u||\partial_t u|dx&=2\int_{\mathbb{R}}\left|\frac{\kappa u\partial_x^2 u}{\sqrt{D_1}}\right|\left|\sqrt{D_1}\partial_t u\right|dx\\&\le\frac{\kappa^2}{D_1}\left\|u(t,\cdot)\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+D_1\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\end{aligned}$$

where D_1 is a positive constant, which will be specified later. It follows from (83) that

$$\begin{aligned}&\frac{d}{dt}\left(\beta^2\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+h^2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)+(2-5D_1)\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&\quad\le\frac{a^2+C}{D_1}\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\alpha^2}{D_1}\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&\qquad+\frac{16\kappa^2}{D_1}\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}+\frac{\kappa^2}{D_1}\left\|u(t,\cdot)\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Taking $D_1=\frac{1}{5}$, we have

$$\begin{aligned}&\frac{d}{dt}\left(\beta^2\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+h^2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)+\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&\quad\le C\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+5\alpha^2\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&\qquad+80\kappa^2\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}+5\kappa^2\left\|u(t,\cdot)\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Integrating on $(0,t)$, by (4), (49) and (65), we get

$$\begin{aligned}&\beta^2\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+h^2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\int_0^t\left\|\partial_t u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\&\quad\le\beta^2\left\|\partial_x^2u_0\right\|^2_{L^2(\mathbb{R})}+h^2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C\int_0^t\left\|u(s,\cdot)\partial_x u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\&\qquad+5\alpha^2\int_0^t\left\|\partial_x^3u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds+80\kappa^2\int_0^t\left\|\partial_x u(s,\cdot)\right\|^4_{L^4(\mathbb{R})}ds\\&\qquad+5\kappa^2\int_0^t\left\|u(s,\cdot)\partial_x^2u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C,\end{aligned}$$

which gives (82).

Finally, arguing as in Lemma 5, we have (56), while, arguing as in Lemma 6, we have (58). □

Lemma 12. *Fixed $T>0$, then*

$$\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\beta^2\int_0^t\left\|\partial_x^2u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds+2h^2\int_0^t\left\|\partial_x u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C\quad(84)$$

for every $0\le t\le T$.

Proof. Multiplying (62) by $2u$, an integration on $\mathbb{R}$ gives

$$\frac{d}{dt}\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=2\int_{\mathbb{R}}u\partial_t u dx$$

$$
\begin{aligned}
&= -4a\int_{\mathbb{R}} u^2\partial_x u dx - 6b\int_{\mathbb{R}} u^3\partial_x u dx - 2\alpha\int_{\mathbb{R}} u\partial_x^3 dx\\
&\quad - 2\beta^2\int_{\mathbb{R}} u\partial_x^4 u dx - 8\kappa\int_{\mathbb{R}} u(\partial_x u)^2 dx - 2\kappa\int_{\mathbb{R}} u^2\partial_x^2 u dx + 2h^2\int_{\mathbb{R}} u\partial_x^2 u dx\\
&= 2\alpha\int_{\mathbb{R}} \partial_x u\partial_x^2 u dx + 2\beta^2\int_{\mathbb{R}} \partial_x u\partial_x^3 u dx - 4\kappa\int_{\mathbb{R}} u(\partial_x u)^2 dx - 2h^2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&= -2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 4\kappa\int_{\mathbb{R}} u(\partial_x u)^2 dx - 2h^2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
\frac{d}{dt}\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2h^2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}&\\
= -4\kappa\int_{\mathbb{R}} u(\partial_x u)^2 dx.&
\end{aligned}
\tag{85}
$$

Thanks to (47) and (64),

$$
4|\kappa|\int_{\mathbb{R}} |u|(\partial_x u)^2 dx \le 4|\kappa|\left\|u\right\|_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C.
$$

Consequently, by (85),

$$
\frac{d}{dt}\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2h^2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C.
$$

Integrating on $(0,t)$, by (4), we get

$$
\begin{aligned}
\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + 2h^2\int_0^t\left\|\partial_x u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds&\\
\le \left\|u_0\right\|^2_{L^2(\mathbb{R})} + Ct \le C,&
\end{aligned}
$$

which gives (84). □

Finally, arguing as in Section 2, we have Theorem 1.

Acknowledgments

GMC is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). GMC has been partially supported by the Project funded under the National Recovery and Resilience Plan (NRRP), Mission 4 Component 2 Investment 1.4 -Call for tender No. 3138 of 16/12/2021 of Italian Ministry of University and Research funded by the European Union -NextGenerationEUoAward Number: CN000023, Concession Decree No. 1033 of 17/06/2022 adopted by the Italian Ministry of University and Research, CUP: D93C22000410001, Centro Nazionale per la Mobilità Sostenibile, the Italian Ministry of Education, University and Research under the Programme Department of Excellence Legge 232/2016 (Grant

No. CUP - D93C23000100001), and the Research Project of National Relevance "Evolution problems involving interacting scales" granted by the Italian Ministry of Education, University and Research (MIUR Prin 2022, project code 2022M9BKBC, Grant No. CUP D53D23005880006).
GMC expresses its gratitude to the HIAS - Hamburg Institute for Advanced Study for their warm hospitality.

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