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SPATIAL BEHAVIOUR IN TYPE III THERMOELASTICITY WITH TWO POROUS STRUCTURES

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Dedicated to Professor Marin Marin on the occasion of his 70th anniversary

Abstract

This article is about the spatial behaviour in one-dimensional type III thermoelasticity with two voids structures, with porous dissipation in one of the voids components. After deriving a preliminary integral identity of Lagrange-Brun type, we prove the main results with the help of a time-weighted function.

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1 Introduction

This article is based on the new mathematical model from [3] for the one-dimensional type III thermoelasticity with two voids structures, with porous dissipation in one of the voids components. Other similar mathematical models can be found in [1], [4], [5].

The system of equations for this mathematical model are given in [3]

$$\rho\ddot{u} = t_x, \tag{1}$$

$$J_1\ddot{\phi}_1 = h_{1,x} + g_1, \tag{2}$$

$$J_2\ddot{\phi}_2 = h_{2,x} + g_2, \tag{3}$$

$$\rho\dot{\eta} = q_x. \tag{4}$$

The constitutive equations are given in [3]

$$t = \mu u_x + \gamma_1\phi_1 + \gamma_2\phi_2 - \beta\theta, \tag{5}$$

$$h_1 = b_{11}\phi_{1,x} + b_{12}\phi_{2,x} + m_1\psi_x, \tag{6}$$

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$$h_2 = b_{12}\phi_{1,x} + b_{22}\phi_{2,x} + m_2\psi_x, \tag{7}$$

$$g_1 = -\gamma_1 u_x + d_1\theta - \xi_{11}\phi_1 - \xi_{12}\phi_2 - \xi^*\dot{\phi}_1, \tag{8}$$

$$g_2 = -\gamma_2 u_x + d_2\theta - \xi_{12}\phi_1 - \xi_{22}\phi_2, \tag{9}$$

$$\rho\eta = \beta u_x + a\theta + d_1\phi_1 + d_2\phi_2, \tag{10}$$

$$q = k\psi_x + m_1\phi_{1,x} + m_2\phi_{2,x} + k^*\theta_x. \tag{11}$$

As usual, ρ is the mass density, J_i $(i = 1, 2)$ are the products of the mass density by the equilibrated inertias, t is the stress, h_i are the equilibrated stresses, g_i are the equilibrated body forces, q is the heat flux, η is the entropy, u is the displacement, ϕ_i are the volume fractions, ψ is the thermal displacement, θ is the temperature.

The field equation for the one-dimensional problem are given in [3] and are obtained by replacing the constitutive equations (5)-(11) into the system of equations (1)-(4)

$$\begin{cases} \rho\ddot{u} = \mu u_{xx} + \gamma_1\phi_{1,x} + \gamma_2\phi_{2,x} - \beta\dot{\psi}_x \\ J_1\ddot{\phi}_1 = b_{11}\phi_{1,xx} + b_{12}\phi_{2,xx} + m_1\psi_{xx} - \xi_{11}\phi_1 - \xi_{12}\phi_2 + d_1\dot{\psi} - \gamma_1 u_x - \xi^*\dot{\phi}_1 \\ J_2\ddot{\phi}_2 = b_{12}\phi_{1,xx} + b_{22}\phi_{2,xx} + m_2\psi_{xx} - \xi_{12}\phi_1 - \xi_{22}\phi_2 + d_2\dot{\psi} - \gamma_2 u_x \\ a\ddot{\psi} = m_1\phi_{1,xx} + m_2\phi_{2,xx} + k\psi_{xx} - d_1\dot{\phi}_1 - d_2\dot{\phi}_2 - \beta\dot{u}_x + k^*\theta_{xx} \end{cases}. \tag{12}$$

As in [3], we assume that

$$J_i > 0 (i = 1, 2), a > 0, \rho > 0, \xi^* > 0, k^* > 0, \tag{13}$$

and that the two matrices below are positive definite

$$M_1 = \begin{pmatrix} b_{11} & b_{12} & m_1 \\ b_{12} & b_{22} & m_2 \\ m_1 & m_2 & k \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} \mu & \gamma_1 & \gamma_2 \\ \gamma_1 & \xi_{11} & \xi_{12} \\ \gamma_2 & \xi_{12} & \xi_{22} \end{pmatrix}. \tag{14}$$

The boundary conditions are

$$\begin{aligned} & u(0, t) = u(\pi, t) = 0, \\ & \phi_{1,x}(0, t) = \phi_{1,x}(\pi, t) = \phi_{2,x}(0, t) = \phi_{2,x}(\pi, t) = 0, \\ & \psi_x(0, t) = \psi_x(\pi, t) = 0. \end{aligned} \tag{15}$$

and the initial conditions are

$$\begin{aligned} & u(x, 0) = u_0(x), \dot{u}(x, 0) = v_0(x), \phi_1(x, 0) = \phi_{10}(x), \dot{\phi}_1(x, 0) = \varphi_{10}(x), \\ & \phi_2(x, 0) = \phi_{20}(x), \dot{\phi}_2(x, 0) = \varphi_{20}(x), \psi(x, 0) = \psi_0(x), \dot{\psi}(x, 0) = \theta_0(x) \end{aligned} \tag{16}$$

for $x \in [0, \pi]$.

2 Preliminary results

In proving the main results, we follow [2].

First, we define the following quadratic form

$$W = \frac{1}{2}\mu u_x u_x + \gamma_1 \phi_1 u_x + \gamma_2 \phi_2 u_x + \frac{1}{2} b_{11} \phi_{1,x}^2 + \\ + b_{12}\phi_{2,x}\phi_{1,x} + m_1 \psi_x \phi_{1,x} + \frac{1}{2}\xi_{11}\phi_1^2 + \xi_{12}\phi_1\phi_2 + \\ + \frac{1}{2} b_{22}\phi_{2,x}^2 + m_2 \psi_x \phi_{2,x} + \frac{1}{2}\xi_{22}\phi_2^2 + \frac{1}{2} k \psi_x^2. \quad (17)$$

The quadratic form W is positive definite, so there exist constants $\mu_m > 0$ and $\mu_M > 0$ such that

$$\mu_m \left(u_x u_x + \phi_1^2 + \phi_2^2 + \phi_{1,x}^2 + \phi_{2,x}^2 + \psi_x^2 \right) \leq 2W \leq \\ \leq \mu_M \left(u_x u_x + \phi_1^2 + \phi_2^2 + \phi_{1,x}^2 + \phi_{2,x}^2 + \psi_x^2 \right). \quad (18)$$

We define the state of strain by

$$E := \{u_x, \phi_1, \phi_2, \phi_{1,x}, \phi_{2,x}, \psi_x\}. \quad (19)$$

Let $\mathcal{E}$ be the vector space of all E of the form (19). The magnitude of $E \in \mathcal{E}$ is

$$|E| := (E \cdot E)^{\frac{1}{2}} = \left(u_x^2 + \phi_1^2 + \phi_2^2 + \phi_{1,x}^2 + \phi_{2,x}^2 + \psi_x^2 \right)^{\frac{1}{2}}. \quad (20)$$

Let

$$s(E) = \mu u_x + \gamma_1 \phi_1 + \gamma_2 \phi_2, \quad (21)$$

$$h_1(E) = b_{11}\phi_{1,x} + b_{12}\phi_{2,x} + m_1 \psi_x, \quad (22)$$

$$h_2(E) = b_{12}\phi_{1,x} + b_{22}\phi_{2,x} + m_2 \psi_x, \quad (23)$$

$$G_1(E) = -\gamma_1 u_x - \xi_{11}\phi_1 - \xi_{12}\phi_2, \quad (24)$$

$$G_2(E) = -\gamma_2 u_x - \xi_{12}\phi_1 - \xi_{22}\phi_2, \quad (25)$$

$$Q(E) = k\psi_x + m_1 \phi_{1,x} + m_2 \phi_{2,x}. \quad (26)$$

Then

$$t = s - \beta\theta, \quad (27)$$

$$g_1 = G_1 + d_1\theta - \xi^* \dot{\phi}_1, \quad (28)$$

$$g_2 = G_2 + d_2\theta, \quad (29)$$

$$q = Q + k^* \theta_x. \quad (30)$$

We define $S(E)$, which will be useful in proving the main result about the spatial behaviour of the solutions.

$$S(E) = \left\{ s(E), G_1(E), G_2(E), \frac{1}{\sqrt{\kappa_1^0}} h_1(E), \frac{1}{\sqrt{\kappa_2^0}} h_2(E), Q(E) \right\} \in \mathcal{E}. \quad (31)$$

The magnitude of $S(E)$ is given by

$$|S(E)| = \left\{ s(E)^2 + G_1(E)^2 + G_2(E)^2 + \frac{1}{\kappa_1^0} h_1(E)^2 + \frac{1}{\kappa_2^0} h_2(E)^2 + Q(E)^2 \right\}^{\frac{1}{2}}. \tag{32}$$

We define the bilinear form

$$\begin{aligned} \mathcal{F}\left(E^{(1)}, E^{(2)}\right) := \frac{1}{2} \Big[& \mu u_x^{(1)} u_x^{(2)} + \gamma_1 \left(\phi_1^{(1)} u_x^{(2)} + \phi_1^{(2)} u_x^{(1)} \right) + \\ & + \gamma_2 \left(\phi_2^{(1)} u_x^{(2)} + \phi_2^{(2)} u_x^{(1)} \right) + b_{11} \phi_{1,x}^{(1)} \phi_{1,x}^{(2)} + b_{12} \left(\phi_{2,x}^{(1)} \phi_{1,x}^{(2)} + \phi_{2,x}^{(2)} \phi_{1,x}^{(1)} \right) + \\ & + m_1 \left(\psi_x^{(1)} \phi_{1,x}^{(2)} + \psi_x^{(2)} \phi_{1,x}^{(1)} \right) + \xi_{11} \phi_1^{(1)} \phi_1^{(2)} + \xi_{12} \left(\phi_1^{(1)} \phi_2^{(2)} + \phi_1^{(2)} \phi_2^{(1)} \right) + \\ & + b_{22} \phi_{2,x}^{(1)} \phi_{2,x}^{(2)} + m_2 \left(\psi_x^{(1)} \phi_{2,x}^{(2)} + \psi_x^{(2)} \phi_{2,x}^{(1)} \right) + \xi_{22} \phi_2^{(1)} \phi_2^{(2)} + k \psi_x^{(1)} \psi_x^{(2)}, \end{aligned} \tag{33}$$

for all $E^{(\alpha)} = \left\{ u_x^{(\alpha)}, \phi_1^{(\alpha)}, \phi_2^{(\alpha)}, \phi_{1,x}^{(\alpha)}, \phi_{2,x}^{(\alpha)}, \psi_x^{(\alpha)} \right\} \in \mathcal{E},\ \alpha = 1, 2.$

We deduce that

$$\mathcal{F}\left(E^{(1)}, E^{(2)}\right) = \mathcal{F}\left(E^{(2)}, E^{(1)}\right), \forall E^{(1)}, E^{(2)} \in \mathcal{E}. \tag{34}$$

Furthermore, we obtain

$$\mathcal{F}(E, E) = W(E), \forall E \in \mathcal{E}. \tag{35}$$

By the Cauchy-Schwarz inequality, we obtain

$$\mathcal{F}\left(E^{(1)}, E^{(2)}\right) \le \left[W\left(E^{(1)}\right) \right]^{\frac{1}{2}} \left[W\left(E^{(2)}\right) \right]^{\frac{1}{2}}, \forall E^{(1)}, E^{(2)} \in \mathcal{E}. \tag{36}$$

We deduce

$$\begin{aligned} |S(E)|^2 = & \left(\mu u_x + \gamma_1 \phi_1 + \gamma_2 \phi_2 \right) s + \left(-\gamma_1 u_x - \xi_{11} \phi_1 - \xi_{12} \phi_2 \right) G_1 + \\ & + (-\gamma_2 u_x - \xi_{12} \phi_1 - \xi_{22} \phi_2) G_2 + \frac{1}{\kappa_1^0} (b_{11} \phi_{1,x} + b_{12} \phi_{2,x} + m_1 \psi_x) h_1 + \\ & + \frac{1}{\kappa_2^0} (b_{12} \phi_{1,x} + b_{22} \phi_{2,x} + m_2 \psi_x) h_2 + (k \psi_x + m_1 \phi_{1,x} + m_2 \phi_{2,x}) Q = \\ & = 2 \mathcal{F}\left(E, \tilde{S}(E)\right), \end{aligned} \tag{37}$$

where

$$\tilde{S}(E) = \left\{ s(E), -G_1(E), -G_2(E), \frac{1}{\sqrt{\kappa_1^0}} h_1(E), \frac{1}{\sqrt{\kappa_2^0}} h_2(E), Q(E) \right\}. \tag{38}$$

Then

$$|S(E)|^2 \le 2 \left[W(E) \right]^{\frac{1}{2}} \left[W(\tilde{S}(E)) \right]^{\frac{1}{2}} \le\le 2 \left[W(E) \right]^{\frac{1}{2}} \left(\frac{\mu_M}{2} |S(E)|^2 \right)^{\frac{1}{2}}. \tag{39}$$

It follows that

$$|S(E)|^2 \leq 2\mu_M W(E). \tag{40}$$

This leads to

$$s(E)^2 + G_1(E)^2 + G_2(E)^2 + \frac{1}{\kappa_1^0} h_1(E)^2 + \frac{1}{\kappa_2^0} h_2(E)^2 + Q(E)^2 \leq 2\mu_M W(E), \forall E \in \mathcal{E}. \tag{41}$$

Let $\varepsilon > 0$. For every second-order tensor we have the inequality

$$(L_{ij} + M_{ij})(L_{ij} + M_{ij}) \leq (1+\varepsilon) L_{ij} L_{ij} + \left(1 + \frac{1}{\varepsilon}\right) M_{ij} M_{ij}. \tag{42}$$

Below we derive an integral identity of Lagrange-Brun type. This is useful in showing the main results about the spatial behaviour of the solutions of the initial boundary value problem. The lemma below shows a conservation law of total energy which has a weight depending on time.

Lemma 1. *We consider that $P \subset B$ is a regular region which has regular boundary ∂P. If the relations (1)-(4) and (5)-(11) hold, then*

$$\begin{aligned}
&\int_P e^{-\lambda t} \left[\frac{1}{2}\rho \dot{u}(t)\dot{u}(t) + \frac{1}{2} J_1 \dot{\phi}_1^2(t) + \frac{1}{2} J_2 \dot{\phi}_2^2(t) + W(E(t)) + \frac{1}{2} a\theta^2(t)\right] dv + \\
&\quad + \int_0^t \int_P e^{-\lambda s} \left[\frac{\lambda}{2}\rho \dot{u}(s)\dot{u}(s) + \frac{\lambda}{2} J_1 \dot{\phi}_1^2(s) + + \frac{\lambda}{2} J_2 \dot{\phi}_2^2(s) + \lambda W(E(s)) + \right. \\
&\qquad \left. + \frac{\lambda}{2} a\theta^2(s) + k^* \theta_x(s)\theta_x(s) + \xi^* \dot{\phi}_1(s)\dot{\phi}_1(s)\right] dv ds \\
&= \int_P \left[\frac{1}{2}\rho \dot{u}(0)\dot{u}(0) + \frac{1}{2} J_1 \dot{\phi}_1^2(0) + + \frac{1}{2} J_2 \dot{\phi}_2^2(0) + W(E(0)) + \frac{1}{2} a\theta^2(0)\right] dv + \\
&\qquad + \int_0^t \int_{\partial P} e^{-\lambda s} \cdot \left[tn\dot{u} + h_1 n\dot{\phi}_1 + h_2 n\dot{\phi}_2 + qn\theta\right] da ds,
\end{aligned} \tag{43}$$

for $t \in [0, \infty)$ and $\lambda > 0$ a given parameter.

Proof. First, we multiply the equation (1) by $\dot{u}$ and we obtain

$$\rho \ddot{u}\dot{u} = (t\dot{u})_x - t\dot{u}_x. \tag{44}$$

Then, we replace the constitutive equation (5) and we get

$$\frac{1}{2}\frac{d}{ds}(\rho \dot{u}\dot{u}) = (t\dot{u})_x - \mu u_x \dot{u}_x - \gamma_1 \phi_1 \dot{u}_x - \gamma_2 \phi_2 \dot{u}_x + \beta\theta \dot{u}_x. \tag{45}$$

We multiply the equation (2) by $\dot{\phi}_1$ and we deduce that

$$J_1 \ddot{\phi}_1 \dot{\phi}_1 = (h_1 \dot{\phi}_1)_x - h_1 \dot{\phi}_{1,x} + g_1 \dot{\phi}_1. \tag{46}$$

In the equation above, we replace the constitutive equations (6) and (8). So, we obtain

$$\frac{1}{2}\frac{d}{ds}\left(J_1\dot{\phi}_1^2\right) = (h_1\dot{\phi}_1)_x - b_{11}\phi_{1,x}\dot{\phi}_{1,x} - b_{12}\phi_{2,x}\dot{\phi}_{1,x} - m_1\psi_x\dot{\phi}_{1,x} - \\ - \gamma_1 u_x\dot{\phi}_1 + d_1\theta\dot{\phi}_1 - \xi_{11}\phi_1\dot{\phi}_1 - \xi_{12}\phi_2\dot{\phi}_1 - \xi^*\dot{\phi}_1\dot{\phi}_1. \quad (47)$$

Similarly, we multiply the equation (3) by $\dot{\phi}_2$, then use the constitutve equations (7) and (9) in order to obtain

$$\frac{1}{2}\frac{d}{ds}\left(J_2\dot{\phi}_2^2\right) = \left(h_2\dot{\phi}_2\right)_x - b_{12}\phi_{1,x}\dot{\phi}_{2,x} - b_{22}\phi_{2,x}\dot{\phi}_{2,x} - \\ - m_2\psi_x\dot{\phi}_{2,x} - \gamma_2 u_x\dot{\phi}_2 + d_2\theta\dot{\phi}_2 - \xi_{12}\phi_1\dot{\phi}_2 - \xi_{22}\phi_2\dot{\phi}_2. \quad (48)$$

Finally, we multiply the equation (4) by θ and obtain

$$\rho\dot{\eta}\theta = (q\theta)_x - q\theta_x. \quad (49)$$

Then, we use the constitutive equations (10), (11) and deduce that

$$\frac{1}{2}\frac{d}{ds}\left(a\theta^2\right) = -\beta\dot{u}_x\theta - d_1\dot{\phi}_1\theta - d_2\dot{\phi}_2\theta + (q\theta)_x - \\ - k\psi_x\theta_x - m_1\phi_{1,x}\theta_x - m_2\phi_{2,x}\theta_x - k^*\theta_x\theta_x. \quad (50)$$

Then, we add the formulas (45), (47), (48) and (50). So, we obtain

$$\frac{d}{ds}\left\{\frac{1}{2}\rho\dot{u}\dot{u} + \frac{1}{2}J_1\dot{\phi}_1^2 + \frac{1}{2}J_2\dot{\phi}_2^2 + W + \frac{1}{2}a\theta^2\right\} + \\ + k^*\theta_x\theta_x + \xi^*\dot{\phi}_1\dot{\phi}_1 = \left(t\dot{u} + h_1\dot{\phi}_1 + h_2\dot{\phi}_2 + q\theta\right)_x. \quad (51)$$

In the equation above, we used, for example, the fact that

$$\frac{d}{ds}\left(\gamma_1\phi_1 u_x\right) = \gamma_1\dot{\phi}_1 u_x + \gamma_1\phi_1\dot{u}_x. \quad (52)$$

Then, we multiply the equation (51) by $e^{-\lambda s}$, integrate the resulting equation over $P \times [0,t]$ and use the divergence theorem. Therefore, we obtain

$$\int_0^t\int_P \frac{d}{ds}\left\{e^{-\lambda s}\left[\frac{1}{2}\rho\dot{u}\dot{u} + \frac{1}{2}J_1\dot{\phi}_1^2 + \frac{1}{2}J_2\dot{\phi}_2^2 + W + \frac{1}{2}a\theta^2\right]\right\} dvds - \\ - \int_0^t\int_P -\lambda e^{-\lambda s}\left[\frac{1}{2}\rho\dot{u}\dot{u} + \frac{1}{2}J_1\dot{\phi}_1^2 + \frac{1}{2}J_2\dot{\phi}_2^2 + W + \frac{1}{2}a\theta^2\right] dvds + \\ + \int_0^t\int_P e^{-\lambda s}\left[k^*\theta_x\theta_x + \xi^*\dot{\phi}_1\dot{\phi}_1\right] dvds = \\ = \int_0^t\int_{\partial P} e^{-\lambda s}\left[tn\dot{u} + h_1 n\dot{\phi}_1 + h_2 n\dot{\phi}_2 + qn\theta\right] dads, \quad (53)$$

for $t \in [0,\infty)$. □

3 Spatial behaviour

We consider the following function, which is useful in proving the results about the spatial behaviour of the solution.

$$I(r,t) = -\int_0^t \int_{S_r} e^{-\lambda s} \Big[tn(s)\dot{u}(s) + h_1(s)n\dot{\phi}_1(s) + \\ + h_2(s)n\dot{\phi}_2(s) + q(s)n\theta(s) \Big] dads, \quad r \geq 0, t \in [0,T]. \quad (54)$$

In the theorem below we study the spatial behaviour of the solution.

Theorem 1. *Let $\hat{D}_T$ be the bounded support of the external given data in the problem $\mathcal{P}$ on the time interval $[0,T]$. Then, for each $t \in [0,T]$ we have the following properties:*

i) For $0 \leq r_2 < r_1$,

$$\begin{aligned} I(r_1,t) - I(r_2,t) = &-\int_{B(r_1,r_2)} e^{-\lambda t} \Big[\frac{1}{2}\rho\dot{u}(t)\dot{u}(t) + \frac{1}{2}J_1\dot{\phi}_1^2(t) + \\ &+ \frac{1}{2}J_2\dot{\phi}_2^2(t) + W(E(t)) + \frac{1}{2}a\theta^2(t) \Big] dv - \\ &- \int_0^t \int_{B(r_1,r_2)} e^{-\lambda s} \Big[\frac{\lambda}{2}\rho\dot{u}(s)\dot{u}(s) + \frac{\lambda}{2}J_1\dot{\phi}_1^2(s) + \frac{\lambda}{2}J_2\dot{\phi}_2^2(s) + \\ &+ \lambda W(E(s)) + \frac{\lambda}{2}a\theta^2(s) + k^*\theta_x(s)\theta_x(s) + \xi^*\dot{\phi}_1(s)\dot{\phi}_1(s) \Big] dvds; \end{aligned} \quad (55)$$

ii) $I(r,t)$ is a continuous differentiable function on r, and

$$\begin{aligned} \frac{\partial I}{\partial r}(r,t) = &-\int_{S_r} e^{-\lambda t} \Big[\frac{1}{2}\rho\dot{u}(t)\dot{u}(t) + \frac{1}{2}J_1\dot{\phi}_1^2(t) + \frac{1}{2}J_2\dot{\phi}_2^2(t) + \\ &+ W(E(t)) + \frac{1}{2}a\theta^2(t) \Big] da - \int_0^t \int_{S_r} e^{-\lambda s} \Big[\frac{\lambda}{2}\rho\dot{u}(s)\dot{u}(s) + \\ &+ \frac{\lambda}{2}J_1\dot{\phi}_1^2(s) + \frac{\lambda}{2}J_2\dot{\phi}_2^2(s) + \lambda W(E(s)) + \frac{\lambda}{2}a\theta^2(s) + \\ &+ k^*\theta_x(s)\theta_x(s) + \xi^*\dot{\phi}_1(s)\dot{\phi}_1(s) \Big] dads; \end{aligned} \quad (56)$$

iii) $I(r,t)$ is a nonincreasing function with respect to r;
iv) $I(r,t)$ satisfies the first-order differential inequality

$$\frac{\lambda}{c}|I(r,t)| + \frac{\partial I}{\partial r}(r,t) \leq 0 \quad r \geq 0, \quad (57)$$

where

$$c^2 = \frac{\lambda(1+\varepsilon)\mu_M}{\rho[\lambda - 2\varepsilon(1+\varepsilon)\mu_M]} \quad (58)$$

and ε_0 is the positive root of the algebraic equation

$$\begin{aligned}&\varepsilon^2 \cdot 2\lambda\mu_M\gamma + \varepsilon \dot{} 2\lambda\mu_M\left(\gamma - \rho_0 a_0 \lambda\right) + \\ &+ \rho(\lambda^3 - 2\mu_M\beta^2 + 2\lambda^2\beta^2) = 0,\end{aligned} \tag{59}$$

$$\gamma = \rho_0(2d_1^2 - \rho\lambda) - 2\rho\beta^2. \tag{60}$$

Proof. iv)

$$\begin{aligned}&|I(r,t)| \leq \int_0^t \int_{S_r} e^{-\lambda s}\left\{\frac{\varepsilon_1}{2\rho_0}\left[t^2(s) + \frac{1}{\kappa_1^0}h_1^2(s) + \frac{1}{\kappa_2^0}h_2^2(s)\right] + \right.\\ &+ \frac{1}{2\varepsilon_1}\rho\left[\dot{u}^2(s) + \kappa_1^0\dot{\phi}_1^2(s) + \kappa_2^0\dot{\phi}_2^2(s)\right] + \\ &\left. + \frac{\varepsilon_2}{2a_0}q^2(s) + \frac{1}{2\varepsilon_2}a\theta^2(s)\right\} dads\end{aligned} \tag{61}$$

$$\begin{aligned}&t^2(s) + \frac{1}{\kappa_1^0}h_1^2(s) + \frac{1}{\kappa_2^0}h_2^2(s) = \\ &= (s - \beta\theta)(s - \beta\theta) + \frac{1}{\kappa_1^0}h_1^2(s) + \frac{1}{\kappa_2^0}h_2^2(s) \leq \\ &\leq (1+\varepsilon)s^2 + \left(1 + \frac{1}{\varepsilon}\right)\beta^2\theta^2 + \frac{1}{\kappa_1^0}h_1^2(s) + \frac{1}{\kappa_2^0}h_2^2(s) \leq \\ &\leq (1+\varepsilon)2\mu_M W(E) + \left(1 + \frac{1}{\varepsilon}\right)\beta^2\theta^2\end{aligned} \tag{62}$$

$$\begin{aligned}&q^2(s) = (Q + k^*\theta_x)(Q + k^*\theta_x) \leq \\ &\leq (1+\varepsilon)Q^2 + \left(1 + \frac{1}{\varepsilon}\right)k^{*2}\theta_x\theta_x\end{aligned} \tag{63}$$

$$\begin{aligned}&g_1^2(s) = \left(G_1 + d_1\theta - \xi^*\dot{\phi}_1\right)\left(G_1 + d_1\theta - \xi^*\dot{\phi}_1\right) \leq \\ &\leq (1+\varepsilon)(G_1 + d_1\theta)^2 + \left(1 + \frac{1}{\varepsilon}\right)\xi^{*2}\dot{\phi}_1\dot{\phi}_1 \leq \\ &\leq (1+\varepsilon)^2 G_1^2 + (1+\varepsilon)\left(1 + \frac{1}{\varepsilon}\right)d_1^2\theta^2 + \left(1 + \frac{1}{\varepsilon}\right)\xi^{*2}\dot{\phi}_1\dot{\phi}_1\end{aligned} \tag{64}$$

$$\begin{aligned}&q^2(s) + g_1^2(s) \leq (1+\varepsilon)^2 2\mu_M W(E) + (1+\varepsilon)\left(1 + \frac{1}{\varepsilon}\right)d_1^2\theta^2 + \\ &+ \left(1 + \frac{1}{\varepsilon}\right)k^{*2}\theta_x\theta_x + \left(1 + \frac{1}{\varepsilon}\right)\xi^{*2}\dot{\phi}_1\dot{\phi}_1\end{aligned} \tag{65}$$

$$
\begin{aligned}
|I(r,t)| \leq \int_0^t \int_{S_r} e^{-\lambda s} \Bigg\{ \frac{1}{\lambda\varepsilon_1} \cdot \frac{\lambda\rho}{2} \left[\dot{u}^2(s) + \kappa_1^0 \dot{\phi}_1^2(s) + \kappa_2^0 \dot{\phi}_2^2(s)\right] + \\
+ \frac{\varepsilon_1}{2\rho_0\lambda} \left[\lambda(1+\varepsilon)2\mu_M W(E) + \lambda\left(1+\frac{1}{\varepsilon}\right)\beta^2\theta^2\right] + \\
+ \frac{\varepsilon_2}{2a_0\lambda} \left[\lambda(1+\varepsilon)^2 2\mu_M W(E) + \lambda(1+\varepsilon)\left(1+\frac{1}{\varepsilon}\right) d_1^2\theta^2\right] + \\
+ \frac{\varepsilon_2}{2a_0}\left(1+\frac{1}{\varepsilon}\right) k^{*2}\theta_x\theta_x + \frac{\varepsilon_2}{2a_0}\left(1+\frac{1}{\varepsilon}\right)\xi^{*2}\dot{\phi}_1\dot{\phi}_1 + \\
+ \frac{1}{\lambda\varepsilon_2}\frac{\lambda}{2} a\theta^2(s) \Bigg\} dads
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
\frac{c}{\lambda} = \frac{1}{\lambda\varepsilon_1} = \frac{\varepsilon_1}{2\rho_0\lambda}(1+\varepsilon)2\mu_M + \frac{\varepsilon_2}{2a_0\lambda}(1+\varepsilon)^2 2\mu_M = \\
= \frac{\varepsilon_2}{2a_0}\left(1+\frac{1}{\varepsilon}\right) = \frac{1}{\lambda\varepsilon_2} + \frac{\varepsilon_1}{2\rho_0\lambda}\left(1+\frac{1}{\varepsilon}\right)\beta^2\frac{2}{a_0} + \\
+ \frac{\varepsilon_2}{2a_0\lambda}(1+\varepsilon)\left(1+\frac{1}{\varepsilon}\right) d_1^2 \frac{2}{a_0}
\end{aligned}
\tag{67}
$$

□

4 Conclusions

We studied the spatial behaviour in type III thermoelasticity with two voids structures in the one-dimensional case. The model was introduced in [3]. After deriving a result of Lagrange-Brun type, we studied the spatial behaviour of the solution with the help of a time-weighted function, as in [2].

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