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# EXACT SOLITARY WAVE SOLUTIONS OF TIME FRACTIONAL NONLINEAR EVOLUTION MODELS: A HYBRID ANALYTIC APPROACH

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*Dedicated to Professor Marin Marin on the occasion of his 70th anniversary*

**Abstract**

In this article we propose efficient techniques for solving fractional differential equations such as KdV-Burgers, Kadomtsev-Petviashvili, Zakharov-Kuznetsov with less computational efforts and high accuracy for both numerical and analytical purposes. The general $exp_a$-function method is employed to reckon new exact solitary wave solutions of time fractional nonlinear evolution equations (NLEEs) stemming from mathematical physics. Fractional complex transformation in conjunction with modified Riemann-Liouville operator is used to tackle the fractional sense of the accompanying problems. A comparison with existing conventional exp-function method and improved exp-function method shows that the proposed recipe is more productive in terms of obtaining analytical solutions. The graphical depictions of extracted information show a strong relationship among fractional order outcomes with those of classical ones.

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# 1 Introduction

In recent decades, fractional calculus has become one of the most appealing research areas due to its numerous applications in mathematics and applied sciences. The fractional ordinary differential equations (FODEs) and fractional partial differential equations (FPDEs) can serve as the governing equations for many real-life phenomena. These equations stem from non-integer order derivatives. Fractional calculus generalizes the concept of classical calculus and opens a new avenue to ponder. They are used to model viscoelasticity and viscoplasticity (Physics), polymers and proteins (Chemistry), the transmission of ultrasound waves (Electrical engineering), and human tissue modeling subject to mechanical loads (Biomedical science) and possess key role in many other scientific fields. Moreover, fractional nonlinear evolution equations (FNLEEs) describe the motion of isolated waves, associated with memory effects due to the presence of fractional derivatives therein. Thus, for describing the structure of complex physical phenomena, solutions to these nonlinear PDEs are more important their integer order counterparts.

The nonlinear evolution equation (NLEE) is a partial differential equation (PDE) with time $t$ and space $x$ as independent variables. These equations cover several physical phenomena including solitons, vibrations and propagation of waves of finite speed [9] etc. In recent years, a great deal of research has been conducted to solve NLEEs of fractional and integer order. These equations stem from solid-state physics, plasma physics, biomechanics, nonlinear optics, control theory, mathematical finance and other fields of applied sciences. The literature incorporates methods such as F-expansion [29], sine-cosine [6], extended tanh-function [2], Hirota bilinear transformation [14], homogeneous balance [28, 33], Jacobi elliptic function [4], truncated painleve expansion [31], homotopy perturbation [23], generalized rational function expansion [5] and numerous other for solving NLEEs. The general $exp_a$-function method [3] is a general version of the exp-function method, wherein, an arbitrary base "$a$", ($a \neq 1$), is considered, contrary to the conventional base "$e$". In this paper, we apply the said method to look for exact solutions of time-the fractional NLEEs by using Maple software. The idea of B. Zheng [34] is used for compatibility of the method with the fractional order sense, along with the application of modified Riemann-Liouville operator [18] and fractional complex transformation [21]. These two entities are utilized to transform governing problems into the corresponding ordinary differential equations (ODEs), which are then solved by using general $exp_a$-function method for constructing exact solitary wave solutions. Graphical illustration of the outcomes and a comparative description of the proposed method with exp-function method [8, 13] and improved exp-function method [17] is presented to demonstrate the efficacy of general $exp_a$-function method. Few core investigations can be found in references [1, 7, 10, 11, 15, 20, 22, 24–27].

Subsequent to a careful review of the literature, authors learned that exact solitary wave solutions of time-fractional nonlinear evolution equations by means of general $exp_a$-function method have not been conducted thus far. Also, the current effort is aimed to derive the numerical analytical solutions of such highly complex problems. This article comprises six sections. Section 2 is dedicated to an extensive mathematical formulation. The General $exp_a$-function Method is discussed in Section 3. Section four is devoted to obtained analytical results. Section five offers discussion whereas the last Section concludes a summary of outcomes.

## 2 Formulation of the problem

The Jumarie's modified Riemann-Liouville operator [19] is defined as

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\tau)^{-\alpha-1} (f(\tau) - f(0))\, d\tau, \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} (f(\tau) - f(0))\, d\tau, 0 < \alpha < 1, \\ [f^{(n)}(t)]^{\alpha-n}, n \le \alpha < n+1, n \ge 1. \end{cases} \tag{1}$$

The fractional complex transformation is defined as

$$\xi = L_1 \frac{t^\alpha}{\Gamma(\alpha+1)} + L_2 \frac{x^\alpha}{\Gamma(\alpha+1)} + L_3 \frac{y^\alpha}{\Gamma(\alpha+1)} + \ldots, \tag{2}$$

here the variable $\xi$ is a complex variable, $L_n$, $n \in \mathbb{N}$, are constants, $t$ is time, whereas, $x$ and $y$ are differentiable special coordinates. This transformation changes the given fractional partial differential equation (PDE) into its corresponding ordinary differential equation (ODE) for convenience of the solution process. Fractional complex transformation can be articulated into different forms subject to the simplicity and formulation of the governing equation.

The time fractional KdV-Burgers equation [30] reads

$$u_t + uu_x + u_{xx} + u_{xxx} = 0. \tag{3}$$

Eq. (3) governs weak effects of dispersion $u_{xxx}$, nonlinear advection $uu_x$ and dissipation $u_{xx}$ observed in a good deal of wave phenomena. Moreover, it reflects approximations of long wavelengths, wherein, the nonlinear advection influence is counterbalanced by the dispersion [12]. The fractional order form of Eq. (3) is interpreted as

$${}_J D_t^\alpha u + uu_x + u_{xx} + u_{xxx} = 0, \quad 0 < \alpha \le 1, \tag{4}$$

wherein, ${}_J D_t^\alpha$ typifies modified Riemann-Liouville derivative.

The time fractional KP equation is defined as

$$(u_t + \nu_1 uu_x + \nu_2 u_{xxx})_x + \nu_3 u_{yy} = 0, \tag{5}$$

which governs the waves of shallow water which include weakly nonlinear restoring forces.

The said equations yield soliton solutions and are generalized description of the Korteweg-de Vries (KdV) equation [16, 32]. The fractional order form of Eq. (5) is interpreted as

$$({}_J D_t^\alpha u + \nu_1 uu_x + \nu_2 u_{xxx})_x + \nu_3 u_{yy} = 0, \quad 0 < \alpha \le 1. \tag{6}$$

The time fractional Zakharov-Kuznetsov (ZK) equation [16, 32] is defined as

$$u_t + \nu_1' u u_x + \nu_2' u_{xxx} + \nu_3' u_{xyy} = 0, \tag{7}$$

and arises in the behavior of ion-acoustic waves (showing weakly nonlinear behavior) in plasma holding the ions in their cold state and isothermal electrons in their hot state subject to the presence of uniform magnetic field. The fractional sense of time fractional- ZK equation is as follows

$${}_J D_t^{\alpha} u + \nu_1' u u_x + \nu_2' u_{xxx} + \nu_3' u_{xyy} = 0, \quad 0 < \alpha \leq 1. \tag{8}$$

## 3 General $\exp_a$-function method

Consider general time-fractional PDE of the form

$$V\left(u, \frac{\partial^{\alpha} u}{\partial t^{\alpha}}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots\right) = 0, \quad \alpha \in R, \tag{9}$$

where, $u$ is unknown function and $V$ is a polynomial of $u$, its time fractional derivative and classical partial derivatives. The fractional complex transformation (2) changes the Eq. (9) into the ODE

$$W\left(u_1, u', u'', u''', \ldots\right) = 0. \tag{10}$$

The resulted ODE is integrated once or more times while neglecting integration constant. The general $exp_a$-function method assumes following ansatz

$$u_1(\xi) = \frac{\sum_{i=-c_1}^{c_2} A_i a^{i\xi}}{\sum_{j=-c_3}^{c_4} B_j a^{j\xi}}, \tag{11}$$

wherein, $c_n$ are positive integers, $A_i$ and $B_j$ are constants to be determined. The positive real number "$a$" ($a \neq 1$) is an arbitrary fixed number. Alternative form of (11) is

$$u_1(\xi) = \frac{A_{c_1} a^{c_1 \xi} + \ldots + A_{-c_2} a^{-c_2 \xi}}{B_{c_3} a^{c_3 \xi} + \ldots + B_{-c_4} a^{-c_4 \xi}}, \tag{12}$$

for determining the analytic solution of a nonlinear problems. The application of balancing method leads to $c_3 = c_1$ and $c_2 = c_4$ so that the presumed solution (12) takes the form

$$u_1(\xi) = \frac{A_{c_1} a^{c_1 \xi} + \ldots + A_{-c_2} a^{-c_2 \xi}}{B_{c_1} a^{c_1 \xi} + \ldots + B_{-c_2} a^{-c_2 \xi}}, \tag{13}$$

on multiplying its numerator and denominator by $a^{-c\xi}$ will give

$$u_1(\xi) = \frac{A_c + A_{c+1} a^{\xi} + A_{-d} a^{-(d+c)\xi}}{B_c + B_{c+1} a^{\xi} + B_{-d} a^{-(d+c)\xi}}, \tag{14}$$

Which is equivalent to

$$u_1(\xi) = \frac{A_0 + A_1 a^{\xi} + \ldots + A_k a^{k\xi}}{B_0 + B_1 a^{\xi} + \ldots + B_k a^{k\xi}}, \tag{15}$$

where, $k \geq 0$ and for $k = 2$, above ansatz (15) reduces to

$$u_1(\xi) = \frac{A_0 + A_1 a^{\xi} + A_2 a^{2\xi}}{B_0 + B_1 a^{\xi} + B_2 a^{2\xi}}. \tag{16}$$

Using the simplified ansatz (16) in Eq. (10) forms the exp-function polynomial, whose coefficients are equated to zero, resulting in a system of equation. Solution of this particular system provides various results for which the final corresponding exact solutions are established.

# 4 Solution of the problems

This section consists of four subsections with different cases:

## 4.1 Solution of time fractional KdV-Burgers equation

By using the complex transformation

$$u = u_1(\xi), \text{ where } \xi = -L_1 \frac{t^{\alpha}}{\Gamma(\alpha+1)} + L_2 x, \tag{17}$$

combined with the modified Riemann-Liouville derivative, converts time fractional KdV-Burgers Eq. (4) into the ODE

$$-L_1(u_1)' + L_2 u_1 (u_1)' + L_2^2 (u_1)'' + L_2^3 (u_1)^{(3)} = 0, \tag{18}$$

which is integrated while neglecting constant of integration and then the ansatz (16) is substituted in the integrated equation, which leads to

$$\frac{E_1 a^{\xi} + E_2 a^{2\xi} + E_3 a^{3\xi} + E_4 a^{4\xi} + E_5 a^{5\xi} + E_6 a^{6\xi}}{(B_0 + B_1 a^{\xi} + B_2 a^{2\xi})^3} = 0. \tag{19}$$

Equalizing the coefficients of $a^{n\xi}$ $(n = 1, \ldots, 6.)$ to zero, gives a system of equations, whose solution comes up with the following cases

$$\begin{aligned} \text{Case a:} \quad & L_1 = \frac{1}{2}\frac{L_2 A_1}{B_1},\ L_2 = L_2, \\ & A_0 = \frac{A_1 B_0}{B_1},\ A_1 = A_1,\ A_2 = 0, \\ & B_0 = B_0,\ B_1 = B_1,\ B_2 = 0. \end{aligned} \tag{20}$$

$$\begin{aligned} \text{Case b:} \quad & L_1 = -L_2^2 ln\,|a| \left(L_2 ln\,|a| - 1\right),\ L_2 = L_2, \\ & B_0 = B_0, B_1 = B_1,\ B_2 = 0, \\ & A_0 = 2L_2 ln\,|a|\, B_0 - 2L_2^2 \left(ln\,|a|\right)^2 B_0, \\ & A_1 = 2B_1 L_2 ln\,|a| - 2B_1 L_2^2 \left(ln\,|a|\right)^2,\ A_2 = 0. \end{aligned} \tag{21}$$

$$\begin{aligned}\text{Case c:}\quad & L_1 = -2L_2^2 ln\left|a\right|\left(2L_2 ln\left|a\right| - 1\right),\ L_2 = L_2,\\ & B_0 = B_0,\ B_1 = 0,\ B_2 = 0,\\ & A_0 = 4L_2 ln\left|a\right| B_0 - 8L_2^2\left(ln\left|a\right|\right)^2 B_0,\\ & A_1 = 0, A_2 = 4L_2 ln\left|a\right| B_2 - 8L_2^2\left(ln\left|a\right|\right)^2 B_2.\end{aligned} \tag{22}$$

$$\begin{aligned}\text{Case d:}\quad & L_1 = \frac{1}{2}\frac{L_2A_2}{B_2},\ L_2 = L_2,\ A_0 = \frac{A_2B_0}{B_2}, A_1 = \frac{A_2B_1}{B_2},\\ & A_2 = A_2,\ B_0 = B_0,\ B_1 = B_1,\ B_2 = B_2.\end{aligned} \tag{23}$$

$$\begin{aligned}\text{Case e:}\quad & L_1 = -\frac{3}{2L_2^2 ln\left|a\right|\left(L_2 ln\left|a\right| - 1\right)},\ L_2 = L_2,\\ & B_0 = B_0,\ B_1 = B_1,\ B_2 = B_2,\\ & A_0 = 3L_2 ln\left|a\right| B_0 - 3L_2^2\left(ln\left|a\right|\right)^2 B_0,\\ & A_1 = -3L_2 ln\left|a\right|\left(L_2 ln\left|a\right| - 1\right) B_1,\\ & A_2 = 3L_2 ln\left|a\right| B_2 - 3L_2^2\left(ln\left|a\right|\right)^2 B_2.\end{aligned} \tag{24}$$

$$\begin{aligned}\text{Case f:}\quad & L_1 = -L_2^2 ln\left|a\right| + L_2^2\left(ln\left|a\right|\right)^2,\ L_2 = L_2,\\ & B_0 = B_0,\ B_1 = B_1,\ B_2 = 0,\\ & A_0 = -2L_2 ln\left|a\right| B_0 + 2L_2^2\left(ln\left|a\right|\right)^2 B_0,\\ & A_1 = -2L_2 ln\left|a\right| + 2L_2^2\left(ln\left|a\right|\right)^2 B_1,\ A_2 = 0.\end{aligned} \tag{25}$$

Solutions for above cases (20)$-$(25) will respectively be

$$\begin{aligned}u_{1,1}(x,t) &= \frac{\frac{A_1B_0}{B_1} + A_1 a^{\omega_1 t^\alpha + L_2 x}}{B_0 + B_1 a^{\omega_1 t^\alpha + L_2 x}},\\ \omega_1 &= -\frac{L_2A_1}{2B_1\Gamma\left(\alpha+1\right)},\end{aligned} \tag{26}$$

$$\begin{aligned}u_{1,2}(x,t) &= \frac{2L_2 ln\left|a\right|\left(1 - L_2 ln\left|a\right|\right)\left(B_0 + B_1 a^{\omega_2 t^\alpha + L_2 x}\right)}{\left(B_0 + B_1 a^{\omega_2 t^\alpha + L_2 x}\right)},\\ \omega_2 &= \frac{L_2^2 ln\left|a\right|\left(L_2 ln\left|a\right| - 1\right)}{\Gamma\left(\alpha+1\right)},\end{aligned} \tag{27}$$

$$\begin{aligned}u_{1,3}(x,t) &= \frac{4L_2 ln\left|a\right|\left(1 - 2L_2 ln\left|a\right|\right)\left(B_0 + B_2 a^{\omega_3 t^\alpha + 2L_2 x}\right)}{\left(B_0 + B_2 a^{\omega_3 t^\alpha + 2L_2 x}\right)},\\ \omega_3 &= \frac{4L_2^2 ln\left|a\right|\left(2L_2 ln\left|a\right| - 1\right)}{\Gamma\left(\alpha+1\right)},\end{aligned} \tag{28}$$

$$\begin{aligned}u_{1,4}(x,t) &= \frac{\frac{A_2B_0}{B_2} + \frac{A_2B_1}{B_2} a^{\frac{1}{2}\omega_4 t^\alpha + L_2 x} + A_2 a^{\omega_4 t^\alpha + 2L_2 x}}{B_0 + B_1 a^{\frac{1}{2}\omega_4 t^\alpha + L_2 x} + B_2 a^{\omega_4 t^\alpha + 2L_2 x}},\\ \omega_4 &= \frac{L_2A_2}{B_2\Gamma\left(\alpha+1\right)},\end{aligned} \tag{29}$$

$$\begin{aligned}u_{1,5}(x,t) &= \frac{3L_2 ln\left|a\right|\left(1 - L_2 ln\left|a\right|\right)\left(B_0 + B_1 a^{K_1} + B_2 a^{M_1}\right)}{\left(B_0 + B_1 a^{K_1} + B_2 a^{M_1}\right)},\\ K_1 &= \frac{3}{2}\omega_2 t^\alpha + L_2 x, M_1 = 3\omega_2 t^\alpha + 2L_2 x,\end{aligned} \tag{30}$$

$$u_{1,6}(x,t) = \frac{2L_2 ln\,|a|\,(1 - L_2 ln\,|a|)\left(B_0 + B_1 a^{K_2} + B_2 a^{M_2}\right)}{\left(B_0 + B_1 a^{K_2} + B_2 a^{M_2}\right)}, \quad K_2 = \omega_2 t^\alpha + L_2 x, M_2 = 2\left(\omega_2 t^\alpha + L_2 x\right). \tag{31}$$

## 4.2 Solution of time fractional Kadomtsev-Petviashvili (KP)

By using the complex transformation

$$u = u_1(\xi), \xi = -L_1 \frac{t^\alpha}{\Gamma(\alpha+1)} + L_2 x + L_3 y, \tag{32}$$

and modified Riemann-Liouville derivative, Eq. (6) is transformed into the ODE

$$-L_2 L_1 (u_1)'' + \nu_1 L_2^2 \left(u_1'\right)^2 + u_1 (u_1)'') + \nu_2 L_2^4 (u_1)^{(4)} + \nu_3 L_3^2 (u_1)'' = 0, \tag{33}$$

After integrating twice using similar procedure as discussed before, following cases are figured out.

Case a: $$L_1 = \frac{1}{2}\frac{2\nu_3 L_3^2 B_1 + A_1 \nu_1 L_2^2}{L_2 B_1},\ L_2 = L_2,\ L_3 = L_3, \quad A_0 = \frac{A_1 B_0}{B_1},\ A_1 = A_1,\ A_2 = 0, \quad B_0 = B_0,\ B_1 = B_1,\ B_2 = 0. \tag{34}$$

Case b: $$L_1 = \frac{\nu_3 L_3^2 - \nu_2 L_2^4 (ln\,|a|)^2}{L_2},\ L_2 = L_2,\ L_3 = L_3, \quad A_0 = A_0,\ A_1 = -2\frac{\nu_2 L_2^2 (ln\,|a|)^2 B_1}{\nu_1},\ A_2 = -2\frac{\nu_2 L_2^2 (ln\,|a|)^2 B_2}{\nu_1}, \quad B_0 = -\frac{1}{2}\frac{A_0 \nu_1}{\nu_2 L_2^2 (ln\,|a|)^2},\ B_1 = B_1,\ B_2 = B_2, \tag{35}$$

Case c: $$L_1 = \frac{\nu_3 L_3^2 - \nu_2 L_2^4 (ln\,|a|)^2}{L_2}, L_2 = L_2, L_3 = L_3, \quad A_0 = -\frac{1}{2}\frac{\nu_2 L_2^2 (ln\,|a|)^2 B_1^2}{B_2 \nu_1}, A_1 = 4\frac{\nu_2 L_2^2 (ln\,|a|)^2 B_1}{\nu_1}, \quad A_2 = -2\frac{\nu_2 L_2^2 (ln\,|a|)^2 B_2}{\nu_1}, B_0 = \frac{1}{4}\frac{B_1^2}{B_2}, B_1 = B_1, B_2 = B_2. \tag{36}$$

Case d: $$L_1 = L_1, L_2 = L_2, L_3 = L_3, B_0 = B_0, B_1 = B_1, B_2 = B_2, \quad A_0 = -2\frac{B_0\left(-L_2 L_1 + \nu_3 L_3^2\right)}{\nu_1 L_2^2}, A_1 = -2\frac{\left(-L_2 L_1 + \nu_3 L_3^2\right) B_1}{\nu_1 L_2^2} \quad A_2 = -2\frac{\left(-L_2 L_1 + \nu_3 L_3^2\right) B_2}{\nu_1 L_2^2}. \tag{37}$$

Solutions for above cases (34)−(37) are respectively

$$\begin{aligned} u_{2,1}(x,t) &= \frac{\frac{A_1B_0}{B_1} + A_1 a^{\omega_5 t^\alpha + L_2 x + L_3 y}}{B_0 + B_1 a^{\omega_5 t^\alpha + L_2 x + L_3 y}}, \\ \omega_5 &= -\frac{1}{2}\frac{\left(2\nu_3 L_3^2 B_1 + A_1 \nu_1 L_2^2\right)}{L_2 B_1 \Gamma(\alpha+1)}, \end{aligned} \tag{38}$$

$$\begin{aligned} u_{2,2}(x,t) &= \frac{A_0 - 2\nu_2 L_2^2 (ln\,|a|)^2 \left(\frac{B_1 a^{K_3} - B_2 a^{M_3}}{\nu_1}\right)}{-\frac{1}{2}\frac{A_0\nu_1}{\nu_2 L_2^2 (ln|a|)^2} + B_1 a^{K_3} + B_2 a^{M_3}}, \\ K_3 &= \omega_6 t^\alpha + L_2 x + L_3 y, M_3 = -2\left(\omega_6 t^\alpha - L_2 x - L_3 y\right), \\ \omega_6 &= -\frac{\left(\nu_3 L_3^2 - \nu_2 L_2^4 (ln\,|a|)^2\right)}{L_2 \Gamma(\alpha+1)}, \end{aligned} \tag{39}$$

$$u_{2.3}(x,t) = \frac{\left(\nu_2\left(-\frac{L_2^2 (ln|a|)^2 B_1^2}{2B_2\nu_1}\right) + 2L_2^2 \frac{\left((ln|a|)^2 (2B_1 a^{K_3} - B_2 a^{M_3}\right)}{\nu_1}\right)}{\frac{1}{4}\frac{{B_1}^2}{B_2} + B_1 a^{K_3} + B_2 a^{M_3}}, \tag{40}$$

$$\begin{aligned} u_{2.4}(x,t) &= \left(\frac{2\left(L_2 L_1 - \nu_3 L_3^2\right)}{\nu_1 L_2^2}\right)\left(\frac{B_0 + B_1 a^{K_4} + B_2 a^{M_4}}{B_0 + B_1 a^{K_4} + B_2 a^{M_4}}\right), \\ K_4 &= \omega_7 t^\alpha + L_2 x + L_3 y, M_4 = 2\omega_7 t^\alpha + L_2 x + My, \omega_7 = \frac{-L_1}{\Gamma(\alpha+1)}. \end{aligned} \tag{41}$$

### 4.3 Solution of time fractional Zakharov-Kuznetsov (ZK)

By using the complex transformation (32) and modified Riemann-Liouville derivative, ZK Eq. (8) is converted into the ODE

$$-L_1(u_1)' + \nu_1' L_2 u_1 (u_1)' + \nu_2' L_2^3 (u_1)''' + \nu_3' L_2 L_3^2 u_1''' = 0, \tag{42}$$

which is integrated once and using similar procedure as discussed before, following cases are figured out.

$$\begin{aligned} \text{Case a:} \quad & L_1 = \frac{1}{2}\nu_1' L_2 \frac{A_1}{B_1},\ L_2 = L_2,\ L_3 = L_3, \\ & A_0 = \frac{A_1 B_0}{B_1},\ A_1 = A_1,\ A_2 = 0, \\ & B_0 = B_0,\ B_1 = B_1,\ B_2 = 0. \end{aligned} \tag{43}$$

$$\begin{aligned} \text{Case b:} \quad & L_1 = \nu_2' L_2^3 (ln\,|a|)^2 + \nu_3' L_3^2 L_2 (ln\,|a|)^2,\ L_2 = L_2,\ L_3 = L_3, \\ & A_0 = 2\frac{(ln\,|a|)^2 B_0 \left(\nu_2' L_2^2 + \nu_3' L_3^2\right)}{\nu_1'}, \\ & A_1 = 2\frac{B_1 (ln\,|a|)^2 \left(\nu_2' L_2^2 + \nu_3' L_3^2\right)}{\nu_1'},\ A_2 = 0, \\ & B_0 = B_0,\ B_1 = B_1,\ B_2 = 0. \end{aligned} \tag{44}$$

$$\text{Case c:} \quad L_1 = \frac{1}{2}\nu_1' L_2 \frac{A_2}{B_2},\ L_2 = L_2,\ L_3 = L_3, \\ A_0 = \frac{A_2 B_0}{B_2},\ A_1 = \frac{A_2 B_1}{B_2},\ A_2 = A_2, \\ B_0 = B_0,\ B_1 = B_1,\ B_2 = B_2. \tag{45}$$

Solutions for above cases (43)−(45) are respectively

$$u_{3,1}(x,t) = \frac{\frac{A_1 B_0}{B_1} + A_1 a^{\omega_8 t^\alpha + L_2 x + L_3 y}}{B_0 + B_1 a^{\omega_8 t^\alpha + L_2 x + L_3 y}}, \\ \omega_8 = -\frac{1}{2}\frac{\nu_1' L_2 A_1}{B_1 \Gamma(\alpha+1)}, \tag{46}$$

$$u_{3,2}(x,t) = \frac{2(ln|a|)^2 \left(\nu_2' L_2^2 + \nu_3' L_3^2\right)\left(B_0 + B_1 a^{\omega_9 t^\alpha + L_2 x + L_3 y}\right)}{\nu_1'\left(B_0 + B_1 a^{\omega_9 t^\alpha + L_2 x + L_3 y}\right)}, \\ \omega_9 = -\frac{\left(\nu_2' L_2^3 (ln|a|)^2 + \nu_3' L_3^2 L_2 (ln|a|)^2\right)}{\Gamma(\alpha+1)}, \tag{47}$$

$$u_{3.3}(x,t) = \frac{\left(\frac{A_2 B_0}{B_2} + \frac{A_2 B_1 a^{\frac{1}{2}\omega_{10} t^\alpha + L_2 x + L_3 y}}{B_2} + A_2 a^{\omega_{10} t^\alpha + 2L_2 x + 2L_3 y}\right)}{\left(B_0 + B_1 a^{\frac{1}{2}\omega_{10} t^\alpha + L_2 x + L_3 y} + B_2 a^{\omega_{10} t^\alpha + 2L_2 x + 2L_3 y}\right)}, \\ \omega_{10} = -\frac{\nu_1' L_2 A_2}{B_2 \Gamma(\alpha+1)}. \tag{48}$$

## 4.4 Solutions by means of exp-function method and improved exp-function method

$$u_{4,1}(x,t) = \frac{\frac{a_1 b_0}{b_1} + a_1 e^{\omega_{11} t^\alpha + L_2 x}}{b_0 + b_1 e^{\omega_{11} t^\alpha + L_2 x}}, \\ \omega_{11} = \frac{-L_2 a_1}{2 b_1 \Gamma(\alpha+1)}, \tag{49}$$

$$u_{4,2}(x,t) = -2L_2(L_2 - 1)\frac{b_0 - b_1 e^{\omega_{12} t^\alpha + L_2 x}}{b_0 + b_1 e^{\omega_{12} t^\alpha + L_2 x}}, \\ \omega_{12} = \frac{L_2^2 (L_2 - 1)}{\Gamma(\alpha+1)}, \tag{50}$$

$$u_{4,3}(x,t) = \frac{a_{-1} e^{\omega_{13} t^\alpha - L_2 x} + \frac{a_{-1} b_0}{b_{-1}} + \frac{b_1 a_{-1}}{b_{-1}} e^{-(\omega_{13} t^\alpha - L_2 x)}}{b_{-1} e^{\omega_{13} t^\alpha - L_2 x} + b_0 + b_1 e^{-(\omega_{13} t^\alpha - L_2 x)}}, \\ \omega_{13} = \frac{L_2 a_{-1}}{2 b_{-1} \Gamma(\alpha+1)}, \tag{51}$$

$$u_{4,4}(x,t) = -2L_2 \frac{b_0(L_2-1) + (L_2+1)\left(b_{-1}e^{-(\omega_{14}t^\alpha + L_2x)} + b_1 e^{\omega_{14}t^\alpha + L_2x}\right)}{b_{-1}e^{-(\omega_{14}t^\alpha + L_2x)} + b_0 + b_1 e^{\omega_{14}t^\alpha + L_2x}},$$
$$\omega_{14} = \frac{L_2^2(L_2+1)}{\Gamma(\alpha+1)}, \tag{52}$$

$$u_{5,1}(x,t) = \frac{a_{-1}e^{\omega_{15}t^\alpha - L_2x - L_3y} + \frac{a_{-1}b_0}{b_{-1}}}{b_{-1}e^{\omega_{15}t^\alpha - L_2x - L_3y} + b_0},$$
$$\omega_{15} = \frac{(\nu_1 L_2^2 a_{-1} + 2\nu_3 L_3^2 b_{-1})}{2L_2 b_{-1}\Gamma(\alpha+1)}, \tag{53}$$

$$u_{5,2}(x,t) = \frac{\frac{a_1 b_{-1}}{b_1}e^{\omega_{16}t^\alpha - L_2x - L_3y} + \frac{a_1 b_0}{b_1} + a_1 e^{-(\omega_{16}t^\alpha - L_2x - L_3y)}}{b_{-1}e^{\omega_{16}t^\alpha - L_2x - L_3y} + b_0 + b_1 e^{-(\omega_{16}t^\alpha - L_2x - L_3y)}},$$
$$\omega_{16} = \frac{(a_1\nu_1 L_2^2 + 2\nu_3 L_3^2 b_1)}{2L_2 b_1\Gamma(\alpha+1)}, \tag{54}$$

$$u_{5,3}(x,t) = \frac{-2\nu_2 L_2^2\left(b_{-1}e^{\omega_{17}t^\alpha - L_2x - L_3y} + a_0 + b_1 e^{-(\omega_{17}t^\alpha - L_2x - L_3y)}\right)}{\nu_1\left(b_{-1}e^{\omega_{17}t^\alpha - L_2x - L_3y} + \frac{\nu_1 a_0}{4\nu_2 L_2^2} + b_1 e^{-(\omega_{17}t^\alpha - L_2x - L_3y)}\right)},$$
$$\omega_{17} = \frac{\nu_3 L_3^2 - \nu_2 L_2^2}{L_2\Gamma(\alpha+1)}, \tag{55}$$

$$u_{6,1}(x,t) = \frac{\frac{a_1 b_0}{b_1} + a_1 e^{-(\omega_{18}t^\alpha - L_2x - L_3y)}}{b_0 + b_1 e^{-(\omega_{18}t^\alpha - L_2x - L_3y)}},$$
$$\omega_{18} = \frac{-\nu_1' L_2 a_1}{2b_1\Gamma(\alpha+1)}, \tag{56}$$

$$u_{6,2}(x,t) = \frac{a_{-1}e^{\omega_{19}t^\alpha - L_2x - L_3y} + \frac{a_{-1}b_0}{b_{-1}} + \frac{a_{-1}b_1}{b_{-1}}e^{-(\omega_{19}t^\alpha - L_2x - L_3y)}}{b_{-1}e^{\omega_{19}t^\alpha - L_2x - L_3y} + b_0 + b_1 e^{-(\omega_{19}t^\alpha - L_2x - L_3y)}},$$
$$\omega_{19} = \frac{\nu_1' L_2 a_{-1}}{2b_{-1}\Gamma(\alpha+1)}. \tag{57}$$

Here, Eqs. (49)−(52) , Eqs. (53)−(55) and Eqs. (56)−(57) represent the solutions of KdV-Burgers equation, KP equation and ZK equation respectively.

The solutions computed by improved exp-function method are as follows

$$u_{7,1}(x,t) = -4L_2(2L_2-1)\frac{b_0 - e^{\omega_{20}t^\alpha + 2L_2x}}{b_0 + e^{\omega_{20}t^\alpha + 2L_2x}},$$
$$\omega_{20} = \frac{4L_2^2(2L_2-1)}{\Gamma(\alpha+1)}, \tag{58}$$

$$u_{7,2}(x,t) = \frac{a_2 b_0 + a_2 b_1 e^{\omega_{21}t^\alpha + L_2x} + a_2 e^{\omega_{21}t^\alpha + 2L_2x}}{b_0 + b_1 e^{\omega_{21}t^\alpha + L_2x} + e^{\omega_{21}t^\alpha + 2L_2x}},$$
$$\omega_{21} = \frac{-L_2 a_2}{2\Gamma(\alpha+1)}, \tag{59}$$

$$u_{7,3}(x,t) = -3L_2(L_2-1)\frac{b_0 - b_1 e^{\frac{3}{2}\omega_{12}t^\alpha + L_2x} + e^{3\omega_{12}t^\alpha + 2L_2x}}{b_0 + b_1 e^{\frac{3}{2}\omega_{12}t^\alpha + L_2x} + e^{3\omega_{12}t^\alpha + 2L_2x}}, \tag{60}$$

$$u_{7,4}(x,t) = -2L_2(L_2-1)\frac{b_1e^{\omega_{12}t^\alpha+L_2x} - e^{2(\omega_{12}t^\alpha+L_2x)}}{b_1e^{\omega_{12}t^\alpha+L_2x} + e^{2(\omega_{12}t^\alpha+L_2x)}}, \tag{61}$$

$$\begin{aligned} u_{8,1}(x,t) &= \frac{\frac{a_1b_0}{b_1} + a_1e^{\omega_{22}t^\alpha+L_2x+L_3y}}{b_0+b_1e^{\omega_{22}t^\alpha+L_2x+L_3y}}, \\ \omega_{22} &= \frac{-(a_1\nu_1L_2^2+2\nu_3L_3^2b_1)}{2L_2b_1\Gamma(\alpha+1)}, \end{aligned} \tag{62}$$

$$\begin{aligned} u_{8,2}(x,t) &= \frac{\frac{a_2b_0}{b_2} + \frac{a_2b_1}{b_2}e^{-\frac{1}{2}(\omega_{23}t^\alpha+L_2x+L_3y)} + a_2e^{-(\omega_{23}t^\alpha-2L_2x-2L_3y)}}{b_0+b_1e^{-\frac{1}{2}(\omega_{23}t^\alpha-L_2x-L_3y)}+b_2e^{-(\omega_{23}t^\alpha-2L_2x-2L_3y)}}, \\ \omega_{23} &= \frac{a_2\nu_1L_2^2+2\nu_3L_3^2b_2}{L_2B_2\Gamma(\alpha+1)}, \end{aligned} \tag{63}$$

$$\begin{aligned} u_{8,3}(x,t) &= \frac{\frac{-\nu_1a_1^2}{32\nu_2L_2^2b_2} + a_1e^{-(\omega_{24}t^\alpha-L_2x-L_3y)} - \frac{2\nu_2L_2^2b_2}{\nu_1}e^{-2(\omega_{24}t^\alpha-L_2x-L_3y)}}{\frac{\nu_1^2a_1^2}{64\nu_2^2L_2^4b_2} + \frac{\nu_1a_1}{4\nu_2L_2^2}e^{-(\omega_{24}t^\alpha-L_2x-L_3y)}+b_2e^{-2(\omega_{24}t^\alpha-L_2x-L_3y)}}, \\ \omega_{24} &= \frac{\nu_3L_3^2-\nu_2L_2^4}{L_2\Gamma(\alpha+1)}, \end{aligned} \tag{64}$$

$$u_{9,1}(x,t) = \frac{\frac{a_1b_0}{b_1} + a_1e^{\omega_{18}+L_2x+L_3y}}{b_0+b_1e^{\omega_{18}+L_2x+L_3y}}, \tag{65}$$

$$\begin{aligned} u_{9,2}(x,t) &= \frac{\frac{a_2b_0}{b_2} + \frac{a_2b_1}{b_2}e^{-\frac{1}{2}(\omega_{25}-L_2x-L_3y)} + a_2e^{-(\omega_{25}-L_2x-L_3y)}}{b_0+b_1e^{-\frac{1}{2}(\omega_{25}-L_2x-L_3y)}+b_2e^{-(\omega_{25}-L_2x-L_3y)}}, \\ \omega_{25} &= \frac{\nu_1'L_2a_2t^\alpha}{b_2\Gamma(\alpha+1)}. \end{aligned} \tag{66}$$

In which, Eqs. (58)-(61), Eqs. (62)-(64) and Eqs. (65)-(66) represent the solutions of KdV-Burgers equation, KP equation and ZK equation respectively.

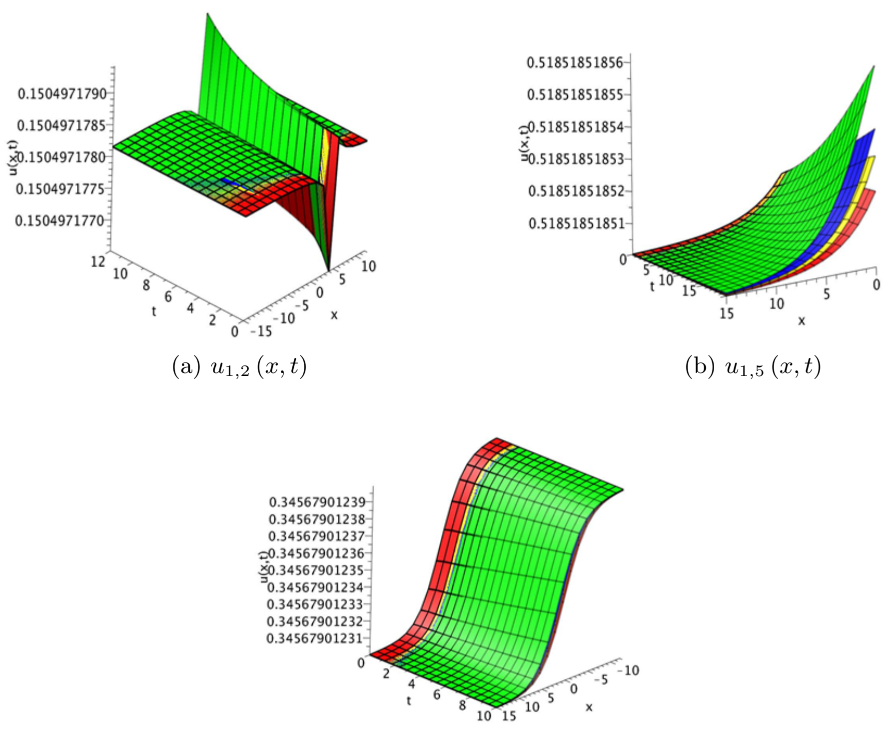
# 5 Discussion

Three dimensional multiple plot of solutions of the investigated physical models are constructed by considering the orders $\alpha = 0.5,\ 0.75,\ 0.85$ and 1, depicted in red, yellow, blue and green colors respectively. The attained solutions are observed to be of 2-soliton type, namely singular and dark-bright soliton (shown in Figure 1(a)). Another 2-soliton type soliton is also observed in Figure 3 known as periodic and singular soliton. Further, the graphical results comprise 1-soliton solutions (shown in Figures: 1(b), 2(a)), kink solution (shown in Figure 1(c)) and kink-like solution (shown in Figure 2(b)). Apart from Figure 3, all the other Figures show that the solution curves of the said orders are very close to each other, reflect subsequent memory patterns and are thus more important for dealing with the physical aspects of governing equations.

Table 1 shows the comparative description of general $exp_a$-function method, exp-function method and improved exp-function method regarding the number of verified solutions, established for each investigated governing time fractional

NLEEs by using symbolic computation. The general $exp_a$-function method is observed to be more dominant tool for its provision of more solutions as compared to the aforesaid methods.

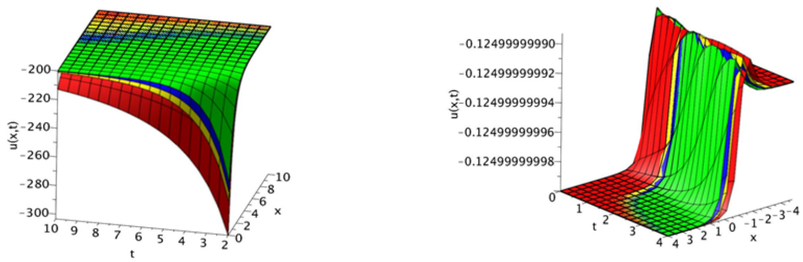
In all the cases, applications of the problems are highlighted so that the linked physical facets can be better comprehended. In some cases, newly computed solutions are also validated by comparative analysis with already existing solutions (for limiting cases of the problems). This study reveals a strong connection among fractional and classical response (solution) curves of the governing equations in each case and can be very supportive to unveil several unreported physical aspects.




(a) $u_{1,2}(x,t)$ (b) $u_{1,5}(x,t)$

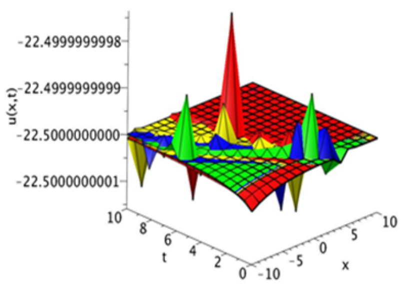
(c) $u_{1,6}(x,t)$

Figure 1: Pictorial view of solitons of fractional order KdV- Burgers Eq. (4).




(a) $u_{2,3}(x,t)$ (b) $u_{2,4}(x,t)$

Figure 2: Pictorial view of solutions of fractional order KP Eq. (6)




(a) $u_{3,3}(x,t)$

Figure 3: Pictorial view of solutions of fractional order ZK Eq. (8).

| | exp-function method | Improved exp - function method | General $exp_a$ - function method |
|---|---|---|---|
| KdV Equation | 04 | 04 | 06 |
| KP Equation | 03 | 03 | 04 |
| ZK Equation | 02 | 02 | 03 |

## 6 Conclusion

Exact, semi analytical and even numerical solutions of the problem of fractional sense are not very common in the existing literature. In order to fill the said gap, the general $exp_a$-function method is observed to provide more play types of new and verified closed from solutions as compared to the conventional exp-function method and improved exp-function method. Moreover, graphical representation of solution for different fractional orders along with the classical ones show that the recommended analytical tool is valid, reliable and effective. It is to be highlighted that fractional order NLEEs are rarely tackled by using the general $exp_a$-function method in the literature and therefore it can contribute an essential role in future study due its promising applicability.

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