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PARAMETRIZED TRIGONOMETRIC DERIVED L_p DEGREE OF APPROXIMATION BY VARIOUS SMOOTH INTEGRAL OPERATORS

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Dedicated to Professor Marin Marin on the occasion of his 70th anniversary

Abstract

In this work we continue with the study of smooth Gauss-Weierstrass, Poisson-Cauchy and trigonometric singular integral operators that started in [Anastassiou, G.A., *Intelligent Mathematics: Computational Analysis*, Springer, Heidelberg, New York, Chapter 12, 2011], see there chapters 10-14. This time the foundation of our research is a trigonometric Taylor's formula. We prove the parametrized univariate L_p convergence of our operators to the unit operator with rates via Jackson type parametrized inequalities involving the first L_p modulus of continuity. Of interest here is a residual appearing term. Note that our operators are not in general positive.

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Key words: Gauss-Weierstrass, L_p modulus of continuity, Poisson-Cauchy and Trigonometric smooth singular integrals parametrized approximation, trigonometric Taylor formula.

1 Introduction

We are motivated by [2], [3] chapters 10-14, and [4], [1]. We use a trigonometric new Taylor formula from [4], see also [1]. Here we consider some very general operators, the smooth Gauss-Weierstrass, Poisson-Cauchy and trigonometric singular integral operators over the real line and we study further their L_p, $p \geq 1$, parametrized convergence properties quantitatively. We establish related parametrized inequalities involving the first L_p, $p \geq 1$, modulus of continuity with respect to L_p, $p \geq 1$, norm. We provide detailed proofs.

Other important motivating articles on the topic are [6]-[10].

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For the history of the topic we mention about the monograph [5] of 2012, which was the first complete source to deal exclusively with the classic theory of the approximation of singular integrals to the identity-unit operator. The authors there studied quantitatively the basic approximation properties of the general Picard, Gauss-Weierstrass and Poisson-Cauchy singular integral operators over the real line, which are not positive linear operators. In particular they studied the rate of convergence of these operators to the unit operator, as well as the related simultaneous approximation. This is given via inequalities and with the use of higher order modulus of smoothness of the high order derivative of the involved function. Some of these inequalities are proven to be sharp. Also, they studied the global smoothness preservation property of these operators. Furthermore they gave asymptotic expansions of Voronovskaya type for the error of approximation. They continued with the study of related properties of the general fractional Gauss-Weierstrass and Poisson-Cauchy singular integral operators. These properties were studied with respect to L_p norm, $1 \leq p \leq \infty$. The case of Lipschitz type functions approximation was studied separately and in detail. Furthermode they presented the corresponding general approximation theory of general singular integral operators with lots of applications to, the under focused till then, trigonometric singular integral.

2 Part I: On Gauss-Weierstrass smooth singular integrals

By [1], [4], for $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta\left(\alpha^2-\beta^2\right) \neq 0$, and $f \in C^4(\mathbb{R})$, $a, x \in \mathbb{R}$, we have the following general trigonometric Taylor formula:

$$
\begin{aligned}
f(x)-f(a) &= f'(a)\left(\frac{\beta^3 \sin(\alpha(x-a))-\alpha^3 \sin(\beta(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right) \\
&+ f''(a)\left(\frac{\cos(\alpha(x-a))-\cos(\beta(x-a))}{\beta^2-\alpha^2}\right) \\
&+ f'''(a)\left(\frac{\beta \sin(\alpha(x-a))-\alpha \sin(\beta(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right) \\
&+ \frac{2\left(f''''(a)+\left(\alpha^2+\beta^2\right) f''(a)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\left(\beta^2 \sin^2\left(\frac{\alpha(x-a)}{2}\right)-\alpha^2 \sin^2\left(\frac{\beta(x-a)}{2}\right)\right) \\
&+ \frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)} \int_a^x \left[\left(f''''(t)+\left(\alpha^2+\beta^2\right) f''(t)+\alpha^2\beta^2 f(t)\right)\right. \\
&\qquad \left. -\left(f''''(a)+\left(\alpha^2+\beta^2\right) f''(a)+\alpha^2\beta^2 f(a)\right)\right] \\
&\qquad [\beta \sin(\alpha(x-t))-\alpha \sin(\beta(x-t))]\, dt.
\end{aligned} \tag{1}
$$

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, we set

$$
\alpha_j := \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j=1,\ldots,r. \\ 1-\sum\limits_{j=1}^{r}(-1)^{r-j}\binom{r}{j} j^{-n}, & n=0, \end{cases} \tag{2}
$$

that is

$$\sum_{j=0}^{r} \alpha_j = 1. \tag{3}$$

Here we consider all $f, f', f'', f''', f^{(4)} \in L_p(\mathbb{R}) \cap C(\mathbb{R})$, $1 \leq p < \infty$.

For $x \in \mathbb{R}$, $\xi > 0$ we consider the Lebesgue integrals, so called smooth Gauss-Weierstrass operators

$$W_{r,\xi}(f, x) = \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x + jt) \right) e^{-\frac{t^2}{\xi}} dt, \tag{4}$$

see [5], $W_{r,\xi}$ are not in general positive operators, see [5].

We notice by

$$\frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{\xi}} dt = 1, \tag{5}$$

that

$$W_{r,\xi}(c, x) = c, \text{ where } c \text{ is a constant} \tag{6}$$

and

$$W_{r,\xi}(f, x) - f(x) = \frac{1}{\sqrt{\pi\xi}} \left(\sum_{j=0}^{r} \alpha_j \int_{-\infty}^{\infty} (f(x + jt) - f(x)) e^{-\frac{t^2}{\xi}} dt \right). \tag{7}$$

Denote by

$$\omega_1(f, h)_p := \sup_{\substack{t \in \mathbb{R} \\ |t| \leq h}} \|f(x + t) - f(x)\|_{p,x}, \tag{8}$$

the first L_p modulus of smoothness of f, $1 \leq p < \infty$.

By (1) we get that

$$\begin{aligned} f(x + jt) - f(x) = {} & f'(x) \left(\frac{\beta^3 \sin(\alpha jt) - \alpha^3 \sin(\beta jt)}{\alpha\beta(\beta^2 - \alpha^2)} \right) \\ & + f''(x) \left(\frac{\cos(\alpha jt) - \cos(\beta jt)}{\beta^2 - \alpha^2} \right) \\ & + f'''(x) \left(\frac{\beta \sin(\alpha jt) - \alpha \sin(\beta jt)}{\alpha\beta(\beta^2 - \alpha^2)} \right) \\ & + \frac{2\left(f^{(4)}(x) + \left(\alpha^2 + \beta^2\right) f''(x)\right)}{(\alpha\beta)^2 (\beta^2 - \alpha^2)} \left(\beta^2 \sin^2\left(\frac{\alpha jt}{2}\right) - \alpha^2 \sin^2\left(\frac{\beta jt}{2}\right) \right) \\ & + \frac{1}{\alpha\beta(\beta^2 - \alpha^2)} \int_x^{x+jt} \left[\left(f^{(4)}(s) + \left(\alpha^2 + \beta^2\right) f''(s) + \alpha^2\beta^2 f(s) \right) \right. \\ & \left. - \left(f^{(4)}(x) + \left(\alpha^2 + \beta^2\right) f''(x) + \alpha^2\beta^2 f(x) \right) \right] \\ & [\beta \sin(\alpha(x + jt - s)) - \alpha \sin(\beta(x + jt - s))] \, ds, \end{aligned} \tag{9}$$

or better,

$$\begin{aligned} f(x + jt) - f(x) = {} & f'(x) \left(\frac{\beta^3 \sin(\alpha jt) - \alpha^3 \sin(\beta jt)}{\alpha\beta(\beta^2 - \alpha^2)} \right) \\ & + f''(x) \left(\frac{\cos(\alpha jt) - \cos(\beta jt)}{\beta^2 - \alpha^2} \right) \end{aligned}$$

$$+ f'''(x)\left(\frac{\beta\sin(\alpha jt)-\alpha\sin(\beta jt)}{\alpha\beta(\beta^2-\alpha^2)}\right) \tag{10}$$
$$+ \frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2(\beta^2-\alpha^2)}\left(\beta^2\sin^2\left(\frac{\alpha jt}{2}\right)-\alpha^2\sin^2\left(\frac{\beta jt}{2}\right)\right)$$
$$+ \frac{1}{\alpha\beta(\beta^2-\alpha^2)}\int_0^{jt}\left[\left(f^{(4)}(x+z)+\left(\alpha^2+\beta^2\right)f''(x+z)+\alpha^2\beta^2 f(x+z)\right)\right.$$
$$\left. -\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2 f(x)\right)\right]$$
$$[\beta\sin(\alpha(jt-z))-\alpha\sin(\beta(jt-z))]\,dz.$$

Furthermore it holds

$$\sum_{j=0}^{r}\alpha_j[f(x+jt)-f(x)] = \left(\frac{f'(x)}{\alpha\beta(\beta^2-\alpha^2)}\right)\left[\beta^3\sum_{j=0}^{r}\alpha_j\sin(\alpha jt)-\alpha^3\sum_{j=0}^{r}\alpha_j\sin(\beta jt)\right]$$
$$+\left(\frac{f''(x)}{\beta^2-\alpha^2}\right)\left[\sum_{j=0}^{r}\alpha_j\cos(\alpha jt)-\sum_{j=0}^{r}\alpha_j\cos(\beta jt)\right] \tag{11}$$
$$+\left(\frac{f^{(3)}(x)}{\alpha\beta(\beta^2-\alpha^2)}\right)\left[\beta\sum_{j=0}^{r}\alpha_j\sin(\alpha jt)-\alpha\sum_{j=0}^{r}\alpha_j\sin(\beta jt)\right]$$
$$+\left(\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2(\beta^2-\alpha^2)}\right)\left(\beta^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\alpha jt}{2}\right)-\alpha^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\beta jt}{2}\right)\right)$$
$$+\frac{1}{\alpha\beta(\beta^2-\alpha^2)}\sum_{j=0}^{r}\alpha_j\int_0^{jt}\left[\left(f^{(4)}(x+z)+\left(\alpha^2+\beta^2\right)f''(x+z)+\alpha^2\beta^2 f(x+z)\right)\right.$$
$$\left.-\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2 f(x)\right)\right]$$
$$[\beta\sin(\alpha(jt-z))-\alpha\sin(\beta(jt-z))]\,dz,$$

or better

$$\sum_{j=0}^{r}\alpha_j[f(x+jt)-f(x)] =$$
$$\left(\frac{f'(x)}{\alpha\beta(\beta^2-\alpha^2)}\right)\left[\beta^3\sum_{j=0}^{r}\alpha_j\sin(\alpha jt)-\alpha^3\sum_{j=0}^{r}\alpha_j\sin(\beta jt)\right]$$
$$+\left(\frac{f''(x)}{\beta^2-\alpha^2}\right)\left[\sum_{j=0}^{r}\alpha_j\cos(\alpha jt)-\sum_{j=0}^{r}\alpha_j\cos(\beta jt)\right] \tag{12}$$
$$+\left(\frac{f^{(3)}(x)}{\alpha\beta(\beta^2-\alpha^2)}\right)\left[\beta\sum_{j=0}^{r}\alpha_j\sin(\alpha jt)-\alpha\sum_{j=0}^{r}\alpha_j\sin(\beta jt)\right]$$
$$+\left(\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2(\beta^2-\alpha^2)}\right)\left(\beta^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\alpha jt}{2}\right)-\alpha^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\beta jt}{2}\right)\right)$$

$$+\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\sum_{j=0}^{r}\alpha_j j\int_0^t\Big[\Big(f^{(4)}(x+jw)+\left(\alpha^2+\beta^2\right)f''(x+jw)+\alpha^2\beta^2 f(x+jw)\Big)$$
$$-\Big(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2 f(x)\Big)\Big]$$
$$\left[\beta\sin(\alpha j(t-w))-\alpha\sin(\beta j(t-w))\right]dw.$$

We call

$$R:=R(t):=\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\sum_{j=0}^{r}\alpha_j j$$
$$\int_0^t\Big[\Big(f^{(4)}(x+jw)+\left(\alpha^2+\beta^2\right)f''(x+jw)+\alpha^2\beta^2 f(x+jw)\Big)\tag{13}$$
$$-\Big(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2 f(x)\Big)\Big]$$
$$\left[\beta\sin(\alpha j(t-w))-\alpha\sin(\beta j(t-w))\right]dw,\quad\forall\ t\in\mathbb{R}.$$

We set

$$E_1(x):=W_{r,\xi}(f,x)-f(x)$$
$$-\left(\frac{f'(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta^3\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\sin(\alpha jt)e^{-\frac{t^2}{\xi}}dt\right)\right.$$
$$\left.-\alpha^3\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\sin(\beta jt)e^{-\frac{t^2}{\xi}}dt\right)\right]$$
$$-\left(\frac{f''(x)}{\beta^2-\alpha^2}\right)\left[\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\cos(\alpha jt)e^{-\frac{t^2}{\xi}}dt\right)-\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\cos(\beta jt)e^{-\frac{t^2}{\xi}}dt\right)\right]$$
$$-\left(\frac{f^{(3)}(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\sin(\alpha jt)e^{-\frac{t^2}{\xi}}dt\right)\right.$$
$$\left.-\alpha\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\sin(\beta jt)e^{-\frac{t^2}{\xi}}dt\right)\right]$$
$$-\left(\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\right)\left[\beta^2\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{\alpha jt}{2}\right)e^{-\frac{t^2}{\xi}}dt\right)\right.\tag{14}$$
$$\left.-\alpha^2\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{\beta jt}{2}\right)e^{-\frac{t^2}{\xi}}dt\right)\right]$$
$$=\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}R(t)e^{-\frac{t^2}{\xi}}dt.$$

Next we simplify $E_1(x)$:
We observe that

$$\int_{-\infty}^{\infty}\sin(\alpha jt)e^{-\frac{t^2}{\xi}}dt=\int_{-\infty}^{0}\sin(\alpha jt)e^{-\frac{t^2}{\xi}}dt+\int_{0}^{\infty}\sin(\alpha jt)e^{-\frac{t^2}{\xi}}dt.\tag{15}$$

Notice $-\infty \leq t \leq 0 \Rightarrow \infty \geq -t \geq 0$. So that

$$\int_{-\infty}^{0} \sin(\alpha jt)\, e^{-\frac{\left|t^2\right|}{\xi}} dt = -\int_{-\infty}^{0} \sin(\alpha j(-(-t)))\, e^{-\frac{t^2}{\xi}} d(-t)$$

$$= -\int_{-\infty}^{0} (-\sin(\alpha j(-t)))\, e^{-\frac{t^2}{\xi}} d(-t) = \int_{-\infty}^{0} \sin(\alpha j(-t))\, e^{-\frac{t^2}{\xi}} d(-t) = \tag{16}$$

$$\int_{\infty}^{0} \sin(\alpha jt)\, e^{-\frac{t^2}{\xi}} dt = -\int_{0}^{\infty} \sin(\alpha jt)\, e^{-\frac{t^2}{\xi}} dt.$$

So that

$$\int_{-\infty}^{\infty} \sin(\alpha jt)\, e^{-\frac{t^2}{\xi}} dt = 0,$$

and all sine integrals in (14) are zeros.

Furthermore we have that

$$\int_{-\infty}^{\infty} \sin^2\left(\frac{\alpha jt}{2}\right) e^{-\frac{t^2}{\xi}} dt = 2\int_{0}^{\infty} \sin^2\left(\frac{\alpha jt}{2}\right) e^{-\frac{t^2}{\xi}} dt$$

$$= 2\sqrt{\xi}\int_{0}^{\infty} \sin^2\left(\left(\frac{\alpha j\sqrt{\xi}}{2}\right)\frac{t}{\sqrt{\xi}}\right) e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2} d\frac{t}{\sqrt{\xi}}$$

($\frac{t}{\sqrt{\xi}} =: x$, and $\frac{\alpha j\sqrt{\xi}}{2} = \beta_1$)

$$= 2\sqrt{\xi}\int_{0}^{\infty} \sin^2(\beta_1 x)\, e^{-x^2} dx = 2\sqrt{\xi}\frac{1}{4}\sqrt{\pi}e^{-\beta_1^2}\left(e^{\beta_1^2} - 1\right) \tag{17}$$

$$= \frac{\sqrt{\pi\xi}}{2}\left(1 - e^{-\beta_1^2}\right) = \frac{\sqrt{\pi\xi}}{2}\left(1 - e^{-\frac{\alpha^2 j^2\xi}{4}}\right).$$

That is

$$\int_{-\infty}^{\infty} \sin^2\left(\frac{\alpha jt}{2}\right) e^{-\frac{t^2}{\xi}} dt = \frac{\sqrt{\pi\xi}}{2}\left(1 - e^{-\frac{\alpha^2 j^2\xi}{4}}\right), \tag{18}$$

and similarly,

$$\int_{-\infty}^{\infty} \sin^2\left(\frac{\beta jt}{2}\right) e^{-\frac{t^2}{\xi}} dt = \frac{\sqrt{\pi\xi}}{2}\left(1 - e^{-\frac{\beta^2 j^2\xi}{4}}\right). \tag{19}$$

Next, we treat

$$\int_{-\infty}^{\infty} \cos(\alpha jt)\, e^{-\frac{t^2}{\xi}} dt = 2\int_{0}^{\infty} \cos(\alpha jt)\, e^{-\frac{t^2}{\xi}} dt =$$

$$= 2\sqrt{\xi}\int_{0}^{\infty} \cos\left(\left(\frac{\alpha j\sqrt{\xi}}{2}\right)\frac{t}{\sqrt{\xi}}\right) e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2} d\frac{t}{\sqrt{\xi}} \tag{20}$$

($\frac{t}{\sqrt{\xi}} =: x$, and $\frac{\alpha j\sqrt{\xi}}{2} =: \beta_1$)

$$= 2\sqrt{\xi}\int_{0}^{\infty} \cos(\beta_1 x)\, e^{-x^2} dx = 2\sqrt{\xi}\frac{1}{2}\sqrt{\pi}e^{-\frac{\beta_1^2}{4}} = \sqrt{\pi\xi}e^{-\frac{\alpha^2 j^2\xi}{16}}.$$

That is

$$\int_{-\infty}^{\infty} \cos(\alpha jt)\, e^{-\frac{t^2}{\xi}} dt = \sqrt{\pi\xi}e^{-\frac{\alpha^2 j^2\xi}{16}}, \tag{21}$$

and similarly,

$$\int_{-\infty}^{\infty} \cos(\beta j t)\, e^{-\frac{t^2}{\xi}} dt = \sqrt{\pi\xi} e^{-\frac{\beta^2 j^2 \xi}{16}}. \tag{22}$$

Hence, we have the simplified expression,

$$E_1(x) = W_{r,\xi}(f,x) - f(x) - \left(\frac{f''(x)}{\beta^2-\alpha^2}\right)\left[\sum_{j=0}^{r} \alpha_j \left(e^{-\frac{\alpha^2 j^2 \xi}{16}} - e^{-\frac{\beta^2 j^2 \xi}{16}}\right)\right] -$$

$$\left(\frac{f^{(4)}(x) + (\alpha^2+\beta^2) f''(x)}{(\alpha\beta)^2(\beta^2-\alpha^2)}\right)\left[\beta^2 \sum_{j=0}^{r} \alpha_j \left(1 - e^{-\frac{\alpha^2 j^2 \xi}{4}}\right) - \alpha^2 \sum_{j=0}^{r} \alpha_j \left(1 - e^{-\frac{\beta^2 j^2 \xi}{4}}\right)\right]$$

$$= \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} R(t)\, e^{-\frac{t^2}{\xi}} dt. \tag{23}$$

It follows a parametrized L_p $(p>1)$ approximation result for $W_{r,\xi}$, $\xi > 0$.

Theorem 1. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2-\beta^2) \neq 0$, and let all $f, f', f'', f^{(3)}, f^{(4)} \in L_p(\mathbb{R}) \cap C(\mathbb{R})$, $1 < p < \infty$. Here $q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; $\xi > 0$, $x \in \mathbb{R}$. Then*

$$\|E_1(x)\|_p = \Bigg\| W_{r,\xi}(f,x) - f(x) - \frac{f''(x)}{(\beta^2-\alpha^2)}\left[\sum_{j=0}^{r} \alpha_j \left(e^{-\frac{\alpha^2 j^2 \xi}{16}} - e^{-\frac{\beta^2 j^2 \xi}{16}}\right)\right] -$$

$$\left(\frac{f^{(4)}(x) + (\alpha^2+\beta^2) f''(x)}{(\alpha\beta)^2(\beta^2-\alpha^2)}\right)\left[\beta^2 \sum_{j=0}^{r} \alpha_j \left(1 - e^{-\frac{\alpha^2 j^2 \xi}{4}}\right) - \alpha^2 \sum_{j=0}^{r} \alpha_j \left(1 - e^{-\frac{\beta^2 j^2 \xi}{4}}\right)\right]\Bigg\|_p$$

$$\leq \frac{\left(\sum\limits_{j=1}^{r} |\alpha_j| j^2\right)^{\frac{1}{q}} 4 \cdot 2^{\frac{1}{\beta}}}{(\sqrt{\pi})^{\frac{1}{p}} |\beta^2-\alpha^2| (q+1)^{\frac{1}{q}} (p+1)^{\frac{1}{p}}} \left[\sum_{j=1}^{r} |\alpha_j| j^{p+2} \left(\frac{1}{(3p+1)} + \Gamma(3p+1)\right)\right]^{\frac{1}{p}} \tag{24}$$

$$\omega_1\left(f^{(4)} + (\alpha^2+\beta^2) f'' + \alpha^2\beta^2 f, \sqrt{\xi}\right)_p \cdot \xi =: \Phi_1(\xi) \to 0,$$

as $\xi \to 0$.

Above Γ stands for the gamma function.

Proof. Call the function

$$F := f^{(4)} + (\alpha^2+\beta^2) f'' + \alpha^2\beta^2 f. \tag{25}$$

Then, we get

$$R = R(t) = \frac{1}{\alpha\beta(\beta^2-\alpha^2)} \sum_{j=0}^{r} \alpha_j j \int_0^t [F(x+jw) - F(x)] [\beta \sin(\alpha j (t-w)) \tag{26}$$

$$-\alpha \sin(\beta j (t-w))] dw, \forall\, t \in \mathbb{R}.$$

We isolate and study

$$I := \int_0^t [F(x+jw) - F(x)] [\beta \sin(\alpha j (t-w)) - \alpha \sin(\beta j (t-w))] dw, \quad \forall\, t \in \mathbb{R}. \tag{27}$$

For $t<0$, we have that

$$
\begin{aligned}
|I| &= \left| \int_t^0 \left[F(x+jw)-F(x)\right]\left[\beta\sin(\alpha j(t-w))-\alpha\sin(\beta j(t-w))\right]dw \right| \\
&\le \int_t^0 |F(x+jw)-F(x)|\,|\beta\sin(\alpha j(t-w))-\alpha\sin(\beta j(t-w))|\,dw \qquad (28)\\
&\le 2|\alpha||\beta| j\int_t^0 |F(x+jw)-F(x)|(w-t)\,dw \\
&= -2|\alpha||\beta| j\int_t^0 |F(x-j(-w))-F(x)|(-t-(-w))\,d(-w) \\
&\quad (t\le w\le 0 \Rightarrow -t\ge -w=:\theta\ge 0) \\
&= -2|\alpha||\beta| j\int_{-t}^0 |F(x-j\theta)-F(x)|(-t-\theta)\,d\theta \\
&= 2|\alpha||\beta| j\int_0^{-t} |F(x-j\theta)-F(x)|(-t-\theta)\,d\theta \\
&= 2|\alpha\beta| j\int_0^{|t|} |F(x+sign(t)j\theta)-F(x)|(|t|-\theta)\,d\theta.
\end{aligned}
$$

So, we have proved that

$$
|I|\le 2|\alpha\beta| j\int_0^{|t|} |F(x+sign(t)j\theta)-F(x)|(|t|-\theta)\,d\theta,\quad \forall\, t\in\mathbb{R}, \tag{29}
$$

and, by (26),

$$
|R(t)|\le \frac{2}{|\beta^2-\alpha^2|}\sum_{j=0}^{r}|\alpha_j| j^2\int_0^{|t|}|F(x+jsign(t)\theta)-F(x)|(|t|-\theta)\,d\theta, \tag{30}
$$

$\forall\, t\in\mathbb{R}$.

By (14), we have

$$
E_1(x)=\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}R(t)\,e^{-\frac{t^2}{\xi}}\,dt. \tag{31}
$$

Hence it holds ($p,q>1:\frac{1}{p}+\frac{1}{q}=1$),

$$
\begin{aligned}
\int_{-\infty}^{\infty}|E_1(x)|^p\,dx &= \int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{\pi\xi}}\left|\int_{-\infty}^{\infty}R(t)\,e^{-\frac{t^2}{\xi}}dt\right|\right)^p dx \\
&\le \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|\frac{e^{-\frac{t^2}{\xi}}}{\sqrt{\pi\xi}}dt\right)^p dx \\
&\le \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|^p\,e^{-\frac{t^2}{\xi}}\,dt\right)dx \\
&\overset{(30)}{\le} \frac{1}{\sqrt{\pi\xi}}\frac{2^p}{|\beta^2-\alpha^2|^p}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left[\sum_{j=1}^{r}|\alpha_j| j^2\right.\right. \qquad (32)\\
&\quad \left.\left.\cdot\left(\int_0^{|t|}|F(x+jsign(t)\theta)-F(x)|(|t|-\theta)\,d\theta\right)\right]^p e^{-\frac{t^2}{\xi}}\,dt\right)dx
\end{aligned}
$$

$$\overset{\text{(by Jensen inequality)}}{\leq} \frac{1}{\sqrt{\pi\xi}} \frac{2^p}{|\beta^2-\alpha^2|^p} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(\sum_{j=1}^{r} |\alpha_j| j^2 \right)^{p-1} \right.$$

$$\left. \cdot \left[\sum_{j=1}^{r} |\alpha_j| j^2 \left(\int_0^{|t|} |F(x+jsign(t)\theta) - F(x)| (|t|-\theta) d\theta \right)^p \right] e^{-\frac{t^2}{\xi}} dt \right) dx$$

$$\leq \frac{2^p \left(\sum_{j=1}^{r} |\alpha_j| j^2 \right)^{p-1}}{\sqrt{\pi\xi} |\beta^2-\alpha^2|^p} \left[\sum_{j=1}^{r} |\alpha_j| j^2 \right.$$

$$\cdot \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(\int_0^{|t|} |F(x+jsign(t)\theta) - F(x)|^p d\theta \right) \right.$$

$$\left. \left. \cdot \left(\int_0^{|t|} (|t|-\theta)^q d\theta \right)^{\frac{p}{q}} e^{-\frac{t^2}{\xi}} dt \right) dx \right]$$

$$= \frac{2^p \left(\sum_{j=1}^{r} |\alpha_j| j^2 \right)^{p-1}}{\sqrt{\pi\xi} |\beta^2-\alpha^2|^p (q+1)^{p-1}} \left[\sum_{j=1}^{r} |\alpha_j| j^2 \right. \tag{33}$$

$$\cdot \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(\int_0^{|t|} |F(x+jsign(t)\theta) - F(x)|^p d\theta \right) \right.$$

$$\left. \left. \cdot |t|^{(q+1)(p-1)} e^{-\frac{t^2}{\xi}} dt \right) dx \right]$$

$$\left(\text{call } c := \frac{2^p \left(\sum_{j=1}^{r} |\alpha_j| j^2 \right)^{p-1}}{\sqrt{\pi\xi} |\beta^2-\alpha^2|^p (q+1)^{p-1}} \right)$$

$$= c \left[\sum_{j=1}^{r} |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(\int_{-\infty}^{\infty} |F(x+jsign(t)\theta) - F(x)|^p dx \right) d\theta \right) \right.$$

$$\left. \cdot |t|^{(q+1)(p-1)} e^{-\frac{t^2}{\xi}} dt \right]$$

$$\leq c \left[\sum_{j=1}^{r} |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \omega_1 \left(F, \sqrt{\xi} \frac{j\theta}{\sqrt{\xi}} \right)_p^p d\theta \right) |t|^{(q+1)(p-1)} e^{-\frac{t^2}{\xi}} dt \right] \tag{34}$$

$$\leq c\omega_1 \left(F, \sqrt{\xi} \right)_p^p \left[\sum_{j=1}^{r} |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(1 + \frac{j\theta}{\sqrt{\xi}} \right)^p d\theta \right) |t|^{(q+1)(p-1)} e^{-\frac{t^2}{\xi}} dt \right]$$

$$= c\omega_1 \left(F, \sqrt{\xi} \right)_p^p \frac{\sqrt{\xi}}{(p+1)} \left[\sum_{j=1}^{r} |\alpha_j| j \int_{-\infty}^{\infty} \left[\left(1 + \frac{j}{\sqrt{\xi}} |t| \right)^{p+1} - 1 \right] \right.$$

$$\left. \cdot |t|^{(q+1)(p-1)} e^{-\frac{t^2}{\xi}} dt \right]$$

$$\leq \frac{2^p c\sqrt{\xi}}{(p+1)}\omega_1\left(F,\sqrt{\xi}\right)_p^p\left[\sum_{j=1}^r|\alpha_j|\,j\,\frac{j^{p+1}}{\left(\sqrt{\xi}\right)^{p+1}}\int_{-\infty}^{\infty}|t|^{p+1}\,|t|^{(q+1)(p-1)}\,e^{-\frac{t^2}{\xi}}dt\right]$$

$$=\frac{2^{p+1}c\left(\sqrt{\xi}\right)^{-p}}{(p+1)}\omega_1\left(F,\sqrt{\xi}\right)_p^p\left[\sum_{j=1}^r|\alpha_j|\,j^{p+2}\int_0^{\infty}t^{(q+1)(p-1)+(p+1)}e^{-\frac{t^2}{\xi}}dt\right] \quad (35)$$

$$=\frac{2^{p+1}c\left(\sqrt{\xi}\right)^{-p}}{(p+1)}\omega_1\left(F,\sqrt{\xi}\right)_p^p\left(\sqrt{\xi}\right)^{3p+1}$$

$$\left[\sum_{j=1}^r|\alpha_j|\,j^{p+2}\int_0^{\infty}\left(\frac{t}{\sqrt{\xi}}\right)^{3p}e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2}d\frac{t}{\sqrt{\xi}}\right]$$

$$=\frac{2^{p+1}c\left(\sqrt{\xi}\right)^{2p+1}}{(p+1)}\omega_1\left(F,\sqrt{\xi}\right)_p^p\left[\sum_{j=1}^r|\alpha_j|\,j^{p+2}\int_0^{\infty}x^{3p}e^{-x^2}dx\right]$$

$$=\frac{2^{p+1}c\left(\sqrt{\xi}\right)^{2p+1}}{(p+1)}\omega_1\left(F,\sqrt{\xi}\right)_p^p\left[\sum_{j=1}^r|\alpha_j|\,j^{p+2}\left(\int_0^1x^{3p}e^{-x^2}dx+\int_1^{\infty}x^{3p}e^{-x^2}dx\right)\right] \quad (36)$$

$$\leq\frac{2^{p+1}c\left(\sqrt{\xi}\right)^{2p+1}}{(p+1)}\omega_1\left(F,\sqrt{\xi}\right)_p^p\left[\sum_{j=1}^r|\alpha_j|\,j^{p+2}\left(\int_0^1x^{3p}dx+\int_1^{\infty}x^{3p}e^{-x}dx\right)\right]$$

$$\leq\frac{2^{p+1}c\left(\sqrt{\xi}\right)^{2p+1}}{(p+1)}\omega_1\left(F,\sqrt{\xi}\right)_p^p\left[\sum_{j=1}^r|\alpha_j|\,j^{p+2}\left(\frac{1}{(3p+1)}+\int_0^{\infty}x^{3p}e^{-x}dx\right)\right]$$

$$=\frac{2^{p+1}c\left(\sqrt{\xi}\right)^{2p+1}}{(p+1)}\omega_1\left(F,\sqrt{\xi}\right)_p^p\left[\sum_{j=1}^r|\alpha_j|\,j^{p+2}\left(\frac{1}{(3p+1)}+\Gamma(3p+1)\right)\right] \quad (37)$$

$$=\frac{\left(\sum_{j=1}^r|\alpha_j|\,j^2\right)^{p-1}2^{2p+1}}{\sqrt{\pi}\,|\beta^2-\alpha^2|^p\,(q+1)^{p-1}\,(p+1)}\xi^p\omega_1\left(F,\sqrt{\xi}\right)_p^p$$

$$\cdot\left[\sum_{j=1}^r|\alpha_j|\,j^{p+2}\left(\frac{1}{(3p+1)}+\Gamma(3p+1)\right)\right]$$

$$=\frac{\left(\sum_{j=1}^r|\alpha_j|\,j^2\right)^{p-1}2^{2p+1}}{\sqrt{\pi}\,|\beta^2-\alpha^2|^p\,(q+1)^{p-1}\,(p+1)}\left[\sum_{j=1}^r|\alpha_j|\,j^{p+2}\left(\frac{1}{(3p+1)}+\Gamma(3p+1)\right)\right]$$

$$\cdot\,\omega_1\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2f,\sqrt{\xi}\right)_p^p\xi^p, \quad (38)$$

proving the claim. □

It follows a consequence.

Corollary 1. *(to Theorem 1) It holds*

$$\left\|W_{r,\xi}(f)-f\right\|_p \leq \Phi_1(\xi)+\frac{\|f''\|_p}{|\beta^2-\alpha^2|}\left[\sum_{j=0}^{r}|\alpha_j|\left|e^{-\frac{\alpha^2 j^2\xi}{16}}-e^{-\frac{\beta^2 j^2\xi}{16}}\right|\right] \tag{39}$$

$$+\left(\frac{\left\|f^{(4)}\right\|_p+\left(\alpha^2+\beta^2\right)\|f''\|_p}{(\alpha\beta)^2|\beta^2-\alpha^2|}\right)\left|\beta^2\sum_{j=0}^{r}\alpha_j\left(1-e^{-\frac{\alpha^2 j^2\xi}{4}}\right)-\alpha^2\sum_{j=0}^{r}\alpha_j\left(1-e^{-\frac{\beta^2 j^2\xi}{4}}\right)\right|\to 0,$$

as $\xi\to 0$.

Above $\Phi_1(\xi)$ *is as in (24).*

Proof. Directly from (24). □

Next comes the L_1 approximation by $W_{r,\xi}$.

Theorem 2. *Let* $\alpha,\beta\in\mathbb{R}$ *with* $\alpha\beta\left(\alpha^2-\beta^2\right)\neq 0$, *and let all* $f,f',f'',f^{(3)},f^{(4)}\in L_1(\mathbb{R})\cap C(\mathbb{R})$; $\xi>0$, $x\in\mathbb{R}$. *Then*

$$\|E_1(x)\|_1=\Bigg\|W_{r,\xi}(f,x)-f(x)-\frac{f''(x)}{(\beta^2-\alpha^2)}\left[\sum_{j=0}^{r}\alpha_j\left(e^{-\frac{\alpha^2 j^2\xi}{16}}-e^{-\frac{\beta^2 j^2\xi}{16}}\right)\right]-$$

$$\left(\frac{f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)}{(\alpha\beta)^2(\beta^2-\alpha^2)}\right)\left[\beta^2\sum_{j=0}^{r}\alpha_j\left(1-e^{-\frac{\alpha^2 j^2\xi}{4}}\right)-\alpha^2\sum_{j=0}^{r}\alpha_j\left(1-e^{-\frac{\beta^2 j^2\xi}{4}}\right)\right]\Bigg\|_1$$

$$\leq\left(\frac{4}{\sqrt{\pi}|\beta^2-\alpha^2|}\right)\left[\sum_{j=1}^{r}|\alpha_j|j^2\left(\frac{7}{3}+j\frac{25}{8}\right)\right]\omega_1\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f,\sqrt{\xi}\right)_1\cdot\xi$$

$$=:\Psi_1(\xi)\to 0, \tag{40}$$

as $\xi\to 0$.

Proof. We have (F is as in (25))

$$\int_{-\infty}^{\infty}|E_1(x)|\,dx\overset{(31)}{=}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{\pi\xi}}\left|\int_{-\infty}^{\infty}R(t)e^{-\frac{t^2}{\xi}}dt\right|\right)dx$$

$$\leq\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|\frac{e^{-\frac{t^2}{\xi}}}{\sqrt{\pi\xi}}dt\right)dx \tag{41}$$

$$\overset{(30)}{\leq}\frac{1}{\sqrt{\pi\xi}}\frac{2}{|\beta^2-\alpha^2|}\int_{-\infty}^{\infty}\Bigg(\int_{-\infty}^{\infty}\Bigg[\sum_{j=1}^{r}|\alpha_j|j^2\Bigg(\int_0^{|t|}|F(x+jsign(t)\theta)$$

$$-F(x)|(|t|-\theta)\,d\theta\Bigg)\Bigg]e^{-\frac{t^2}{\xi}}dt\Bigg)dx$$

$$\leq\frac{21}{\sqrt{\pi\xi}|\beta^2-\alpha^2|}\left[\sum_{j=1}^{r}|\alpha_j|j^2\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}|F(x+jsign(t)\theta)-F(x)|d\theta\right)|t|e^{-\frac{t^2}{\xi}}dt\right)dx\right]$$

$$=\frac{21}{\sqrt{\pi\xi}|\beta^2-\alpha^2|}\left[\sum_{j=1}^{r}|\alpha_j|j^2\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(\int_{-\infty}^{\infty}|F(x+jsign(t)\theta)-F(x)|dx\right)d\theta\right)|t|e^{-\frac{t^2}{\xi}}dt\right]$$

$$\leq \frac{2}{\sqrt{\pi\xi}\left|\beta^2-\alpha^2\right|}\left[\sum_{j=1}^{r}\left|\alpha_j\right| j^2 \int_{-\infty}^{\infty}\left(\int_0^{|t|}\omega_1\left(F,\sqrt{\xi}\frac{j\theta}{\sqrt{\xi}}\right)_1 d\theta\right)|t|\, e^{-\frac{t^2}{\xi}}dt\right]$$

$$\leq \frac{2}{\sqrt{\pi\xi}\left|\beta^2-\alpha^2\right|}\omega_1\left(F,\sqrt{\xi}\right)_1\left[\sum_{j=1}^{r}\left|\alpha_j\right| j^2 \int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(1+\frac{j\theta}{\sqrt{\xi}}\right)d\theta\right)|t|\, e^{-\frac{t^2}{\xi}}dt\right]$$

$$= \frac{2}{\sqrt{\pi\xi}\left|\beta^2-\alpha^2\right|}\omega_1\left(F,\sqrt{\xi}\right)_1\left[\sum_{j=1}^{r}\left|\alpha_j\right| j^2 \int_{-\infty}^{\infty}\left(|t|+\frac{j}{2\sqrt{\xi}}t^2\right)|t|\, e^{-\frac{t^2}{\xi}}dt\right]$$

$$= \frac{4}{\sqrt{\pi\xi}\left|\beta^2-\alpha^2\right|}\omega_1\left(F,\sqrt{\xi}\right)_1\left[\sum_{j=1}^{r}\left|\alpha_j\right| j^2 \int_0^{\infty}\left(t+\frac{j}{2\sqrt{\xi}}t^2\right)te^{-\frac{t^2}{\xi}}dt\right] \tag{42}$$

$$= \frac{4}{\sqrt{\pi\xi}\left|\beta^2-\alpha^2\right|}\omega_1\left(F,\sqrt{\xi}\right)_1\left[\sum_{j=1}^{r}\left|\alpha_j\right| j^2 \left(\int_0^{\infty}t^2e^{-\frac{t^2}{\xi}}dt+\frac{j}{2\sqrt{\xi}}\int_0^{\infty}t^3e^{-\frac{t^2}{\xi}}dt\right)\right]$$

$$= \frac{4}{\sqrt{\pi\xi}\left|\beta^2-\alpha^2\right|}\omega_1\left(F,\sqrt{\xi}\right)_1\left[\sum_{j=1}^{r}\left|\alpha_j\right| j^2 \left(\left(\sqrt{\xi}\right)^3\int_0^{\infty}\left(\frac{t}{\sqrt{\xi}}\right)^2 e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2}d\frac{t}{\sqrt{\xi}}\right.\right.$$

$$\left.\left.+\ \frac{j}{2\sqrt{\xi}}\left(\sqrt{\xi}\right)^4\int_0^{\infty}\left(\frac{t}{\sqrt{\xi}}\right)^3 e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2}d\frac{t}{\sqrt{\xi}}\right)\right]$$

$$= \frac{4\xi}{\sqrt{\pi}\left|\beta^2-\alpha^2\right|}\omega_1\left(F,\sqrt{\xi}\right)_1\left[\sum_{j=1}^{r}\left|\alpha_j\right| j^2 \left(\int_0^{\infty}x^2e^{-x^2}dx+\ \frac{j}{2}\int_0^{\infty}x^3e^{-x^2}dx\right)\right] \tag{43}$$

$$\leq \frac{4\xi}{\sqrt{\pi}\left|\beta^2-\alpha^2\right|}\omega_1\left(F,\sqrt{\xi}\right)_1\left[\sum_{j=1}^{r}\left|\alpha_j\right| j^2 \left[\left(\int_0^1 x^2dx+\int_0^{\infty}x^2e^{-x}dx\right)\right.\right.$$

$$\left.\left.+\frac{j}{2}\left(\int_0^1 x^3dx+\int_0^{\infty}x^3e^{-x}dx\right)\right]\right]$$

$$= \frac{4\xi}{\sqrt{\pi}\left|\beta^2-\alpha^2\right|}\omega_1\left(F,\sqrt{\xi}\right)_1\left[\sum_{j=1}^{r}\left|\alpha_j\right| j^2 \left[\left(\frac{1}{3}+\Gamma\left(3\right)\right)+\frac{j}{2}\left(\frac{1}{4}+\Gamma\left(4\right)\right)\right]\right]$$

$$= \frac{4\xi}{\sqrt{\pi}\left|\beta^2-\alpha^2\right|}\omega_1\left(F,\sqrt{\xi}\right)_1\left[\sum_{j=1}^{r}\left|\alpha_j\right| j^2 \left(\frac{7}{3}+j\frac{25}{8}\right)\right] \tag{44}$$

$$= \frac{4\xi}{\sqrt{\pi}\left|\beta^2-\alpha^2\right|}\omega_1\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f,\sqrt{\xi}\right)_1\left[\sum_{j=1}^{r}\left|\alpha_j\right| j^2 \left(\frac{7}{3}+j\frac{25}{8}\right)\right].$$

The claim is proved. □

We give

Corollary 2. *(to Theorem 2) It holds*

$$\left\|W_{r,\xi}\left(f\right)-f\right\|_1 \leq \Psi_1\left(\xi\right)+\frac{\left\|f''\right\|_1}{\left|\beta^2-\alpha^2\right|}\left[\sum_{j=0}^{r}\left|\alpha_j\right|\left(e^{-\frac{\alpha^2 j^2\xi}{16}}-e^{-\frac{\beta^2 j^2\xi}{16}}\right)\right] \tag{45}$$

$$+\left(\frac{\left\|f^{(4)}\right\|_1+\left(\alpha^2+\beta^2\right)\|f''\|_1}{(\alpha\beta)^2\left|\beta^2-\alpha^2\right|}\right)\left|\beta^2\sum_{j=0}^{r}\alpha_j\left(1-e^{-\frac{\alpha^2j^2\xi}{4}}\right)-\alpha^2\sum_{j=0}^{r}\alpha_j\left(1-e^{-\frac{\beta^2j^2\xi}{4}}\right)\right|\to 0$$

as $\xi\to 0$.

Above $\Psi_1(\xi)$ *is as in (40).*

Proof. Directly from (40). □

3 Part II: On the smooth Poisson-Cauchy singular integral operators ([5])

Let $\overline{\alpha}\in\mathbb{N}$, $\overline{\beta}>\frac{1}{2\overline{\alpha}}$ and $f\in C^2(\mathbb{R})$. We define for $x\in\mathbb{R}$, $\xi>0$ the Lebesgue integral

$$M_{r,\xi}(f;x)=W\int_{-\infty}^{\infty}\frac{\sum_{j=0}^{r}\alpha_j f(x+jt)}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}dt, \tag{46}$$

where the constant is defined as

$$W=\frac{\Gamma\left(\overline{\beta}\right)\overline{\alpha}\xi^{2\overline{\alpha}\overline{\beta}-1}}{\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{1}{2\overline{\alpha}}\right)}. \tag{47}$$

We assume that $M_{r,\xi}(f;x)\in\mathbb{R}$ for all $x\in\mathbb{R}$. We will use also that

$$M_{r,\xi}(f;x)=W\sum_{j=0}^{r}\alpha_j\left(\int_{-\infty}^{\infty}f(x+jt)\frac{1}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}dt\right). \tag{48}$$

We notice by $W\int_{-\infty}^{\infty}\frac{1}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}dt=1$ that $M_{r,\xi}(c;x)=c$, c constant, and

$$M_{r,\xi}(f;x)-f(x)=W\left(\sum_{j=0}^{r}\alpha_j\int_{-\infty}^{\infty}\left[f(x+jt)-f(x)\right]\frac{1}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}dt\right). \tag{49}$$

We set

$$\begin{aligned}
E_2(x) &:= M_{r,\xi}(f,x)-f(x)\\
&-\left(\frac{f'(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta^3\sum_{j=0}^{r}\alpha_jW\left(\int_{-\infty}^{\infty}\sin(\alpha jt)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right.\\
&\qquad\left.-\alpha^3\sum_{j=0}^{r}\alpha_jW\left(\int_{-\infty}^{\infty}\sin(\beta jt)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right] \qquad (50)\\
&-\left(\frac{f''(x)}{\beta^2-\alpha^2}\right)\left[\sum_{j=0}^{r}\alpha_jW\left(\int_{-\infty}^{\infty}\cos(\alpha jt)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right.\\
&\qquad\left.-\sum_{j=0}^{r}\alpha_jW\left(\int_{-\infty}^{\infty}\cos(\beta jt)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right]
\end{aligned}$$

$$-\left(\frac{f^{(3)}(x)}{\alpha\beta(\beta^2-\alpha^2)}\right)\left[\beta\sum_{j=0}^{r}\alpha_j W\left(\int_{-\infty}^{\infty}\sin(\alpha jt)\frac{dt}{(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}})^{\overline{\beta}}}\right)\right.$$
$$\left.-\alpha\sum_{j=0}^{r}\alpha_j W\left(\int_{-\infty}^{\infty}\sin(\beta jt)\frac{dt}{(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}})^{\overline{\beta}}}\right)\right]$$
$$-\left(\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2(\beta^2-\alpha^2)}\right)$$
$$\cdot\left[\beta^2\sum_{j=0}^{r}\alpha_j W\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{\alpha jt}{2}\right)\frac{dt}{(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}})^{\overline{\beta}}}\right)\right.$$
$$\left.-\alpha^2\sum_{j=0}^{r}\alpha_j W\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{\beta jt}{2}\right)\frac{dt}{(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}})^{\overline{\beta}}}\right)\right]$$
$$=M_{r,\xi}(f,x)-f(x)$$
$$-\left(\frac{2f''(x)}{\beta^2-\alpha^2}\right)W\left[\sum_{j=0}^{r}\alpha_j\left(\int_{0}^{\infty}(\cos(\alpha jt)-\cos(\beta jt))\frac{dt}{(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}})^{\overline{\beta}}}\right)\right] \tag{51}$$
$$-\left(\frac{4\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2(\beta^2-\alpha^2)}\right)$$
$$\cdot W\left[\beta^2\sum_{j=0}^{r}\alpha_j\left(\int_{0}^{\infty}\sin^2\left(\frac{\alpha jt}{2}\right)\frac{dt}{(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}})^{\overline{\beta}}}\right)\right.$$
$$\left.-\alpha^2\sum_{j=0}^{r}\alpha_j\left(\int_{0}^{\infty}\sin^2\left(\frac{\beta jt}{2}\right)\frac{dt}{(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}})^{\overline{\beta}}}\right)\right]$$
$$=W\int_{-\infty}^{\infty}R(t)\frac{dt}{(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}})^{\overline{\beta}}}. \tag{52}$$

It follows a parametrized L_p $(p>1)$ approximation result for $M_{r,\xi}$, $\xi>0$.

Theorem 3. *Let $\alpha,\beta\in\mathbb{R}$ with $\alpha\beta\left(\alpha^2-\beta^2\right)\neq 0$, and let all $f,f',f'',f^{(3)},f^{(4)}\in L_p(\mathbb{R})\cap C(\mathbb{R})$, $1<p<\infty$. Here $q>1:\frac{1}{p}+\frac{1}{q}=1$; $\xi>0$, $x\in\mathbb{R}$. Assume $\overline{\alpha}\in\mathbb{N}$, $\overline{\beta}>\frac{3p+1}{2\overline{\alpha}}$. Then*

$$\|E_2(x)\|_p=\|M_{r,\xi}(f,x)-f(x)$$
$$-\left(\frac{2f''(x)}{\beta^2-\alpha^2}\right)W\left[\sum_{j=0}^{r}\alpha_j\left(\int_{0}^{\infty}(\cos(\alpha jt)-\cos(\beta jt))\frac{dt}{(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}})^{\overline{\beta}}}\right)\right] \tag{53}$$
$$-\left(\frac{4\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2(\beta^2-\alpha^2)}\right)$$
$$\cdot W\left[\beta^2\sum_{j=0}^{r}\alpha_j\left(\int_{0}^{\infty}\sin^2\left(\frac{\alpha jt}{2}\right)\frac{dt}{(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}})^{\overline{\beta}}}\right)\right.$$

$$- \alpha^2 \sum_{j=0}^{r} \alpha_j \left(\int_0^\infty \sin^2\left(\frac{\beta j t}{2}\right) \frac{dt}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} \right) \Bigg] \Bigg\|_p$$

$$\leq \left[\frac{4\Gamma^{\frac{1}{p}}\left(\frac{3p+1}{2\overline{\alpha}}\right)\Gamma^{\frac{1}{p}}\left(\overline{\beta} - \left(\frac{3p+1}{2\overline{\alpha}}\right)\right)}{(p+1)^{\frac{1}{p}}(q+1)^{\frac{1}{q}}\left|\beta^2-\alpha^2\right|\Gamma^{\frac{1}{p}}\left(\frac{1}{2\overline{\alpha}}\right)\Gamma^{\frac{1}{p}}\left(\overline{\beta} - \frac{1}{2\overline{\alpha}}\right)} \right] \left(\sum_{j=1}^{r} |\alpha_j| j^2 \right)^{\frac{1}{q}} \left(\sum_{j=1}^{r} |\alpha_j| j^{p+2} \right)^{\frac{1}{p}}$$

$$\omega_1\left(f^{(4)} + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f, \xi\right)_p \cdot \xi^2 =: \Phi_2(\xi) \to 0,$$

as $\xi \to 0$.

Proof. By (52), we have ($\overline{\alpha} \in \mathbb{N}$, $\overline{\beta} > \frac{3p+1}{2\overline{\alpha}}$)

$$E_2(x) = W \int_{-\infty}^{\infty} R(t) \frac{1}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} dt.$$

Hence it holds ($p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$),

$$\int_{-\infty}^{\infty} |E_2(x)|^p dx = \int_{-\infty}^{\infty} \left(W \left| \int_{-\infty}^{\infty} R(t) \frac{1}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} dt \right| \right)^p dx$$

$$\leq \int_{-\infty}^{\infty} \left(W \int_{-\infty}^{\infty} |R(t)| \frac{1}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} dt \right)^p dx \tag{54}$$

$$\leq W \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)|^p \frac{1}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} dt \right) dx$$

$$\overset{(30)}{\leq} \frac{2^p W}{\left|\beta^2-\alpha^2\right|^p} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left[\sum_{j=1}^{r} |\alpha_j| j^2 \right.\right.$$

$$\left.\left. \cdot \left(\int_0^{|t|} |F(x + j sign(t)\theta) - F(x)| (|t| - \theta) d\theta \right) \right]^p \frac{1}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} dt \right) dx$$

(by Jensen inequality)

$$\leq \frac{2^p W}{\left|\beta^2-\alpha^2\right|^p} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(\sum_{j=1}^{r} |\alpha_j| j^2 \right)^{p-1} \right.$$

$$\left. \left[\sum_{j=1}^{r} |\alpha_j| j^2 \left(\int_0^{|t|} |F(x + j sign(t)\theta) - F(x)| (|t| - \theta) d\theta \right)^p \right] \frac{1}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} dt \right) dx$$

$$\leq W \left(\sum_{j=1}^{r} |\alpha_j| j^2 \right)^{p-1} \frac{2^p}{\left|\beta^2-\alpha^2\right|^p} \left[\sum_{j=1}^{r} |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \right.\right. \tag{55}$$

$$\left.\left. \left(\int_0^{|t|} |F(x + j sign(t)\theta) - F(x)|^p d\theta \right) \left(\int_0^{|t|} (|t| - \theta)^q d\theta \right)^{\frac{p}{q}} \frac{1}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} dt \right) dx \right]$$

$$= W \left(\sum_{j=1}^{r} |\alpha_j| j^2 \right)^{p-1} \frac{2^p}{\left|\beta^2-\alpha^2\right|^p (q+1)^{p-1}} \left[\sum_{j=1}^{r} |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \right.\right.$$

$$\left(\int_0^{|t|} |F(x + jsign(t)\theta) - F(x)|^p d\theta\right) |t|^{(q+1)(p-1)} \frac{1}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} dt\right) dx\Bigg]$$

(call $\rho := W\left(\sum_{j=1}^{r} |\alpha_j| j^2\right)^{p-1} \frac{2^p}{|\beta^2 - \alpha^2|^p (q+1)^{p-1}}$)

$$= \rho\left[\sum_{j=1}^{r} |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(\int_{-\infty}^{\infty} |F(x + jsign(t)\theta) - F(x)|^p dx\right) d\theta\right)\right.$$

$$\left. \cdot |t|^{(q+1)(p-1)} \frac{1}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} dt\right]$$

$$\leq \rho\left[\sum_{j=1}^{r} |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \omega_1\left(F, \xi\frac{j\theta}{\xi}\right)_p^p d\theta\right) |t|^{(q+1)(p-1)} \frac{1}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} dt\right] \quad (56)$$

$$\leq \rho\omega_1(F, \xi)_p^p \left[\sum_{j=1}^{r} |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(1 + \frac{j\theta}{\xi}\right)^p d\theta\right) |t|^{(q+1)(p-1)} \frac{1}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} dt\right]$$

$$= \rho\omega_1(F, \xi)_p^p \frac{\xi}{(p+1)} \left[\sum_{j=1}^{r} |\alpha_j| j \int_{-\infty}^{\infty} \left[\left(1 + \frac{j}{\xi}|t|\right)^{p+1} - 1\right] |t|^{(q+1)(p-1)} \frac{1}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} dt\right]$$

$$\leq \frac{2^p \rho\xi\omega_1(F, \xi)_p^p}{(p+1)} \left[\sum_{j=1}^{r} |\alpha_j| j \frac{j^{p+1}}{\xi^{p+1}} \int_{-\infty}^{\infty} |t|^{p+1} |t|^{(q+1)(p-1)} \frac{1}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} dt\right] \quad (57)$$

$$= \frac{2^{p+1} \rho\xi^{-p}\omega_1(F, \xi)_p^p}{(p+1)} \left[\sum_{j=1}^{r} |\alpha_j| j^{p+2} \int_0^{\infty} t^{(q+1)(p-1)+(p+1)} \frac{1}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} dt\right]$$

$$= \frac{2^{p+1} \rho\xi^{-p}\omega_1(F, \xi)_p^p}{(p+1)} \left[\sum_{j=1}^{r} |\alpha_j| j^{p+2} \xi^{3p-2\overline{\alpha}\overline{\beta}+1} \int_0^{\infty} \left(\frac{t}{\xi}\right)^{3p} \frac{1}{\left(\left(\frac{t}{\xi}\right)^{2\overline{\alpha}} + 1\right)^{\overline{\beta}}} d\left(\frac{t}{\xi}\right)\right] \quad (58)$$

$$= \frac{2^{p+1} \rho\xi^{2p-2\overline{\alpha}\overline{\beta}+1}\omega_1(F, \xi)_p^p}{(p+1)} \left[\sum_{j=1}^{r} |\alpha_j| j^{p+2} \left[\int_0^{\infty} x^{3p} \frac{1}{\left(x^{2\overline{\alpha}} + 1\right)^{\overline{\beta}}} dx\right]\right]$$

(by [13], p. 397 formula 595)

$$= \frac{2^{p+1} \rho\xi^{2p-2\overline{\alpha}\overline{\beta}+1}\omega_1(F, \xi)_p^p}{(p+1)} \left[\sum_{j=1}^{r} |\alpha_j| j^{p+2}\right] \left[\frac{\Gamma\left(\frac{3p+2}{2\overline{\alpha}}\right) \Gamma\left(\overline{\beta} - \left(\frac{3p+1}{2\overline{\alpha}}\right)\right)}{2\overline{\alpha}\Gamma\left(\overline{\beta}\right)}\right]$$

$$= \frac{2^{p+1} \xi^{2p-2\overline{\alpha}\overline{\beta}+1}}{(p+1)} \left(W\left(\sum_{j=1}^{r} |\alpha_j| j^2\right)^{p-1} \frac{2^p}{|\beta^2 - \alpha^2|^p (q+1)^{p-1}}\right) \omega_1(F, \xi)_p^p$$

$$\cdot \left[\sum_{j=1}^{r} |\alpha_j| j^{p+2}\right] \left[\frac{\Gamma\left(\frac{3p+2}{2\overline{\alpha}}\right) \Gamma\left(\overline{\beta} - \left(\frac{3p+1}{2\overline{\alpha}}\right)\right)}{2\overline{\alpha}\Gamma\left(\overline{\beta}\right)}\right]$$

$$\overset{(47)}{=} \frac{2^{p+1}\xi^{2p-2\overline{\alpha}\overline{\beta}+1}}{(q+1)^{p-1}(p+1)\left|\beta^2-\alpha^2\right|^p}\left(\frac{\Gamma\left(\overline{\beta}\right)\overline{\alpha}\xi^{2\overline{\alpha}\overline{\beta}-1}}{\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{1}{2\overline{\alpha}}\right)}\right)\left(\sum_{j=1}^{r}\left|\alpha_j\right|j^2\right)^{p-1} \tag{59}$$

$$\cdot\left[\sum_{j=1}^{r}\left|\alpha_j\right|j^{p+2}\right]\left(\frac{\Gamma\left(\frac{3p+2}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\left(\frac{3p+1}{2\overline{\alpha}}\right)\right)}{2\overline{\alpha}\Gamma\left(\overline{\beta}\right)}\right)\omega_1\left(F,\xi\right)_p^p$$

$$= \frac{4^p\xi^{2p}}{(q+1)^{p-1}(p+1)\left|\beta^2-\alpha^2\right|^p\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{1}{2\overline{\alpha}}\right)}\left(\sum_{j=1}^{r}\left|\alpha_j\right|j^2\right)^{p-1}\left(\sum_{j=1}^{r}\left|\alpha_j\right|j^{p+2}\right)$$

$$\cdot\Gamma\left(\frac{3p+1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\left(\frac{3p+1}{2\overline{\alpha}}\right)\right)\omega_1\left(F,\xi\right)_p^p$$

$$= \left[\frac{4^p\Gamma\left(\frac{3p+1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\left(\frac{3p+1}{2\overline{\alpha}}\right)\right)}{(p+1)(q+1)^{(p-1)}\left|\beta^2-\alpha^2\right|^p\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{1}{2\overline{\alpha}}\right)}\right]\left(\sum_{j=1}^{r}\left|\alpha_j\right|j^2\right)^{p-1}$$

$$\cdot\left(\sum_{j=1}^{r}\left|\alpha_j\right|j^{p+2}\right)\omega_1\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2f,\xi\right)_p^p\cdot\xi^{2p}, \tag{60}$$

the claim is proved. ∎

Next we give a consequence of Theorem 3.

Corollary 3. *(to Theorem 3) It holds*

$$\left\|M_{r,\xi}(f)-f\right\|_p \leq \Phi_2(\xi)+$$

$$\left\|f''\right\|_p\left(\frac{2W}{\left|\beta^2-\alpha^2\right|}\right)\left[\sum_{j=0}^{r}\left|\alpha_j\right|\int_0^\infty\left|\cos(\alpha jt)-\cos(\beta jt)\right|\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right]+$$

$$\left(\frac{4\left\|f^{(4)}\right\|_p+\left(\alpha^2+\beta^2\right)\left\|f''\right\|_p}{(\alpha\beta)^2\left|\beta^2-\alpha^2\right|}\right)$$

$$\cdot W\left[\beta^2\sum_{j=0}^{r}\left|\alpha_j\right|\left(\int_0^\infty\sin^2\left(\frac{\alpha jt}{2}\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right. \tag{61}$$

$$\left.+\alpha^2\sum_{j=0}^{r}\left|\alpha_j\right|\left(\int_0^\infty\sin^2\left(\frac{\beta jt}{2}\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right]\to 0,$$

as $\xi\to 0$.

Above $\Phi_2(\xi)$ *is as in (53).*

Proof. We have

$$W\int_0^\infty\left|\cos(\alpha jt)-\cos(\beta jt)\right|\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}$$

$$=2W\int_0^\infty\left|\sin\left(\frac{(\alpha+\beta)jt}{2}\right)\right|\left|\sin\left(\frac{(\alpha-\beta)jt}{2}\right)\right|\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}$$

$$\leq 2W\int_0^\infty \frac{|\alpha+\beta|\,jt}{2}\frac{|\alpha-\beta|\,jt}{2}\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}$$

$$= \frac{W\left|\alpha^2-\beta^2\right|j^2}{2}\int_0^\infty \frac{t^2dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}$$

$$= \frac{W\left|\alpha^2-\beta^2\right|j^2}{2}\xi^{3-2\overline{\alpha}\overline{\beta}}\int_0^\infty \frac{x^2dx}{\left(1+x^{2\overline{\alpha}}\right)^{\overline{\beta}}} \tag{62}$$

(by [13], p. 397, formula 595)

$$= \frac{\left|\alpha^2-\beta^2\right|j^2}{2}\left(\frac{\Gamma\left(\overline{\beta}\right)\overline{\alpha}\xi^{2\overline{\alpha}\overline{\beta}-1}}{\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{1}{2\overline{\alpha}}\right)}\right)\xi^{3-2\overline{\alpha}\overline{\beta}}\frac{\Gamma\left(\frac{3}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{3}{2\overline{\alpha}}\right)}{2\overline{\alpha}\Gamma\left(\overline{\beta}\right)} \tag{63}$$

$$= \frac{\left|\beta^2-\alpha^2\right|j^2}{4}\frac{\Gamma\left(\frac{3}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{3}{2\overline{\alpha}}\right)}{\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{1}{2\overline{\alpha}}\right)}\xi^2 \to 0, \quad \text{as } \xi\to 0.$$

Furthermore, we have that

$$W\int_0^\infty \sin^2\left(\frac{\alpha jt}{2}\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}$$

$$\leq W\frac{\alpha^2j^2}{4}\int_0^\infty t^2\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} \tag{64}$$

$$= \left(\frac{\Gamma\left(\overline{\beta}\right)\overline{\alpha}\xi^{2\overline{\alpha}\overline{\beta}-1}}{\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{1}{2\overline{\alpha}}\right)}\right)\frac{\alpha^2j^2}{4}\xi^{3-2\overline{\alpha}\overline{\beta}}\left(\int_0^\infty \frac{x^2dx}{\left(1+x^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)$$

$$= \left(\frac{\Gamma\left(\overline{\beta}\right)\overline{\alpha}\xi^2}{\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{1}{2\overline{\alpha}}\right)}\right)\frac{\alpha^2j^2}{4}\frac{\Gamma\left(\frac{3}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{3}{2\overline{\alpha}}\right)}{2\overline{\alpha}\Gamma\left(\overline{\beta}\right)}$$

$$= \frac{\alpha^2j^2}{8}\left(\frac{\Gamma\left(\frac{3}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{3}{2\overline{\alpha}}\right)}{\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{1}{2\overline{\alpha}}\right)}\right)\xi^2 \to 0, \quad \text{as } \xi\to 0. \tag{65}$$

The proof now is clear. □

Next comes the L_1 approximation by $M_{r,\xi}$.

Theorem 4. *Let $\alpha,\beta\in\mathbb{R}$ with $\alpha\beta\left(\alpha^2-\beta^2\right)\neq 0$, and let all $f, f', f'', f^{(3)}, f^{(4)} \in L_1(\mathbb{R})\cap C(\mathbb{R})$; $\xi>0$, $x\in\mathbb{R}$. Here $\overline{\alpha}\in\mathbb{N}$, $\overline{\beta}>\frac{2}{\overline{\alpha}}$. Then*

$$\begin{aligned}&\left\|E_2(x)\right\|_1 = \Bigg\|M_{r,\xi}(f,x) - f(x) - \\ &\left(\frac{2f''(x)}{\beta^2-\alpha^2}\right)W\left[\sum_{j=0}^r \alpha_j\left(\int_0^\infty \left(\cos(\alpha jt)-\cos(\beta jt)\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right]\\ &\quad-\left(\frac{4\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\right)\\ &\qquad\cdot W\left[\beta^2\sum_{j=0}^r\alpha_j\left(\int_0^\infty \sin^2\left(\frac{\alpha jt}{2}\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right.\\ &\qquad\quad\left.-\alpha^2\sum_{j=0}^r\alpha_j\left(\int_0^\infty \sin^2\left(\frac{\beta jt}{2}\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right]\Bigg\|_1\end{aligned}$$

$$\leq \left(\frac{2}{\left|\beta^2-\alpha^2\right|\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{1}{2\overline{\alpha}}\right)}\right)\left[\sum_{j=1}^{r}\left|\alpha_j\right|j^2\left[\Gamma\left(\frac{3}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{3}{2\overline{\alpha}}\right)\right.\right. \tag{66}$$

$$\left.\left.+\frac{j}{2}\Gamma\left(\frac{2}{\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{2}{\overline{\alpha}}\right)\right]\right]$$

$$\omega_1\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f,\xi\right)_1\cdot\xi^2=:\Psi_2(\xi)\to 0,$$

as $\xi \to 0$.

Proof. We have that (F is as in (25))

$$\int_{-\infty}^{\infty}\left|E_2(x)\right|dx \overset{(52)}{=} W\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}R(t)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right|dx$$

$$\leq W\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left|R(t)\right|\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)dx$$

$$\overset{(30)}{\leq}\frac{2W}{\left|\beta^2-\alpha^2\right|}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left[\sum_{j=1}^{r}\left|\alpha_j\right|j^2\left(\int_0^{|t|}\left|F(x+jsign(t)\theta)-F(x)\right|(|t|-\theta)d\theta\right)\right]\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)dx \tag{67}$$

$$\leq\frac{2W}{\left|\beta^2-\alpha^2\right|}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left[\sum_{j=1}^{r}\left|\alpha_j\right|j^2\left(\int_0^{|t|}\left|F(x+jsign(t)\theta)-F(x)\right|d\theta\right)|t|\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)dx\right]$$

$$=\frac{2W}{\left|\beta^2-\alpha^2\right|}\left[\sum_{j=1}^{r}\left|\alpha_j\right|j^2\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(\int_{-\infty}^{\infty}\left|F(x+jsign(t)\theta)-F(x)\right|dx\right)d\theta\right)\frac{|t|\,dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right]$$

$$\leq\frac{2W}{\left|\beta^2-\alpha^2\right|}\left[\sum_{j=1}^{r}\left|\alpha_j\right|j^2\int_{-\infty}^{\infty}\left(\int_0^{|t|}\omega_1\left(F,\xi\frac{j\theta}{\xi}\right)_1 d\theta\right)\frac{|t|\,dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right]$$

$$\leq\frac{2W}{\left|\beta^2-\alpha^2\right|}\omega_1(F,\xi)_1\left[\sum_{j=1}^{r}\left|\alpha_j\right|j^2\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(1+\frac{j\theta}{\xi}\right)d\theta\right)\frac{|t|\,dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right]$$

$$=\frac{2W}{\left|\beta^2-\alpha^2\right|}\omega_1(F,\xi)_1\left[\sum_{j=1}^{r}\left|\alpha_j\right|j^2\int_{-\infty}^{\infty}\left(|t|+\frac{j}{2\xi}t^2\right)\frac{|t|\,dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right] \tag{68}$$

$$=\frac{4W}{\left|\beta^2-\alpha^2\right|}\omega_1(F,\xi)_1\left[\sum_{j=1}^{r}\left|\alpha_j\right|j^2\int_0^{\infty}\left(t+\frac{j}{2\xi}t^2\right)\frac{tdt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right]$$

$$=\frac{4W}{\left|\beta^2-\alpha^2\right|}\omega_1(F,\xi)_1\left[\sum_{j=1}^{r}\left|\alpha_j\right|j^2\left[\int_0^{\infty}\frac{t^2dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}+\frac{j}{2\xi}\int_0^{\infty}\frac{t^3dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right]\right]$$

$$=\frac{4W}{\left|\beta^2-\alpha^2\right|}\omega_1(F,\xi)_1\left[\sum_{j=1}^{r}\left|\alpha_j\right|j^2\left[\xi^{3-2\overline{\alpha}\overline{\beta}}\int_0^{\infty}\frac{x^2dx}{\left(1+x^{2\overline{\alpha}}\right)^{\overline{\beta}}}+\frac{j}{2\xi}\xi^{4-2\overline{\alpha}\overline{\beta}}\int_0^{\infty}\frac{x^3dx}{\left(1+x^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right]\right] \tag{69}$$

$$=\frac{4W\xi^{3-2\overline{\alpha}\overline{\beta}}}{\left|\beta^2-\alpha^2\right|}\omega_1(F,\xi)_1\left[\sum_{j=1}^{r}\left|\alpha_j\right|j^2\left[\int_0^{\infty}\frac{x^2dx}{\left(1+x^{2\overline{\alpha}}\right)^{\overline{\beta}}}+\frac{j}{2}\int_0^{\infty}\frac{x^3dx}{\left(1+x^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right]\right]$$

(by [13], p. 397, formula 595)

$$= \frac{4}{|\beta^2-\alpha^2|}\left(\frac{\Gamma\left(\overline{\beta}\right)\overline{\alpha}\xi^2}{\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{1}{2\overline{\alpha}}\right)}\right)\omega_1(F,\xi)_1\left[\sum_{j=1}^{r}|\alpha_j|j^2\left[\frac{\Gamma\left(\frac{3}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{3}{2\overline{\alpha}}\right)}{2\overline{\alpha}\Gamma\left(\overline{\beta}\right)}+\frac{j}{2}\frac{\Gamma\left(\frac{2}{\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{2}{\overline{\alpha}}\right)}{2\overline{\alpha}\Gamma\left(\overline{\beta}\right)}\right]\right] \tag{70}$$

$$= \frac{2}{|\beta^2-\alpha^2|\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{1}{2\overline{\alpha}}\right)}\omega_1(F,\xi)_1\xi^2\left[\sum_{j=1}^{r}|\alpha_j|j^2\left[\Gamma\left(\frac{3}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{3}{2\overline{\alpha}}\right)+\frac{j}{2}\Gamma\left(\frac{2}{\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{2}{\overline{\alpha}}\right)\right]\right].$$

The claim is proved. □

It follows the related result.

Corollary 4. *(to Theorem 4) It holds*

$$\|M_{r,\xi}(f)-f\|_1 \le \Psi_2(\xi) + $$

$$+\|f''\|_1\left(\frac{2W}{|\beta^2-\alpha^2|}\right)\left[\sum_{j=0}^{r}|\alpha_j|\int_0^\infty|\cos(\alpha jt)-\cos(\beta jt)|\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right] \tag{71}$$

$$+\left(\frac{4\left\|f^{(4)}\right\|_1+\left(\alpha^2+\beta^2\right)\|f''\|_1}{(\alpha\beta)^2|\beta^2-\alpha^2|}\right)$$

$$\cdot W\left[\beta^2\sum_{j=0}^{r}|\alpha_j|\left(\int_0^\infty\sin^2\left(\frac{\alpha jt}{2}\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right.$$

$$\left.+\alpha^2\sum_{j=0}^{r}|\alpha_j|\left(\int_0^\infty\sin^2\left(\frac{\beta jt}{2}\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right]\to 0$$

as $\xi \to 0$. *Above* $\Psi_2(\xi)$ *is as in (66).*

Proof. Directly from (66), and see also the proof of Corollary 3. □

4 Part III: On the smooth trigonometric singular integral operators ([5])

Let $\xi > 0$, $f \in C^2(\mathbb{R})$, $x \in \mathbb{R}$, $\overline{\beta} \in \mathbb{N}$; we set

$$T_{r,\xi}(f;x) := \frac{1}{\lambda}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(x+jt)\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt, \tag{72}$$

where

$$\lambda := \int_{-\infty}^{\infty}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt = 2\xi^{1-2\overline{\beta}}\int_0^\infty\left(\frac{\sin t}{t}\right)^{2\overline{\beta}}dt \tag{73}$$

(by [11], p. 210, item 1033)

$$= 2\xi^{1-2\overline{\beta}}\pi(-1)^{\overline{\beta}}\,\overline{\beta}\sum_{k=1}^{\overline{\beta}}(-1)^k\frac{k^{2\overline{\beta}-1}}{\left(\overline{\beta}-k\right)!\left(\overline{\beta}+k\right)!}.$$

Denote

$$\lambda_1 := 2\pi \left(-1\right)^{\overline{\beta}} \overline{\beta} \sum_{k=1}^{\overline{\beta}} \left(-1\right)^k \frac{k^{2\overline{\beta}-1}}{\left(\overline{\beta}-k\right)! \left(\overline{\beta}+k\right)!}; \tag{74}$$

that is

$$\lambda = \lambda_1 \xi^{1-2\overline{\beta}}. \tag{75}$$

We suppose that $T_{r,\xi}\left(f;x\right) \in \mathbb{R}$ for all $x \in \mathbb{R}$. Clearly, again it is

$$\frac{1}{\lambda} \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt = 1, \tag{76}$$

and $T_{r,\xi}\left(c;x\right) = c$, c constant, and

$$T_{r,\xi}\left(f;x\right) - f\left(x\right) = \frac{1}{\lambda} \left(\sum_{j=0}^{r} \alpha_j \int_{-\infty}^{\infty} \left[f\left(x+jt\right) - f\left(x\right)\right] \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt \right). \tag{77}$$

We set

$$\begin{aligned}
&E_3\left(x\right) := T_{r,\xi}\left(f,x\right) - f\left(x\right) - \\
&\left(\frac{f'\left(x\right)}{\alpha\beta\left(\beta^2-\alpha^2\right)} \right) \left[\beta^3 \sum_{j=0}^{r} \alpha_j \frac{1}{\lambda} \left(\int_{-\infty}^{\infty} \sin\left(\alpha j t\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt \right) - \right. \\
&\qquad \left. \cdot\, \alpha^3 \sum_{j=0}^{r} \alpha_j \frac{1}{\lambda} \left(\int_{-\infty}^{\infty} \sin\left(\beta j t\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt \right) \right] \\
&- \left(\frac{f''\left(x\right)}{\beta^2-\alpha^2} \right) \left[\sum_{j=0}^{r} \alpha_j \frac{1}{\lambda} \left(\int_{-\infty}^{\infty} \cos\left(\alpha j t\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt \right) \right. \\
&\qquad \left. - \sum_{j=0}^{r} \alpha_j \frac{1}{\lambda} \left(\int_{-\infty}^{\infty} \cos\left(\beta j t\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt \right) \right] \\
&- \left(\frac{f^{(3)}\left(x\right)}{\alpha\beta\left(\beta^2-\alpha^2\right)} \right) \left[\beta \sum_{j=0}^{r} \alpha_j \frac{1}{\lambda} \left(\int_{-\infty}^{\infty} \sin\left(\alpha j t\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt \right) \right. \\
&\qquad \left. - \alpha \sum_{j=0}^{r} \alpha_j \frac{1}{\lambda} \left(\int_{-\infty}^{\infty} \sin\left(\beta j t\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt \right) \right] \\
&- \left(\frac{2\left(f^{(4)}\left(x\right) + \left(\alpha^2+\beta^2\right) f''\left(x\right)\right)}{\left(\alpha\beta\right)^2 \left(\beta^2-\alpha^2\right)} \right)
\end{aligned} \tag{78}$$

$$\cdot\left[\beta^2\sum_{j=0}^{r}\alpha_j\frac{1}{\lambda}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{\alpha jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right.$$

$$\left.-\alpha^2\sum_{j=0}^{r}\alpha_j\frac{1}{\lambda}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{\beta jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right]$$

$$=T_{r,\xi}(f,x)-f(x)-\left(\frac{2f''(x)}{\beta^2-\alpha^2}\right)\frac{1}{\lambda}\left[\sum_{j=0}^{r}\alpha_j\left(\int_{-\infty}^{\infty}(\cos(\alpha jt)-\cos(\beta jt))\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right]$$

$$-\left(\frac{4\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\right)\frac{1}{\lambda}$$

$$\cdot\left[\beta^2\sum_{j=0}^{r}\alpha_j\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{\alpha jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right. \tag{79}$$

$$\left.-\alpha^2\sum_{j=0}^{r}\alpha_j\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{\beta jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right]$$

$$=\frac{1}{\lambda}\int_{-\infty}^{\infty}R(t)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt. \tag{80}$$

It follows a parametrized L_p $(p>1)$ approximation result for $T_{r,\xi}$, $\xi>0$.

Theorem 5. *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\overline{\beta}\in\mathbb{N}$ and $\overline{\beta}>\frac{\lceil 3p\rceil+1}{2}$ ($\lceil\cdot\rceil$ the ceiling of the number), $\xi>0$, λ_1 is as in (74).*

When $\overline{\lambda}\in\mathbb{N}$ is even we define

$$\psi_{1\overline{\lambda}}:=\frac{\pi(-1)^{\frac{2\overline{\beta}-\overline{\lambda}}{2}}\left(2\overline{\beta}\right)!}{2^{\overline{\lambda}+1}\left(2\overline{\beta}-\overline{\lambda}-1\right)!}\left(\sum_{k=1}^{\overline{\beta}}(-1)^k\frac{k^{2\overline{\beta}-\overline{\lambda}-1}}{\left(\overline{\beta}-k\right)!\left(\overline{\beta}+k\right)!}\right),$$

and when $\overline{\lambda}$ is odd we define

$$\psi_{2\overline{\lambda}}:=\frac{(-1)^{\frac{\overline{\lambda}-1}{2}}\left(2\overline{\beta}\right)!}{2^{\overline{\lambda}}\left(2\overline{\beta}-\overline{\lambda}-1\right)!}\left(\sum_{k=1}^{\overline{\beta}}(-1)^{\overline{\beta}-k}\frac{k^{2\overline{\beta}-\overline{\lambda}-1}\ln(2k)}{\left(\overline{\beta}-k\right)!\left(\overline{\beta}+k\right)!}\right),$$

and we set

$$\psi_{\overline{\lambda}}:=\left\{\begin{array}{c}\psi_{1\overline{\lambda}},\ \text{if } \overline{\lambda} \text{ is even,}\\ \psi_{2\overline{\lambda}},\ \text{if } \overline{\lambda} \text{ is odd}\end{array}\right\}.$$

Let $\alpha,\beta\in\mathbb{R}:\alpha\beta\left(\alpha^2-\beta^2\right)\neq 0$; $f,f',f'',f^{(3)},f^{(4)}\in L_p(\mathbb{R})\cap C(\mathbb{R})$; $\xi>0$, $x\in\mathbb{R}$.

Then

$$\|E_3\|_p = \Big\| T_{r,\xi}(f,x) - f(x) - \left(\frac{2f''(x)}{\beta^2-\alpha^2}\right)\frac{1}{\lambda}\left[\sum_{j=0}^{r}\alpha_j\left(\int_0^\infty(\cos(\alpha j t)-\cos(\beta j t))\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right]$$

$$- \left(\frac{4\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\right)\frac{1}{\lambda}$$

$$\cdot\left[\beta^2\sum_{j=0}^{r}\alpha_j\left(\int_0^\infty\sin^2\left(\frac{\alpha j t}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right.$$

$$\left. - \alpha^2\sum_{j=0}^{r}\alpha_j\left(\int_0^\infty\sin^2\left(\frac{\beta j t}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right]\Big\|_p$$

$$\leq \frac{4\cdot 2^{\frac{1}{p}}}{\lambda_1^{\frac{1}{p}}\left|\beta^2-\alpha^2\right|(q+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left(\sum_{j=1}^{r}|\alpha_j|\,j^2\right)^{\frac{1}{q}}\left(\sum_{j=1}^{r}|\alpha_j|\,j^{p+2}\right)^{\frac{1}{p}} \tag{81}$$

$$\left[\left(\frac{-(2\overline{\beta})!}{8\left(2\overline{\beta}-4\right)!}\sum_{k=1}^{\overline{\beta}}(-1)^{\overline{\beta}-k}\frac{k^{2\overline{\beta}-4}\ln(2k)}{(\overline{\beta}-k)!\,(\overline{\beta}+k)!}\right)+\psi_{\lceil 3p\rceil}\right]^{\frac{1}{p}}$$

$$\omega_1\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f,\xi\right)_p\cdot\xi^2 =: \Phi_3(\xi)\to 0, \quad \textit{as } \xi\to 0.$$

Proof. By (80) we have ($\overline{\beta}\in\mathbb{N}:\overline{\beta}>\frac{\lceil 3p\rceil+1}{2}$)

$$E_3(x) = \frac{1}{\lambda}\int_{-\infty}^{\infty}R(t)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt. \tag{82}$$

Hence it holds ($p,q>1:\frac{1}{p}+\frac{1}{q}=1$)

$$\int_{-\infty}^{\infty}|E_3(x)|^p\,dx = \int_{-\infty}^{\infty}\left(\frac{1}{\lambda}\left|\int_{-\infty}^{\infty}R(t)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right|\right)^p dx$$

$$\leq \int_{-\infty}^{\infty}\left(\frac{1}{\lambda}\int_{-\infty}^{\infty}|R(t)|\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)^p dx \tag{83}$$

$$\leq \frac{1}{\lambda}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|^p\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)dx$$

$$\overset{(30)}{\leq} \frac{2^p}{|\beta^2-\alpha^2|^p\,\lambda}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\right.\right.$$

$$\cdot\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\left(\int_0^{|t|}|F(x+jsign(t)\,\theta)-F(x)|\,(|t|-\theta)\,d\theta\right)\right]^p$$

$$\left.\left.\cdot\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)dx\right)$$

$$\overset{\text{(by Jensen's inequality)}}{\leq} \frac{2^p}{|\beta^2-\alpha^2|^p\,\lambda}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}|\alpha_j|\,j^2\right)^{p-1}\right.$$

$$\cdot\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\left(\int_0^{|t|}|F(x+jsign(t)\,\theta)-F(x)|\,(|t|-\theta)\,d\theta\right)^p\right]$$

$$\left.\cdot\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)dx$$

$$\leq \frac{1}{\lambda}\left(\sum_{j=0}^{r}|\alpha_j|\,j^2\right)^{p-1}\frac{2^p}{|\beta^2-\alpha^2|^p}$$

$$\cdot\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}|F(x+jsign(t)\,\theta)-F(x)|^p\,d\theta\right)\right.\right.$$

$$\left.\left.\cdot\left(\int_0^{|t|}(|t|-\theta)^q\,d\theta\right)^{\frac{p}{q}}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)dx\right] \quad (84)$$

$$=\frac{1}{\lambda}\left(\sum_{j=0}^{r}|\alpha_j|\,j^2\right)^{p-1}\frac{2^p}{|\beta^2-\alpha^2|^p\,(q+1)^{p-1}}$$

$$\cdot\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}|F(x+jsign(t)\,\theta)-F(x)|^p\,d\theta\right)\right.\right.$$

$$\left.\left.\cdot\,|t|^{(q+1)(p-1)}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)dx\right]$$

$$\left(\text{call } \mu := \frac{1}{\lambda}\left(\sum_{j=0}^{r}|\alpha_j|\,j^2\right)^{p-1}\frac{2^p}{|\beta^2-\alpha^2|^p\,(q+1)^{p-1}}\right)$$

$$=\mu\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(\int_{-\infty}^{\infty}|F(x+jsign(t)\,\theta)-F(x)|^p\,dx\right)d\theta\right)\right.$$

$$\cdot |t|^{(q+1)(p-1)} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}} dt\Bigg]$$

$$\le \mu\left[\sum_{j=0}^{r}|\alpha_j|\, j^2\int_{-\infty}^{\infty}\left(\int_0^{|t|}\omega_1\left(F,\xi\frac{j\theta}{\xi}\right)_p^p d\theta\right)|t|^{(q+1)(p-1)}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}} dt\right] \tag{85}$$

$$\le \mu\omega_1(F,\xi)_p^p$$

$$\cdot\left[\sum_{j=0}^{r}|\alpha_j|\, j^2\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(1+\frac{j\theta}{\xi}\right)^p d\theta\right)|t|^{(q+1)(p-1)}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}} dt\right]$$

$$= \mu\omega_1(F,\xi)_p^p\,\frac{\xi}{(p+1)}$$

$$\cdot\left[\sum_{j=1}^{r}|\alpha_j|\, j\int_{-\infty}^{\infty}\left(\left(1+\frac{j}{\xi}|t|\right)^{p+1}-1\right)|t|^{(q+1)(p-1)}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}} dt\right]$$

$$\le \frac{2^p\mu\xi\omega_1(F,\xi)_p^p}{(p+1)}\left[\sum_{j=1}^{r}|\alpha_j|\, j\frac{j^{p+1}}{\xi^{p+1}}\int_{-\infty}^{\infty}|t|^{p+1}\,|t|^{(q+1)(p-1)}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}} dt\right] \tag{86}$$

$$= \frac{2^{p+1}\mu\xi^{-p}\omega_1(F,\xi)_p^p}{(p+1)}\left[\sum_{j=1}^{r}|\alpha_j|\, j^{p+2}\int_0^{\infty}t^{(q+1)(p-1)+(p+1)}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}} dt\right]$$

$$= \frac{2^{p+1}\mu\xi^{-p}\omega_1(F,\xi)_p^p}{(p+1)}\left[\sum_{j=1}^{r}|\alpha_j|\, j^{p+2}\xi^{3p-2\overline{\beta}+1}\int_0^{\infty}\left(\frac{t}{\xi}\right)^{3p}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{\left(\frac{t}{\xi}\right)}\right)^{2\overline{\beta}} d\frac{t}{\xi}\right]$$

$$= \frac{2^{p+1}\mu\xi^{2p-2\overline{\beta}+1}\omega_1(F,\xi)_p^p}{(p+1)}\left[\sum_{j=1}^{r}|\alpha_j|\, j^{p+2}\right]\left[\int_0^{\infty}x^{3p}\left(\frac{\sin x}{x}\right)^{2\overline{\beta}}dx\right] \tag{87}$$

$$= \frac{2^{p+1}\mu\xi^{2p-2\overline{\beta}+1}\omega_1(F,\xi)_p^p}{(p+1)}\left[\sum_{j=1}^{r}|\alpha_j|\, j^{p+2}\right]$$

$$\cdot\left[\int_0^{1}x^{3p}\left(\frac{\sin x}{x}\right)^{2\overline{\beta}}dx+\int_1^{\infty}x^{3p}\left(\frac{\sin x}{x}\right)^{2\overline{\beta}}dx\right]$$

$$\le \frac{2^{p+1}\mu\xi^{2p-2\overline{\beta}+1}\omega_1(F,\xi)_p^p}{(p+1)}\left(\sum_{j=1}^{r}|\alpha_j|\, j^{p+2}\right)$$

$$\cdot\left[\int_0^{1}x^{3}\left(\frac{\sin x}{x}\right)^{2\overline{\beta}}dx+\int_1^{\infty}x^{\lceil 3p\rceil}\left(\frac{\sin x}{x}\right)^{2\overline{\beta}}dx\right]$$

$$\leq \frac{2^{p+1}\mu\xi^{2p-2\overline{\beta}+1}\omega_1(F,\xi)_p^p}{(p+1)}\left(\sum_{j=1}^{r}|\alpha_j|\,j^{p+2}\right) \tag{88}$$

$$\cdot\left[\int_0^\infty x^3\left(\frac{\sin x}{x}\right)^{2\overline{\beta}}dx+\int_0^\infty x^{\lceil 3p\rceil}\left(\frac{\sin x}{x}\right)^{2\overline{\beta}}dx\right]$$

(we use [11], p. 210, item 1033)

$$= \frac{2^{p+1}\mu\xi^{2p-2\overline{\beta}+1}\omega_1(F,\xi)_p^p}{(p+1)}\left(\sum_{j=1}^{r}|\alpha_j|\,j^{p+2}\right)$$

$$\cdot\left[\left(\frac{-(2\overline{\beta})!}{8(2\overline{\beta}-4)!}\sum_{k=1}^{\overline{\beta}}(-1)^{\overline{\beta}-k}\frac{k^{2\overline{\beta}-4}\ln(2k)}{(\overline{\beta}-k)!(\overline{\beta}+k)!}\right)+\psi_{\lceil 3p\rceil}\right]$$

$$= \frac{2^{p+1}\xi^{2p}\omega_1(F,\xi)_p^p}{(p+1)}\left(\sum_{j=1}^{r}|\alpha_j|\,j^{p+2}\right)$$

$$\cdot\left[\left(\frac{-(2\overline{\beta})!}{8(2\overline{\beta}-4)!}\sum_{k=1}^{\overline{\beta}}(-1)^{\overline{\beta}-k}\frac{k^{2\overline{\beta}-4}\ln(2k)}{(\overline{\beta}-k)!(\overline{\beta}+k)!}\right)+\psi_{\lceil 3p\rceil}\right]$$

$$\cdot\frac{1}{\lambda_1}\left(\sum_{j=0}^{r}|\alpha_j|\,j^2\right)^{p-1}\frac{2^p}{|\beta^2-\alpha^2|^p(q+1)^{p-1}} \tag{89}$$

$$= \frac{2^{2p+1}}{\lambda_1|\beta^2-\alpha^2|^p(q+1)^{p-1}(p+1)}\left(\sum_{j=0}^{r}|\alpha_j|\,j^2\right)^{p-1}\left(\sum_{j=1}^{r}|\alpha_j|\,j^{p+2}\right)$$

$$\left[\left(\frac{-(2\overline{\beta})!}{8(2\overline{\beta}-4)!}\sum_{k=1}^{\overline{\beta}}(-1)^{\overline{\beta}-k}\frac{k^{2\overline{\beta}-4}\ln(2k)}{(\overline{\beta}-k)!(\overline{\beta}+k)!}\right)+\psi_{\lceil 3p\rceil}\right]\omega_1(F,\xi)_p^p\,\xi^{2p}<\infty.$$

The claim is proved. □

Next we give a consequence of Theorem 5.

Corollary 5. *(to Theorem 5) It holds*

$$\|T_{r,\xi}(f)-f\|_p\leq\Phi_3(\xi)+$$

$$+\|f''\|_p\left(\frac{2\xi^{2\overline{\beta}-1}}{\lambda_1|\beta^2-\alpha^2|}\right)\left[\sum_{j=0}^{r}|\alpha_j|\int_0^\infty|\cos(\alpha jt)-\cos(\beta jt)|\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right]$$

$$+\left(\frac{4\left\|f^{(4)}\right\|_p+\left(\alpha^2+\beta^2\right)\|f''\|_p}{(\alpha\beta)^2|\beta^2-\alpha^2|}\right)\frac{\xi^{2\overline{\beta}-1}}{\lambda_1}$$

$$\cdot\left[\beta^2\sum_{j=0}^{r}|\alpha_j|\left(\int_0^\infty\sin^2\left(\frac{\alpha jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)+\right.$$

$$\alpha^2 \sum_{j=0}^{r} |\alpha_j| \left(\int_0^\infty \sin^2 \left(\frac{\beta j t}{2} \right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt \right) \Bigg] \to 0, \quad (90)$$

as $\xi \to 0$. Above $\Phi_3(\xi)$ is as in (81).

Proof. We have that

$$\frac{\xi^{2\overline{\beta}-1}}{\lambda_1} \int_0^\infty |\cos(\alpha j t) - \cos(\beta j t)| \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt$$

$$= \frac{2\xi^{2\overline{\beta}-1}}{\lambda_1} \int_0^\infty \left| \sin\left(\frac{(\alpha+\beta) jt}{2} \right) \right| \left| \sin\left(\frac{(\alpha-\beta) jt}{2} \right) \right| \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt$$

$$\leq \frac{2\xi^{2\overline{\beta}-1}}{\lambda_1} \int_0^\infty \frac{|\alpha+\beta| jt}{2} \frac{|\alpha-\beta| jt}{2} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt \quad (91)$$

$$= \frac{\xi^{2\overline{\beta}-1}}{\lambda_1} \frac{\left|\alpha^2-\beta^2\right|}{2} j^2 \int_0^\infty t^2 \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt$$

$$= \frac{\xi^{2\overline{\beta}-1}}{\lambda_1} \xi^{3-2\overline{\beta}} \frac{\left|\alpha^2-\beta^2\right|}{2} j^2 \int_0^\infty \left(\frac{t}{\xi}\right)^2 \left(\frac{\sin\left(\frac{t}{\xi}\right)}{\left(\frac{t}{\xi}\right)} \right)^{2\overline{\beta}} d\frac{t}{\xi}$$

$$= \frac{\xi^2}{\lambda_1} \frac{\left|\alpha^2-\beta^2\right|}{2} j^2 \int_0^\infty x^2 \left(\frac{\sin x}{x} \right)^{2\overline{\beta}} dx = \frac{\xi^2}{\lambda_1} \frac{\left|\alpha^2-\beta^2\right|}{2} j^2 \psi_{12} \to 0, \quad \text{as } \xi \to 0.$$

Furthermore, we have that

$$\frac{\xi^{2\overline{\beta}-1}}{\lambda_1} \int_0^\infty \sin^2\left(\frac{\alpha j t}{2} \right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt \leq \frac{\xi^{2\overline{\beta}-1}}{\lambda_1} \frac{\alpha^2 j^2}{4} \int_0^\infty t^2 \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt \quad (92)$$

$$= \frac{\xi^2}{\lambda_1} \frac{\alpha^2 j^2}{4} \psi_{12} \to 0, \quad \text{as } \xi \to 0.$$

The proof now is clear. □

Next comes the corresponding L_1 approximation result.

Theorem 6. *Let $\overline{\beta} \in \mathbb{N} : \overline{\beta} > 2$, and $\alpha, \beta \in \mathbb{R} : \alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$; $f, f', f'', f^{(3)}$, $f^{(4)} \in L_1(\mathbb{R}) \cap C(\mathbb{R})$; $\xi > 0$, $x \in \mathbb{R}$. Then*

$$\|E_3\|_1 = \| T_{r,\xi}(f, x) - f(x) -$$

$$\left(\frac{2f''(x)}{\beta^2 - \alpha^2} \right) \frac{1}{\lambda} \left[\sum_{j=0}^{r} \alpha_j \left(\int_0^\infty (\cos(\alpha j t) - \cos(\beta j t)) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt \right) \right] \quad (93)$$

$$-\left(\frac{4\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\right)\frac{1}{\lambda}$$

$$\cdot\left[\beta^2\sum_{j=0}^{r}\alpha_j\left(\int_0^\infty\sin^2\left(\frac{\alpha jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right.$$

$$\left.\left.-\alpha^2\sum_{j=0}^{r}\alpha_j\left(\int_0^\infty\sin^2\left(\frac{\beta jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right]\right\|_1$$

$$\leq\left[\frac{4}{|\beta^2-\alpha^2|\lambda_1}\right]\left[\sum_{j=1}^{r}|\alpha_j|j^2\left[\psi_{12}+\frac{j}{2}\psi_{23}\right]\right]\omega_1\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f,\xi\right)_1\cdot\xi^2$$

$$=:\Psi_3(\xi)\to 0,$$

as $\xi\to 0$.

Proof. We have that (F is as in (25))

$$\int_{-\infty}^{\infty}|E_3(x)|\,dx\overset{(82)}{=}\int_{-\infty}^{\infty}\left(\frac{1}{\lambda}\left|\int_{-\infty}^{\infty}R(t)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right|\right)dx$$

$$\leq\int_{-\infty}^{\infty}\left(\frac{1}{\lambda}\int_{-\infty}^{\infty}|R(t)|\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)dx$$

$$\overset{(30)}{\leq}\frac{2}{|\beta^2-\alpha^2|\lambda}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left[\sum_{j=1}^{r}|\alpha_j|j^2\left(\int_0^{|t|}|F(x+jsign(t)\theta)\right.\right.\right.$$

$$\left.\left.\left.-F(x)|\left(|t|-\theta\right)d\theta\right)\right]\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)dx \quad (94)$$

$$\leq\frac{2}{|\beta^2-\alpha^2|\lambda}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left[\sum_{j=1}^{r}|\alpha_j|j^2\left(\int_0^{|t|}|F(x+jsign(t)\theta)\right.\right.\right.$$

$$\left.\left.\left.-F(x)|\,d\theta\right)|t|\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)dx\right]$$

$$=\frac{2}{|\beta^2-\alpha^2|\lambda}\left[\sum_{j=1}^{r}|\alpha_j|j^2\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(\int_{-\infty}^{\infty}|F(x+jsign(t)\theta)\right.\right.\right.$$

$$\left.\left.\left.-F(x)|\,dx\right)d\theta\right)|t|\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right]$$

$$\leq \frac{2}{|\beta^2-\alpha^2|\lambda}\left[\sum_{j=1}^{r}|\alpha_j|j^2\int_{-\infty}^{\infty}\left(\int_0^{|t|}\omega_1\left(F,\xi\frac{j\theta}{\xi}\right)_1 d\theta\right)|t|\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right]$$

$$\leq \frac{2}{|\beta^2-\alpha^2|\lambda}\omega_1(F,\xi)_1\left[\sum_{j=1}^{r}|\alpha_j|j^2\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(1+\frac{j\theta}{\xi}\right)d\theta\right)|t|\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right]$$

$$= \frac{2}{|\beta^2-\alpha^2|\lambda}\omega_1(F,\xi)_1\left[\sum_{j=1}^{r}|\alpha_j|j^2\int_{-\infty}^{\infty}\left(|t|+\frac{j}{2\xi}t^2\right)|t|\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right] \quad (95)$$

$$= \frac{4}{|\beta^2-\alpha^2|\lambda}\omega_1(F,\xi)_1\left[\sum_{j=1}^{r}|\alpha_j|j^2\int_{0}^{\infty}\left(t+\frac{j}{2\xi}t^2\right)t\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right]$$

$$= \frac{4}{|\beta^2-\alpha^2|\lambda}\omega_1(F,\xi)_1\sum_{j=1}^{r}|\alpha_j|j^2\left[\int_0^{\infty}t^2\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt+\frac{j}{2\xi}\int_0^{\infty}t^3\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right]$$

$$= \frac{4\xi^{3-2\overline{\beta}}}{|\beta^2-\alpha^2|\lambda}\omega_1(F,\xi)_1\left[\sum_{j=1}^{r}|\alpha_j|j^2\left[\int_0^{\infty}x^2\left(\frac{\sin x}{x}\right)^{2\overline{\beta}}dx+\frac{j}{2}\int_0^{\infty}x^3\left(\frac{\sin x}{x}\right)^{2\overline{\beta}}dx\right]\right]$$

(we use [11], p. 210, formula 1033)

$$= \frac{4\xi^2}{|\beta^2-\alpha^2|\lambda_1}\omega_1(F,\xi)_1\left[\sum_{j=1}^{r}|\alpha_j|j^2\left[\psi_{12}+\frac{j}{2}\psi_{23}\right]\right]<\infty. \quad (96)$$

The claim is proved. □

We finish this work with the following result.

Corollary 6. *(to Theorem 6) It holds*

$$\|T_{r,\xi}(f)-f\|_1 \leq \Psi_3(\xi)$$

$$+\|f''\|_1\left(\frac{2\xi^{2\overline{\beta}-1}}{\lambda_1|\beta^2-\alpha^2|}\right)\left[\sum_{j=0}^{r}|\alpha_j|\int_0^{\infty}|\cos(\alpha jt)-\cos(\beta jt)|\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right] \quad (97)$$

$$+\left(\frac{4\left\|f^{(4)}\right\|_1+\left(\alpha^2+\beta^2\right)\|f''\|_1}{(\alpha\beta)^2|\beta^2-\alpha^2|}\right)\frac{\xi^{2\overline{\beta}-1}}{\lambda_1}\left[\beta^2\sum_{j=0}^{r}|\alpha_j|\left(\int_0^{\infty}\sin^2\left(\frac{\alpha jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right.$$

$$\left.+\alpha^2\sum_{j=0}^{r}|\alpha_j|\left(\int_0^{\infty}\sin^2\left(\frac{\beta jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right]\to 0,$$

as $\xi \to 0$. Above $\Psi_3(\xi)$ is as in (93).

Proof. As similar to Corollary 5 is omitted. □

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