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ASYMPTOTIC PARTITION OF ENERGIES FOR A COSSERAT THERMOELASTIC MEDIUM

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Dedicated to Professor Marin Marin on the occasion of his 70th anniversary

Abstract

The main aim of this study is to obtain a partition of the asymptotic type of energy of a solution for the mixed problem considered in the context of the Cosserat thermoelastic media. The concept of asymptotic equipartition is a notion, frequently used, for differential equations theory. In a simple formulation, this concept is formulated as follows: potential and kinetic energy, for a classical solution with finite energy, tend to become asymptotically equal on average, when time tends to infinity.

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1. Introduction

There is a large number of different works which approaches the asymptotic equipartition in the case of physical systems whose evolution is governed by nondisipative partial differential equations of hyperbolic type or systems of such type of equations. In our paper we study the asymptotic equipartition of energy for a solution of the mixed problem with initial and boundary values within the context of the theory of Cosserat thermoelastic media. In our mixed problem the basic equations are of the hyperbolic type, with dissipation, what we did not find in the works already published on this topic. In this context, we will use a dissipative mechanism in order to prove that the equipartition asymptotic occurs between the mean strain and kinetic energies.

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We will not approach this issue in its abstract version, but we will prefer to make it concrete in the practical situation of the thermoelastic Cosserat bodies.

The structure of our study is following one. First we will state the basic equations and conditions, initial and boundary, for the mixed problem in the context of the linear theory of Cosserat termoelastic bodies. After that, we prove some identities of the Lagrange type and define some Cesaro means of different parts of total energy, associated to the solutions of our problem. Based on previous estimations, we finally will prove some estimate that characterises the asymptotic behaviour in mean of the different parts of the total energy.

It is necessary to specify that many studies have been published that address different variants of Lagrange's identity, such as Levine [6], Rionero and Chirita [12], Gurtin [3], Marin [11] and so on. There are many published works that use means of Cesaro type, in different context, as Levine [6], Day [2] and so on.

2. Main equations and conditions

We will work on D, which is a domain of the Euclidian space R^3, which in its reference configuration is occupied by a homogeneous Cosserat material. It is supposed that the domain D is regular and has the boundary ∂D and the closure $\bar{D}$ so that $D = \partial D \cup \bar{D}$. A rectangular system of axes is used and the Cartesian vector and tensor notation are adopted. Any point in D is identifiable by its coordinates x_j and by $t \in [0, \infty)$, where t is the temporal variable.

If there is no risk of confusion, then the specification of the dependence of a function upon its temporal variable or on the spatial variables can be omitted.

To describe the evolution of a Cosserat thermoelastic medium, the variables v_m, ϕ_m and ϑ will be used, that is, the components of the displacement, the components of the microrotation and the variation of the temperature, respectively.

The strain tensors are introduced by means of the following kinematic relations:

$$\begin{aligned} e_{mn} &= v_{n,m} + \varepsilon_{nmk}\phi_k, \\ \varepsilon_{mn} &= \phi_{n,m}. \end{aligned} \tag{1}$$

The components of stress tensor τ_{mn}, the components of couple stress σ_{mn}, the components of the heat conduction vector q_m and the specific entropy η are introduced by means of the constitutive relations. So, if we suppose that our solid has in each point of the reference state a center of symmetry and is otherwise non-isotropic, then the constitutive relations have the following form:

$$\begin{aligned} \tau_{mn} &= A_{mnkl}e_{kl} + B_{mnkl}\varepsilon_{kl} + \alpha_{mn}(\vartheta + \alpha\dot{\vartheta}), \\ \sigma_{mn} &= B_{mnkl}e_{kl} + C_{mnkl}\varepsilon_{kl} + \beta_{mn}(\vartheta + \alpha\dot{\vartheta}), \\ q_m &= -\vartheta_0 k_{mn}\vartheta_{,n}, \\ \varrho\eta &= a + d\vartheta + h\dot{\vartheta} - \alpha_{mn}e_{mn} - \beta_{mn}\varepsilon_{mn}, \end{aligned} \tag{2}$$

where all these equations having place for $(t, x) \in [0, \infty) \times D$.

The tensors A_{mnkl}, B_{mnkl}, ..., in (2) are constants which satisfy the following symmetry relations:

$$A_{mnkl} = A_{klmn},\ C_{mnkl} = C_{klmn},\ k_{mn} = k_{nm}. \tag{3}$$

In the case that the volume force, the volume body couple and supply of heat are not present, the basic equations in linear theory of the thermoelasticity of a Cosserat body are, (see [5]):

- the motion equations:

$$\tau_{mn,n} = \varrho \ddot{v}_m, \tag{4}$$

$$\sigma_{mn,n} + \varepsilon_{mjk}\tau_{jk} = I_{mn}\ddot{\phi}_n; \tag{5}$$

- the equation of energy:

$$q_{m,m} = -\varrho\vartheta_0\dot{\eta}, \tag{6}$$

where these equations having place for $(t, x) \in [0, \infty) \times D$.

The notations used in the previous relations have the following meanings: ϱ-the constant density in the reference state, ϑ_0-the constant temperature in the initial state, I_{mn}- the inertia tensor and ε_{ijk}-the Ricci's tensor, i.e., the alternating symbol.

In order to designate the differentiation of a function with respect to time t a superposed dot is used, and a subscript preceded by a comma designates the differentiation of a function with respect to the corresponding spatial variable.

The constants ϱ, I_{mn} and ϑ_0 satisfy the conditions:

$$\varrho > 0,\ \vartheta_0 > 0,\ I_{mn} > 0. \tag{7}$$

From the Clausius-Duhem inequality, that is, the inequality of entropy production, the following conditions are obtained:

$$d\alpha - h \geq 0,\ k_{mn}x_m x_n \geq 0,\ \ \forall x_m, \tag{8}$$

from which the positive definition of tensors $A_{mnkl}, C_{mnkl}, k_{mn}$ is deduced, that is:

$$\begin{aligned} &A_{mnkl}x_{mn}x_{kl} \geq k_0 x_{mn}x_{mn},\ k_0 > 0,\ \ \forall x_{mn} = x_{nm},\\ &C_{mnkl}x_{mn}x_{kl} \geq k_1 x_{mn}x_{mn},\ k_1 > 0,\ \ \forall x_{mn} = x_{nm},\\ &k_{mn}x_m x_n \geq k_2 x_m x_m,\ k_2 > 0,\ \ \forall x_m. \end{aligned} \tag{9}$$

According to a suggestion from [11], it can be supposed that:

$$\alpha > 0,\ h > 0,\ d\alpha - h > 0. \tag{10}$$

In order to complete the mixed problem, the following boundary conditions are prescribed:

$$\begin{aligned} &v_m = 0\ on\ [0, \infty) \times \partial D_1,\ \tau_m \equiv \tau_{mk}n_k = 0\ on\ [0, \infty) \times \partial D_1^c,\\ &\phi_m = 0\ on\ [0, \infty) \times \partial D_2,\ \sigma_m \equiv \sigma_{mk}n_k = 0\ on\ [0, \infty) \times \partial D_2^c,\\ &\vartheta = 0\ on\ [0, \infty) \times \partial D_3,\ q \equiv q_k n_k = 0\ on\ [0, \infty) \times \partial D_3^c, \end{aligned} \tag{11}$$

where $\partial D_1, \partial D_2, \partial D_3$ and $\partial D_1^c, \partial D_2^c, \partial D_3^c$ are subsets of ∂D and their complements with respect to ∂D, so that:

$$\partial D_1 \cup \partial D_1^c = \partial D_2 \cup \partial D_2^c = \partial D_3 \cup \partial D_3^c = \partial D,$$
$$\partial D_1 \cap \partial D_1^c = \partial D_2 \cap \partial D_2^c = \partial D_3 \cap \partial D_3^c = \emptyset,$$

and n_i are the components of the unit outward normal to ∂D.

For the same mixed problem, the initial data are attached, in their most general form:

$$\begin{aligned} &v_m(0,x) = v_m^0(x),\ \dot{v}_m(0,x) = v_m^1(x),\ \phi_m(0,x) = \phi_m^0(x), \\ &\dot{\phi}_m(0,x) = \phi_m^1(x),\ \vartheta(0,x) = \vartheta^0(x),\ \dot{\vartheta}(0,x) = \vartheta^1(x). \end{aligned} \tag{12}$$

If we take into account the constitutive relations (2), from equations (4), (5) and (6), we are led to the following system of differential equations:

$$\begin{aligned} \varrho \ddot{v}_m &= A_{mnkl} e_{kl,n} + B_{mnkl} \varepsilon_{kl,n} + \alpha_{mn}(\vartheta_{,n} + \alpha \dot{\vartheta}_{,n}), \\ I_{mn} \ddot{\phi}_n &= B_{mnkl} e_{kl,n} + C_{mnkl} \varepsilon_{kl,n} + \beta_{mn}(\vartheta_{,n} + \alpha \dot{\vartheta}_{,n}) \\ &\quad + \varepsilon_{mjk}(A_{jknl} e_{nl} + B_{jknl} \varepsilon_{nl} + \alpha_{jk}(\vartheta + \alpha \dot{\vartheta})), \\ h \ddot{\vartheta} &= -d \dot{\vartheta} + \alpha_{mn} \dot{e}_{mn} + \beta_{mn} \dot{\varepsilon}_{mn} + k_{mn} \vartheta_{,mn}, \end{aligned} \tag{13}$$

where all these equations having place for $(t,x) \in [0,\infty) \times D$.

An ordered array (v_m, ϕ_m, ϑ) is called a solution of the mixed problem in the thermoelasticity of Cosserat bodies, considered in the cylinder $\Omega_0 = [0,\infty) \times D$, if it satisfies the system of partial differential equations (13) for all $(t,x) \in \Omega_0$, the boundary conditions (11) and the initial data (12).

In the conditions in which it is assumed that $meas\, \partial D_1 = 0$ and $meas\, \partial D_2 = 0$, it was found that there is a family of motions, which are rigid, and a temperature null that satisfies the equations (13) and the boundary relations (11). As such, the initial data $v_m^0,\ \phi_m^0,\ v_m^1,\ \phi_m^1$, can be decomposed as follows:

$$\begin{aligned} &v_m^0 = v_m^{0*} + V_m^0,\ v_m^1 = v_m^{1*} + V_m^1, \\ &\phi_m^0 = \phi_m^{0*} + \Phi_m^0,\ \phi_m^1 = \phi_m^{1*} + \Phi_m^1, \end{aligned} \tag{14}$$

in which $v_m^{0*},\ \phi_m^{0*},\ v_m^{1*},\ \phi_m^{1*}$ can be computed knowing that $V_m^0,\ \Phi_m^0,\ V_m^1,\ \dot{\Phi}_m^1$ verify the following equations:

$$\begin{aligned} &\int_D \varrho V_m^0 dV = 0,\ \int_D \varrho(\varepsilon_{mnk} x_n V_m^0 + \Phi_m^0) dV = 0, \\ &\int_D \varrho V_m^1 dV = 0,\ \int_D \varrho(\varepsilon_{mnk} x_n V_k^1 + \Phi_m^1) dV = 0. \end{aligned} \tag{15}$$

If the case that $meas\, \partial D_1 = 0$ and $meas\, \partial D_2 \neq 0$, then only must be imposed the conditions:

$$\int_D \varrho V_m^0 dV = 0,\ \int_D \varrho V_m^1 dV = 0.$$

In the last situation, that is, *meas* $\partial D_3 = 0$, it was found that there is a set of null motions and constant temperatures, which verify the equations (13) and the boundary conditions (11). This is the reason why the initial temperature data, ϑ^0, $\dot{\vartheta}^0$, can be decomposed as follows:

$$\vartheta^0 = \vartheta^* + T^0, \ \dot{\vartheta}^0 = \dot{\vartheta}^* + T^1, \tag{16}$$

where the constants ϑ^* and $\dot{\vartheta}^*$ can be computed so that:

$$\int_D T^0 dV = 0, \ \int_D T^1 dV = 0. \tag{17}$$

3. Preliminaries

The set of scalar functions which admits derivatives up to the n-th order in D, these being continuous on the domain D, is denoted by $C^n(D)$. We define the norm for a function $u \in C^n(D)$ by:

$$\|u\| = \sum_{k=0}^{n} \sum_{i_1, i_2, \ldots, i_k} \max_D |u_{,i_1 \ldots i_k}|.$$

The set of vector functions with six components, each being an element of $C^n(D)$, is denoted by $\mathbf{C}^n(D)$. For a vector function $\mathbf{w} \in \mathbf{C}^n(D)$, $\mathbf{w} = (w_k), k = \overline{1,6}$, the norm is defined by:

$$\|\mathbf{w}\| = \sum_{k=1}^{6} \|w_k\|_{C^n(D)}.$$

Let us denote by $\|.\|_{W_n(D)}$ the norm induced by the following inner product:

$$(u, v)_{W_n(D)} = \sum_{k=0}^{n} \int_D u_{,i_1 \ldots i_k} v_{,i_1 \ldots i_k} dV.$$

The completion of the space $C^n(D)$ in relation to this norm is a Hilbert space denoted by $W_n(D)$.

Now, we denote by $\|.\|_{\mathbf{W}_n(D)}$ the norm induced by the following inner product:

$$(\mathbf{v}, \mathbf{w})_{\mathbf{W}_n(D)} = \sum_{k=1}^{6} (v_k, w_k)_{W_n(D)}.$$

The completion of the space $\mathbf{C}^n(D)$ in relation to this norm is denoted by $\mathbf{W}_n(D)$. For a Cartesian product of some normed spaces, the norm will be the sum of the norms of the factor spaces.

In what follows, we will use the next notations:

$\hat{C}^1(D) = \{\omega \in C^1(D) : \omega = 0 \text{ on } \partial D_3; \text{ if } meas\, \partial D_3 = 0, \text{ then } \int_D \omega dV = 0\};$
$\hat{\mathbf{C}}^1(D) \equiv \{(v_m, \phi_m) \in \mathbf{C}^1(D) : v_m = 0 \text{ on } \partial D_1,\ \phi_m = 0 \text{ on } \partial D_2;$
if $meas\, \partial D_1 = meas\, \partial D_2 = 0$, then
$\int_D \varrho v_m dV = 0,\ \int_D \varrho(\varepsilon_{mnk} x_n v_k + \phi_m) dv = 0;$
if meas $\partial D_1 = 0$ and meas $\partial D_2 \neq 0 \Rightarrow \int_D \varrho v_m dV = 0\};$
$\hat{W}_1(D) \equiv$ the completion of the space $\hat{C}^1(D)$ by means of $\|.\|_{W_1(D)}$;
$\hat{\mathbf{W}}^1(D) \equiv$ the completion of $\hat{\mathbf{C}}^1(D)$ by means of $\|.\|_{\mathbf{W}_1(D)}$.

In the relations above $W_m(D)$ is the known Sobolev space, see [1], and the notation $\mathbf{W}_n(D) = [W_n(D)]^6$ was used. It should be emphasized that assumption (8) guarantees validity of the Korn's inequality that follows, [4], for all $(w, \psi) \in \hat{\mathbf{W}}_1(D)$,

$$\begin{aligned} &\int_D [A_{mnkl} e_{mn}(w,\psi) e_{kl}(w,\psi) + 2B_{mnkl} e_{mn}(w,\psi) \varepsilon_{kl}(w,\psi) \\ &\quad + C_{mnkl} \varepsilon_{mn}(w,\psi) \varepsilon_{kl}(w,\psi)] dV \geq \\ &\geq m_1 \int_D (w_m w_m + w_{m,n} w_{m,n} + \psi_m \psi_m + \psi_{m,n} \psi_{m,n}) dV, \end{aligned} \tag{18}$$

where $m_1 > 0, m_1 = \text{const.}$ and

$$e_{mn}(w,\psi) = w_{n,m} + \varepsilon_{mnk} \psi_k,$$
$$\varepsilon_{mn}(w,\psi) = \psi_{n,m}.$$

Under the hypothesis (8), for all $\xi \in \hat{W}_1(D)$ the following Poincare's inequality holds

$$\int_D k_{mn} \xi_{,m} \xi_{,n} dV \geq m_2 \int_D \xi^2 dV,\ m_2 > 0. \tag{19}$$

If simultaneously we have $meas\, \partial D_1 = 0$ and $meas\, \partial D_2 = 0$, it will be seen that the decomposition of solution (v_m, ϕ_m, ϑ) is useful, as follows:

$$v_m = v_m^* + t\dot{v}_m^* + w_m,\ \phi_m = \phi_m^* + t\dot{\phi}_m^* + \psi_m,\ \vartheta = \chi, \tag{20}$$

where $((w_m, \psi_m), \chi) \in \hat{\mathbf{W}}_1(D) \times \hat{W}_1(D)$ is a solution of the system of equations (12), with the boundary conditions (11) and satisfying the initial data:

$$v_m = U_m^0,\ \dot{v}_m = \dot{U}_m^0,\ \phi_m = \Phi_m^0,\ \dot{\phi}_m = \dot{\Phi}_m^0,\ \chi = \vartheta^0,\ \dot{\chi} = \dot{\vartheta}^0, \text{ on } D, \text{ at } t = 0.$$

Consider now that $meas\, \partial D_3 = 0$. As such it is possible to use the decompositions (16) and (17) as well as the energy equation (3) in order to obtain a new decomposition of the solution $((v_m, \phi_m), \vartheta)$ as follows:

$$v_m = w_m, \phi_m = \psi_m, \vartheta = \vartheta^* + \frac{h}{d}[1 - e^{-dt/h}]\dot{\vartheta}^* + \chi, \tag{21}$$

where $((v_m, \psi_m), \chi) \in \hat{\mathbf{W}}_1(D) \times \hat{W}_1(D)$ is a solution of the system of equations (12), with the boundary conditions (11) and satisfying the initial data:

$$w_m = V_m^0, \dot{w}_m = V_m^1, \psi_m = \phi_m^0, \dot{\psi}_m = \phi_m^1, \chi = T^0, \dot{\chi} = T^1,$$

on the domain D, at the initial moment $t = 0$.

4. Some auxiliary identities

In the present section a few evolutionary identities, of integral type, will be obtained. These will be the basis of some relations, which in turn, are essential for obtaining the equipartition of the total energy, of asymptotic type. Thus, in the first theorem a law on energy conservation is proven.

Theorem 1. *Consider a solution $((v_m, \phi_m), \vartheta)$ of the mixed problem defined by the equations (13), the boundary conditions (11) and the initial data (12). It is also assumed that the initial data satisfies*

$$(v_m^0, \phi_m^0) \in \mathbf{W}_1(D), (v_m^1, \phi_m^1) \in \mathbf{W}_0(D), \vartheta^0 \in W_1(D), \vartheta^1 \in W_0(D).$$

Then the following law of energy conservation holds:

$$\begin{aligned} E(t) \equiv \frac{1}{2}\int_D [&\varrho \dot{v}_m(t)\dot{v}_m(t) + I_{mn}\dot{\phi}_m(t)\dot{\phi}_n(t) + A_{mnkl}e_{mn}(t)e_{kl}(t) \\ &+ B_{mnkl}e_{mn}(t)\varepsilon_{kl}(t) + C_{mnkl}\varepsilon_{mn}(t)\varepsilon_{kl}(t) + \alpha k_{mn}\vartheta_{,m}(t)\vartheta_{,n}(t) \\ &+ d\vartheta^2(t) + \alpha h\dot{\vartheta}^2(t) + 2h\vartheta(t)\dot{\vartheta}(t) \\ &+ \int_0^t\int_D [k_{mn}\vartheta_{,m}(s)\vartheta_{,n}(s) + (d\alpha - h)\dot{\vartheta}^2(s)]dV ds = E(0), \end{aligned} \tag{22}$$

for $t \in [0, \infty)$.

Proof. Considering the equations $(12)_1$ and $(12)_2$ we get

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}\left[\varrho \dot{v}_m\dot{v}_m + I_{mn}\dot{\phi}_m\dot{\phi}_n\right] &= \left(\dot{v}_m\tau_{nm} + \dot{\phi}_m\sigma_{nm}\right)_{,n} - \\ &-A_{mnkl}e_{mn}\dot{e}_{kl} - B_{mnkl}\left(\varepsilon_{mn}\dot{e}_{kl} + \dot{\varepsilon}_{mn}e_{kl}\right) - C_{mnkl}\varepsilon_{mn}\dot{\varepsilon}_{kl} - \\ &-\alpha_{mn}\left(\vartheta + \alpha\dot{\vartheta}\right)\dot{e}_{mn} - \beta_{mn}\left(\vartheta + \alpha\dot{\vartheta}\right)\dot{\varepsilon}_{mn}. \end{aligned} \tag{23}$$

Now, we take into account the equation of energy $(12)_3$ and obtain:

$$\begin{aligned} &\alpha_{mn}\left(\vartheta + \alpha\dot{\vartheta}\right)\dot{e}_{mn} + \beta_{mn}\left(\vartheta + \alpha\dot{\vartheta}\right)\dot{\varepsilon}_{mn} \\ &\quad = \frac{1}{2}\frac{d}{ds}\left[d\vartheta^2 + \alpha k_{mn}\vartheta_{,m}\vartheta_{,n} + \alpha h\dot{\vartheta}^2 + 2\vartheta\dot{\vartheta}\right] \\ &\quad\quad - \left[k_{mn}\vartheta_{,n}(\vartheta + \alpha\dot{\vartheta})\right]_{,m} + k_{mn}\vartheta_{,m}\vartheta_{,n} + (d\alpha - h)\dot{\vartheta}^2. \end{aligned} \tag{24}$$

Finally, we integrate the equalities (23) and (24), over a cylinder $[0, t] \times D$ and take into account the boundary conditions (11) and the initial conditions (13). Thus, we find the desired conservation law (22). □

Theorem 2. *Consider a solution $((v_m, \phi_m), \vartheta)$ of the mixed problem defined by the equations (13), the boundary conditions (11) and the initial data (12). It is also assumed that the initial data satisfies*

$$\left(v_m^0, \phi_m^0\right) \in \mathbf{W}_1(D), \left(v_m^1, \phi_m^1\right) \in \mathbf{W}_0(D), \vartheta^0 \in W_1(D), \vartheta^1 \in W_0(D).$$

Then the following identity holds

$$
\begin{aligned}
&2\int_D \left[\varrho v_m(t)\dot{v}_m(t)+I_{mn}\phi_m(t)\dot{\phi}_n(t)\right]dV+2\int_D[(d\alpha-h)\vartheta^2(t)\\
&+k_{mn}\left(\int_0^t \vartheta_{,m}(\xi)d\xi\right)\left(\int_0^t \vartheta_{,n}(\xi)d\xi\right)+2\alpha k_{mn}\vartheta_{,m}(t)\left(\int_0^t \vartheta_{,n}(\xi)d\xi\right)]dV\\
&=2\int_0^t\int_D[\varrho\dot{v}_m(s)\dot{v}_m(s)+I_{mn}\dot{\phi}_m(s)\dot{\phi}_n(s)-A_{mnkl}e_{mn}(s)e_{kl}(s)\\
&-2B_{mnkl}e_{mn}(s)\varepsilon_{kl}(s)-C_{mnkl}\varepsilon_{mn}(s)\varepsilon_{kl}(s)-d\vartheta^2(s)-2h\vartheta(s)\dot{\vartheta}(s)\\
&-\alpha h\dot{\vartheta}^2(s)-\alpha k_{mn}\vartheta_{,m}(s)\vartheta_{,n}(s)]dV ds+2\int_D\left[\varrho v_m^0 v_m^1+I_{mn}\phi_m^0\phi_n^1\right]dV\\
&+\int_D(d\alpha-h)(\vartheta^0)^2(t)dV-2\int_0^t\int_D(a-\varrho\eta^0)[\vartheta(s)+\alpha\dot{\vartheta}(s)]dV ds,
\end{aligned}
\tag{25}
$$

where

$$\varrho\eta^0=a+d\vartheta^0+h\vartheta^1-\alpha_{mn}e_{mn}^0-\beta_{mn}\varepsilon_{mn}^0,\ e_{mn}^0=v_{n,m}^0+e_{mnk}^0\phi_k^0,\ \varepsilon_{mn}^0=\phi_{n,m}^0.$$

Proof. First, by using the equations (11), we obtain

$$
\begin{aligned}
&\frac{d}{ds}[\varrho v_m\dot{v}_m+I_{mn}\phi_m\dot{\phi}_n]=(v_m\tau_{nm}+\phi_m\sigma_{nm})_{,n}-A_{mnkl}e_{mn}e_{kl}\\
&-2B_{mnkl}\varepsilon_{mn}e_{kl}-C_{mnkl}\varepsilon_{mn}\varepsilon_{kl}-\alpha_{nm}(\vartheta+\alpha\dot{\vartheta})e_{nm}\\
&-\beta_{nm}(\vartheta+\alpha\dot{\vartheta})\varepsilon_{nm}+\varrho\dot{v}_m\dot{v}_m+I_{mn}\dot{\phi}_m\dot{\phi}_n.
\end{aligned}
\tag{26}
$$

After that, the equation of energy (12) is used to reach:

$$
\begin{aligned}
&\alpha_{nm}(\vartheta+\alpha\dot{\vartheta})e_{nm}+\beta_{nm}(\vartheta+\alpha\dot{\vartheta})\varepsilon_{nm}=\alpha k_{mn}[\dot{\vartheta}_{,m}\int_0^s\vartheta_{,n}(\xi)d\xi+\vartheta_{,m}\vartheta_{,n}]\\
&-\alpha k_{mn}\vartheta_{,m}\vartheta_{,n}+k_{mn}(\vartheta_{,m}+\alpha\dot{\vartheta}_{,m})\int_0^s\vartheta_{,n}(\xi)d\xi-[k_{mn}(\vartheta+\alpha\dot{\vartheta})\int_0^s\vartheta_{,n}(\xi)d\xi]_{,m}\\
&+\alpha k_{mn}[\dot{\vartheta}_{,m}\int_0^s\vartheta_{,n}(\xi)d\xi+\vartheta_{,m}\vartheta_{,n}]+(d\alpha-h)\vartheta\dot{\vartheta}\\
&+k_{mn}\vartheta_{,m}\int_0^s\vartheta_{,n}(\xi)d\xi+d\vartheta^2+\alpha h\dot{\vartheta}^2+2h\vartheta\dot{\vartheta}+(a-\varrho\eta^0)(\vartheta+\alpha\dot{\vartheta}).
\end{aligned}
\tag{27}
$$

From (26) and (27) it results

$$
\begin{aligned}
&\frac{d}{ds}\left[\varrho v_m\dot{v}_m+I_{mn}\phi_m\dot{\phi}_n\right]=(v_m\tau_{nm}+\phi_m\sigma_{nm})_{,n}\\
&-A_{mnkl}e_{mn}e_{kl}-2B_{mnkl}\varepsilon_{mn}e_{kl}-C_{mnkl}\varepsilon_{mn}\varepsilon_{kl}\\
&+\varrho\dot{v}_m\dot{v}_m+I_{mn}\dot{\phi}_m\dot{\phi}_n+[k_{mn}(\vartheta+\alpha\dot{\vartheta})\int_0^s\vartheta_{,n}(\xi)d\xi]_{,m}\\
&-(a-\varrho\eta^0)(\vartheta+\alpha\dot{\vartheta})+\alpha k_{mn}\vartheta_{,m}\vartheta_{,n}-\alpha k_{mn}\left[\dot{\vartheta}_{,m}\int_0^s\vartheta_{,n}(\xi)d\xi+\vartheta_{,m}\vartheta_{,n}\right]\\
&-k_{mn}\vartheta_{,m}\int_0^s\vartheta_{,n}(\xi)d\xi-d\vartheta^2-\alpha h\dot{\vartheta}^2-2h\vartheta\dot{\vartheta}-(d\alpha-h)\vartheta\dot{\vartheta}.
\end{aligned}
\tag{28}
$$

An integration of the identity (28) over $B \times (0,t)$, followed by the use of the boundary conditions (10), the initial conditions (13) and the symmetry relations (5), lead to the identity (25) and the proof of Theorem 2 is complete. □

Theorem 3. *Consider a solution $((v_m,\phi_m),\vartheta)$ of the mixed problem defined by the equations (13), the boundary conditions (11) and the initial data (12). It is also assumed that the initial data satisfies*

$$\left(v_m^0,\phi_m^0\right)\in \mathbf{W}_1(D), \left(v_m^1,\phi_m^1\right)\in \mathbf{W}_0(D), \vartheta^0\in W_1(D), \vartheta^1\in W_0(D).$$

Then the following identity holds

$$\begin{aligned} 2\int_D \Big[\varrho v_m(t)\dot v_m(t) + I_{mn}\phi_m(t)\dot\phi_n(t)\Big]\,dV + \int_D [(d\alpha - h)\vartheta^2(t) \\ + k_{mn}\Big(\int_0^t \vartheta_{,m}(\xi)d\xi\Big)\Big(\int_0^t \vartheta_{,n}(\xi)d\xi\Big) + 2\alpha k_{mn}\vartheta_{,m}(t)\Big(\int_0^t \vartheta_{,n}(\xi)d\xi\Big)]dV \\ = \int_D \varrho[v_m^1 v_m(2t) + v_m^0\dot v_m(2t)] + I_{mn}[\phi_m^1\phi_n(2t) + \phi_m^0\dot\phi_n(2t)]dV \\ + \int_D [(d\alpha - h)\vartheta^0\vartheta(2t) + \alpha k_{mn}\vartheta_{,m}^0\Big(\int_0^{2t}\vartheta_{,n}(\xi)d\xi\Big)]dV \\ + \int_0^t\int_D (a - \varrho\eta^0)\Big(\vartheta(t+s) - \vartheta(t-s) + \alpha\Big[\dot\vartheta(t+s) - \dot\vartheta(t-s)\Big]\Big)dV\,ds. \end{aligned} \tag{29}$$

Proof. Consider $u_m(x,s)$ and $w_m(x,s)$ as twice continuously differentiable functions with respect to time variable s. It is easy to see that

$$\frac{d}{ds}\left[\varrho\left(u_m(s)\dot w_m(s) - \dot u_m(s)w_m(s)\right)\right] = \varrho\left[u_m(s)\ddot w_m(s) - \ddot u_m(s)w_m(s)\right],$$

such that, by integrating over $[0,t]\times D$, it results

$$\begin{aligned} \int_D \varrho\left[u_m(t)\dot w_m(t) - \dot u_m(t)w_m(t)\right]dV = \int_0^t\int_D \varrho[u_m(s)\ddot w_m(s) - \ddot u_m(s)w_m(s)]\,dV\,ds \\ + \int_D \varrho\left[u_m(0)w_m(1) - \dot u_m(0)w_m(0)\right]dV. \end{aligned} \tag{30}$$

By setting $u_m(x,s) = v_m(x,t-s)$, $w_m(x,s) = v_m(x,t+s)$, $s\in[0,t]$, $t\in(0,\infty)$, the relation (30) becomes:

$$\begin{aligned} 2\int_D \varrho v_m(t)\dot v_m(t)dV = \int_D \varrho[v_m^0\dot v_m(2t) + v_m^1 v_m(2t)]dV \\ + \int_0^t\int_D \varrho[v_m(t+s)\ddot v_m(t-s) - v_m(t-s)\ddot v_m(t+s)]dV\,ds, \end{aligned} \tag{31}$$

for $t\in(0,\infty)$.

Similarly, for $t \in (0, \infty)$, we have

$$2\int_D I_{mn}\phi_m(t)\dot{\phi}_n(t)dV = \int_D I_{mn}[\phi_m^0\dot{\phi}_n(2t) + \phi_m^1\phi_n(2t)]dV$$
$$+\int_0^t\int_D I_{mn}[\phi_m(t+s)\ddot{\phi}_n(t-s) - \phi_m(t-s)\ddot{\phi}_n(t+s)]dVds. \tag{32}$$

The inertial terms that appear in the last integrals of the relations (31) and (32) can be eliminated. For this purpose the symmetries (3) and equations (13) are used and obtained:

$$\begin{aligned}
&\varrho[v_m(t+s)\ddot{v}_m(t-s) - v_m(t-s)\ddot{v}_m(t+s)] + I_{mn}[\phi_m(t+s)\ddot{\phi}_n(t-s) \\
&-\phi_m(t-s)\ddot{\phi}_n(t+s)] = [v_m(t+s)\tau_{nm}(t-s) - v_m(t-s)\tau_{nm}(t+s)]_{,n} \\
&\quad + [\phi_m(t+s)\sigma_{nm}(t-s) - \phi_m(t-s)\sigma_{nm}(t+s)]_{,n} \\
&\quad +[\alpha_{nm}e_{nm}(t-s) + \beta_{nm}\varepsilon_{nm}(t-s)][\vartheta(t+s) + \alpha\dot{\vartheta}(t+s)] \\
&\quad -[\alpha_{nm}e_{nm}(t+s) + \beta_{nm}\varepsilon_{nm}(t+s)][\vartheta(t-s) + \alpha\dot{\vartheta}(t-s)].
\end{aligned} \tag{33}$$

Now we consider the enrgy equation $(12)_3$ and deduce:

$$\begin{aligned}
&[\alpha_{nm}e_{nm}(t-s) + \beta_{nm}\varepsilon_{nm}(t-s)]\left[\vartheta(t+s) + \alpha\dot{\vartheta}(t+s)\right] \\
&-[\alpha_{nm}e_{nm}(t+s) + \beta_{nm}\varepsilon_{nm}(t+s)]\left[\vartheta(t-s) + \alpha\dot{\vartheta}(t-s)\right] \\
&+\left(a - \varrho\eta^0\right)\left[\vartheta(t-s) - \vartheta(t+s) + \alpha\left(\dot{\vartheta}(t-s) - \dot{\vartheta}(t+s)\right)\right] \\
&+(d\alpha - h)\left[\vartheta(t-s)\dot{\vartheta}(t+s) - \vartheta(t+s)\dot{\vartheta}(t-s)\right] \\
&+k_{mn}\left[\vartheta_{,m}(t+s)\left(\int_0^{t-s}\vartheta_{,n}(\xi)d\xi\right) - \vartheta_{,m}(t-s)\left(\int_0^{t+s}\vartheta_{,n}(\xi)d\xi\right)\right] \\
&+\alpha k_{mn}\left[\dot{\vartheta}_{,m}(t+s)\left(\int_0^{t-s}\vartheta_{,n}(\xi)d\xi\right) - \vartheta_{,m}(t-s)\vartheta_{,n}(t+s)\right] \\
&+\alpha k_{mn}\left[\dot{\vartheta}_{,m}(t-s)\left(\int_0^{t+s}\vartheta_{,n}(\xi)d\xi\right) - \vartheta_{,m}(t+s)\vartheta_{,n}(t-s)\right] \\
&+\left(k_{mn}\left[\vartheta(t-s) + \alpha\dot{\vartheta}(t-s)\right]\int_0^{t+s}\vartheta_{,n}(\xi)d\xi\right)_{,m} \\
&-\left(k_{mn}\left[\vartheta(t+s) + \alpha\dot{\vartheta}(t+s)\right]\int_0^{t-s}\vartheta_{,n}(\xi)d\xi\right)_{,m}.
\end{aligned} \tag{34}$$

We now substitute (34) into (33) and we use the boundary conditions (10) in order to obtain the following identity

$$2\int_D [\varrho v_m(t)\dot{v}_m(t)+I_{mn}\phi_m(t)\dot{\phi}_n(t)]dV = \int_D \left[\varrho\left(v_m^0\dot{v}_m(2t)+\dot{v}_m^0 v_m(2t)\right)\right.$$
$$+I_{mn}\left(\phi_m^0\dot{\phi}_n(2t)+\dot{\phi}_m^0\phi_n(2t)\right)\Big] dV+\int_0^t\int_D\left(a-\varrho\eta^0\right)[\vartheta(t+s)-\vartheta(t-s)$$
$$+\alpha\left(\dot{\vartheta}(t+s)-\dot{\vartheta}(t-s)\right)\Big] dV ds+\int_0^t\int_D\Big[(d\alpha-h)\frac{d}{ds}(\vartheta(t+s)\vartheta(t-s)) \tag{35}$$
$$+\frac{d}{ds}\left(k_{mn}\int_0^{t+s}\vartheta_{,m}(\xi)d\xi\int_0^{t-s}\vartheta_{,n}(\xi)d\xi\right)+\alpha k_{mn}\vartheta_{,m}(t+s)\int_0^{t-s}\vartheta_{,n}(\xi)d\xi$$
$$\left.+\alpha k_{mn}\vartheta_{,m}(t-s)\int_0^{t+s}\vartheta_{,n}(\xi)d\xi\right] dV ds.$$

If we use the initial data (13) in (35), we obtain the desired identity (29), which concludes the proof of Theorem 3. □

Other applicable results can be found in [9-15].

5. Equipartition of total energy

In order to obtain the asymptotic partition of total energy, the main aim of this section, we will use the estimations (22), (25) and (29) and we will take into account the hypotheses from Section 2.

We start by considering the identity (22) from which we can identify several types of energies, as follows:

$$\begin{aligned}
K_C(t) &\equiv \frac{1}{2t}\int_0^t\int_D[\varrho\dot{v}_m(s)\dot{v}_m(s)+I_{mn}\dot{\phi}_m(s)\dot{\phi}_n(s)]dV ds,\\
L_C(t) &\equiv \frac{1}{2t}\int_0^t\int_D[A_{mnkl}e_{mn}(s)e_{kl}(s)+2B_{mnkl}e_{mn}(s)\varepsilon_{kl}(s)\\
&\qquad +C_{mnkl}\varepsilon_{mn}(s)\varepsilon_{kl}(s)]dV ds,\\
P_C(t) &\equiv \frac{1}{2t}\int_0^t\int_D \alpha Kmn\vartheta_{,m}(s)\vartheta_{,n}(s)dV ds,\\
T_C(t) &\equiv \frac{1}{2t}\int_0^t\int_D d\vartheta^2(s)dV ds,\\
T_{KC}(t) &\equiv \frac{1}{2t}\int_0^t\int_D \alpha h\dot{\vartheta}^2(s)dV ds,\\
S_C(t) &\equiv \frac{1}{2t}\int_0^t\int_0^s\int_D[K_{mn}\vartheta_{,m}(\xi)+(d\alpha-h)\dot{\vartheta}^2(\xi)]dV d\xi ds,
\end{aligned} \tag{36}$$

which are Cesaro means of various energies.

With the help of these Cesaro means, the main result of the present study can now be formulated and proved.

Theorem 4. *It is supposed that are satisfied the hypotheses formulated in Section 2. Then, no matter how the initial data:*

$$(v_m^0, \phi_m^0) \in \mathbf{W}_1(D),\ (v_m^1, \phi_m^1) \in \mathbf{W}_0(D),\ \vartheta^0 \in W_1(D),\ \vartheta^1 \in W_0(D),$$

are chosen, the following two limits occur:

$$\lim_{t\to\infty} P_C(t) = 0,\ \lim_{t\to\infty} T_{KC}(t) = 0. \tag{37}$$

Furthermore, it is proven that:
(i) if meas $\partial D_3 \neq 0$, *then*

$$\lim_{t\to\infty} T_C(t) = 0; \tag{38}$$

(ii) if meas $\partial D_2 = 0$, *then*

$$\lim_{t\to\infty} T_C(t) = \frac{1}{2}\int_D \frac{1}{d}(d\vartheta^* + h\dot{\vartheta}^*)dV; \tag{39}$$

(iii) if meas $\partial D_1 \neq 0$ *and meas* $\partial D_2 \neq 0$, *then*

$$\lim_{t\to\infty} K_C(t) = \lim_{t\to\infty} L_C(t), \tag{40}$$

$$\lim_{t\to\infty} S_C(t) = E(0) - 2\lim_{t\to\infty} K_C(t) = E(0) - 2\lim_{t\to\infty} L_C(t); \tag{41}$$

(iv) if meas $\partial D_1 = 0$ *and meas* $\partial D_2 = 0$, *then*

$$\lim_{t\to\infty} K_C(t) = \lim_{t\to\infty} L_C(t) + \frac{1}{2}\int_D [\varrho \dot{v}_m^* \dot{v}_m^* + I_{mn}\dot{\phi}_m^* \dot{\phi}_n^*]dV, \tag{42}$$

$$\begin{aligned}\lim_{t\to\infty} S_C(t) &= E(0) - 2\lim_{t\to\infty} K_C(t) + \frac{1}{2}\int_D [\varrho \dot{v}_m^* \dot{v}_m^* + I_{mn}\dot{\phi}_m^* \dot{\phi}_n^*]dV \\ &= E(0) - 2\lim_{t\to\infty} L_C(t) - \frac{1}{2}\int_D [\varrho \dot{v}_m^* \dot{v}_m^* + I_{mn}\dot{\phi}_m^* \dot{\phi}_n^*]dV. \end{aligned} \tag{43}$$

Proof. To obtain the limits (37), the hypotheses from Section 2 and the conservation law (22) are used.

First, with the help of the assumptions hypotheses (10), it is obtained:

$$\begin{aligned} &d\vartheta^2(t) + \alpha h\dot{\vartheta}^2(t) + 2h\vartheta(t)\dot{\vartheta}(t) \\ &= \frac{1}{d}\left(d\vartheta(t) + h\dot{\vartheta}(t)\right)^2 + \frac{h}{d}(d\alpha - h)\dot{\vartheta}^2(t) \\ &= \frac{h}{\alpha}\left(\vartheta(t) + \alpha\dot{\vartheta}(t)\right)^2 + \frac{1}{\alpha}(d\alpha - h)\vartheta^2(t) \geq 0. \end{aligned} \tag{44}$$

Then, considering (22) and (36), it is deduced that:

$$T_{KC}(t) \leq \frac{1}{2t} h\alpha\left(d\alpha - h\right)^{-1} E(0), \tag{45}$$

$$P_C(t) \leq \frac{\alpha}{2t} E(0). \tag{46}$$

Finally, the relations (37) are obtained by letting $t \to \infty$ into (45) and (46) and using the criterion of increase for limits.

(i) If it is assumed that *meas* $\partial D_3 \neq 0$, then it is easy to show that $\vartheta \in \hat{W}_1(D)$. After that, with the help of the Poincare's inequality (19) and the equality (22), it is obtained the estimate:

$$\int_0^t \int_D d\vartheta^2(s)dVds \leq \frac{d}{m_2}\int_0^{t^0} \int_D k_{mn}\vartheta_{,m}(s)\vartheta_{,n}(s)dVds \leq \frac{d}{m_2}E(0). \tag{47}$$

If relations (36) and (47) are taken into account, the conclusion (38) is reached.

(ii) Let us assume that *meas* $\partial D_3 = 0$. Based on the decomposition (21) and the fact that $\chi \in \hat{W}_1(D)$, the following identity are deduced:

$$\begin{aligned}\int_D \vartheta^2(t)dV = \int_D (\vartheta^* + \frac{h}{d}\dot{\vartheta}^*)^2 dV + \int_D \chi^2(t)dV\\ -2\int_D \frac{h}{d}(\vartheta^* + \frac{h}{d}\dot{\vartheta}^*)\dot{\vartheta}^* \exp(-\frac{dt}{h})dV + \int_D \frac{h^2}{d^2}(\dot{\vartheta}^*)^2 \exp(-2\frac{dt}{h})dV.\end{aligned} \tag{48}$$

If the relations (36) and (48) are used, the next relation is reached:

$$\begin{aligned}T_C(t) = \frac{1}{2}\int_D \frac{1}{d}(d\vartheta^* + h\dot{\vartheta}^*)^2 dV + \frac{1}{2t}\int_0^t \int_D d\chi^2(s)dVds\\ -\frac{1}{t}[1-\exp(-\frac{d}{h}t)]\int_D \frac{h^2}{d^2}\dot{\vartheta}^*(d\vartheta^* + h\dot{\vartheta}^*)dV\\ +\frac{1}{4t}[1-\exp(-2\frac{d}{h}t)]\int_D \frac{h^3}{d^2}(\dot{\vartheta}^*)^2 dV.\end{aligned} \tag{49}$$

If the Poincare's inequality in (19) is taken into account, then based on the equality (22) and the fact that $\chi \in \hat{W}_1(D)$, it is obtained the following estimate:

$$\begin{aligned}\frac{1}{2t}\int_0^t \int_D d\chi^2(s)dVds \leq \frac{d}{2tm_2}\int_0^t \int_D k_{mn}\chi_{,m}(s)\chi_{,n}(s)dVds\\ = \frac{d}{2tm_2}\int_0^t \int_D k_{mn}\vartheta_{,m}(s)\vartheta_{,n}(s)dVds \leq \frac{d}{2tm_2}E(0).\end{aligned} \tag{50}$$

It is arrived to (40) by letting $t \to \infty$ in (49) and considering (9) and (50).

If the law of conservation (22) and the relation (44) are used and the assumptions in Section 2 are taken into account, the following estimates are reached:

$$\int_D \vartheta^2(t)dV \leq 2\alpha\frac{1}{d\alpha - h}E(0),\ t \in [0,\infty), \tag{51}$$

$$\int_D \left[\varrho\dot{v}_m(t)\dot{v}_m(t) + I_{mn}\dot{\phi}_m(t)\dot{\phi}_n(t)\right] dV \leq 2E(0),\ t \in [0,\infty), \tag{52}$$

$$\int_0^t \int_D k_{mn}\vartheta_{,m}(\tau)\vartheta_{,n}(\tau)dVd\tau \leq E(0),\ t \in [0,\infty), \tag{53}$$

$$\int_0^t \int_D \dot{\vartheta}^2(\tau)dV \leq \frac{1}{d\alpha - h}E(0),\ t \in [0,\infty). \tag{54}$$

Now, we consider the equalities (25) and (29) in order to obtain:

$$\begin{aligned}&\frac{1}{2t}\int_0^t\int_D[\varrho\dot{v}_m(s)\dot{v}_m(s)+I_{mn}\dot{\phi}_m(s)\dot{\phi}_n(s)-A_{mnkl}e_{mn}(s)e_{kl}(s)\\&\qquad-2B_{mnkl}e_{mn}(s)\varepsilon_{kl}(s)-C_{mnkl}\varepsilon_{mn}(s)\varepsilon_{kl}(s)]dVds\\&=\frac{1}{2t}\int_0^t\int_D\Big[d\vartheta^2(s)+2h\vartheta(s)\dot{\vartheta}(s)-\alpha h\dot{\vartheta}^2(s)-\alpha k_{mn}\vartheta_{,m}(s)\vartheta_{,m}(s)\Big]dVds\\&\qquad-\frac{1}{2t}\int_D[\varrho v_m^0v_m^1+I_{mn}\phi_m^0\phi_n^1]dV-\frac{1}{4t}\int_D(d\alpha-h)(\vartheta^0)^2dV\\&+\frac{1}{4t}\int_D[\varrho(v_m^1v_m(2t)+v_m^0\dot{v}_m(2t))+I_{mn}(\phi_m^1\phi_n(2t)+\phi_m^0\dot{\phi}_n(2t))]dV\\&+\frac{1}{2t}\int_0^t\int_D\left(a-\varrho\eta^0\right)\Big(\vartheta(s)+\alpha\dot{\vartheta}(s)\Big)dVds+\frac{1}{4t}\int_D(d\alpha-h)\vartheta^0\vartheta(2t)dV\\&\qquad+\alpha k_{mn}\vartheta_{,m}^0\int_0^{2t}\vartheta_{,n}(\xi)d\xi+\frac{1}{4t}\int_0^t\int_D(a-\varrho\eta^0)[\vartheta(t+s)\\&\qquad-\vartheta(t-s)+\alpha\frac{d}{ds}(\vartheta(t+s)+\vartheta(t-s))]dVds.\end{aligned}\tag{55}$$

If the initial data (13) is considered and the definitions (36) are taken into account, the from previous relation (55) it is deduced:

$$\begin{aligned}K_C(t)-L_C(t)&=\frac{1}{4t}\int_D\left[(d\alpha-h)(\vartheta^0+\alpha(a-\varrho\eta^0)\right]\left[\vartheta(2t)-\vartheta^0\right]dV\\&+\frac{1}{4t}\int_0^{2t}\int_D\alpha k_{mn}\vartheta_{,m}^0\vartheta_{,n}(s)dVds+\frac{1}{t}\int_0^t\int_D h\vartheta(s)\dot{\vartheta}(s)dVds\\&-\frac{1}{2t}\int_D\left[\varrho v_m^0v_m^1+I_{mn}\phi_m^0\phi_n^1\right]dV+T_{KC}(t)-P_C(t)\\&+\frac{1}{4t}\int_D\left[\varrho\left(v_m^1v_m(2t)+v_m^0\dot{v}_m(2t)\right)+I_{mn}\Big(\phi_m^1\phi_n(2t)+\phi_m^0\dot{\phi}_n(2t)\Big)\right]dV\\&+T_C(t)+\frac{1}{4t}\int_0^t\int_D\left(a-\varrho\eta^0\right)[\vartheta(t+s)+\vartheta(s)]dVds.\end{aligned}\tag{56}$$

On the integrals from the right-hand side of the equality (56) the inequality of Schwarz-Cauchy is used. Thus, with the help of the estimations from (45)-(47)

and (51)-(54), the following evaluations are obtained:

$$\left|-\frac{1}{2t}\int_D [\varrho v_m^0 v_m^1 + I_{mn}\phi_m^0\phi_n^1]\,dV\right| \le \frac{1}{4t}\int_D [\varrho\left(v_m^0 v_m^0 + v_m^1 v_m^1\right) + I_{mn}\left(\phi_m^0\phi_n^0 + \phi_m^1\phi_n^1\right)]\,dV;$$

$$\left|\frac{1}{4t}\int_D \left[(d\alpha-h)\left(\vartheta^0 + \alpha(a-\varrho\eta^0)\right)\right]\left[\vartheta(2t)-\vartheta^0\right]dV\right|$$
$$\le \frac{1}{8t}\int_D \left[\left[(d\alpha-h)(\vartheta^0+\alpha(a-\varrho\eta^0)\right]^2 + 2(\vartheta^0)^2\right]dV + \frac{\alpha}{2t(d\alpha-h)}E(0);$$

$$\left|\frac{1}{4t}\int_0^{2t}\int_D \alpha k_{mn}\vartheta^0_{,m}\vartheta_{,n}(s)\,dV\,ds\right| \le \frac{1}{4t}\left(\int_0^{2t}\int_D \alpha k_{mn}\vartheta^0_{,m}\vartheta^0_{,n}\,dV\,ds\right)^{\frac{1}{2}}$$
$$\times\left(\int_0^{2t}\int_D \alpha k_{mn}\vartheta_{,m}(s)\vartheta_{,n}(s)\,dV\,ds\right)^{\frac{1}{2}} \le \frac{1}{2}\left(\frac{\alpha}{2t}E(0)\int_D \alpha k_{mn}\vartheta^0_{,m}\vartheta^0_{,n}\,dV\,ds\right)^{\frac{1}{2}}; \quad (57)$$

$$\left|\frac{1}{t}\int_0^t\int_D h\vartheta(s)\dot\vartheta(s)\,dV\,ds\right| \le \frac{1}{t}\left(\int_0^t\int_D h\vartheta^2(s)\,dV\,ds\right)^{\frac{1}{2}}$$
$$\times\frac{1}{t}\left(\int_0^t\int_D h\dot\vartheta^2(s)\,dV\,ds\right)^{\frac{1}{2}} \le \left(\frac{2\alpha}{t}\right)^{\frac{1}{2}}\frac{h}{d\alpha-h}E(0);$$

$$\left|\frac{1}{4t}\int_D \left[\varrho\left(v_m^1 v_m(2t) + v_m^0\dot v_m(2t)\right) + I_{mn}\left(\phi_m^1\phi_n(2t) + \phi_m^0\dot\phi_n(2t)\right)\right]dV\right|$$
$$\le \frac{1}{8t}\int_D \left[\varrho v_m^0 v_m^0 + I_{mn}\phi_m^0\phi_n^0\right]dV + \frac{1}{4t}E(0).$$

(iii) Now, we suppose that *meas* $\partial D_1 \ne 0$ and *meas* $\partial D_2 \ne 0$. Taking into account that $(v_m, \phi_m) \in \hat{\mathbf{W}}_1(D)$, it is deduced that, for $\tau \in [0,\infty)$, the relations (7), (18), (22) imply that:

$$\int_D [\varrho v_m(\tau)v_m(\tau) + I_{mn}\phi_m(\tau)\phi_n(\tau)]\,dV \le \frac{k}{m_1}\int_D [A_{mnkl}e_{mn}(\tau)e_{kl}(\tau)$$
$$+2B_{mnkl}e_{mn}(\tau)\varepsilon_{kl}(\tau) + C_{mnkl}\varepsilon_{mn}(\tau)\varepsilon_{kl}(\tau)]\,dV \le \frac{2k}{m_1}E(0), \quad (58)$$

and thus it is obtained that:

$$\left|\frac{1}{4t}\int_D \left[\varrho\left(v_m^1 v_m(2t) + v_m^0\dot v_m(2t)\right) + I_{mn}\left(\phi_m^1\phi_n(2t) + \phi_m^0\dot\phi_n(2t)\right)\right]dV\right|$$
$$\le \frac{1}{8t}\int_D \left[\varrho v_m^1 v_m^1 + I_{mn}\phi_m^1\phi_n^1\right]dV + \frac{k}{4tm_1}E(0). \quad (59)$$

Now, it is assumed that *meas* $\partial D_3 \ne 0$. Thus, it is deduced that:

$$\left|T_C(t) + \frac{1}{4t}\int_0^t\int_D \left(a-\varrho\eta^0\right)[\vartheta(t+s)+\vartheta(s)]\,dV\,ds\right| \le T_C(t)$$
$$+\frac{1}{4t}\left(\int_0^t\int_D \left(a-\varrho\eta^0\right)^2 dV\,ds\right)^{\frac{1}{2}}\left(\int_0^t\int_D [\vartheta(t+s)+\vartheta(s)]^2\,dV\,ds\right)^{\frac{1}{2}} \quad (60)$$
$$\le T_C(t) + \left(\frac{1}{2d}\int_D (a-\varrho\eta^0)^2\,dV\right)^{\frac{1}{2}}[T_C(2t)]^{\frac{1}{2}}.$$

In order to obtain the relation (40), we will consider the estimations (57), (59) and (60) and the relations (17) and (49) and we will pass to the limit in (56), with $t \to \infty$.

Let us assume now that $meas\ \partial D_3 = 0$. By using the assumptions (16), (17), (21) and the expression of η^0 (as in Theorem 2), from the equality (49) it is concluded that:

$$\begin{aligned} & T_C(t) + \frac{1}{4t}\int_0^t \int_D \left(a - \varrho\eta^0\right)[\vartheta(t+s) + \vartheta(s)]dV ds \\ & \quad = -\frac{1}{4t}\int_D \frac{h^2}{d^2}\dot{\vartheta}^* \left(d\vartheta^* + \frac{3}{2}h\dot{\vartheta}^*\right)\left[\exp(-2\frac{dt}{h}) - 1\right] dV \\ & \quad + \frac{1}{t}\int_D \frac{h^2}{d^2}\dot{\vartheta}^* \left(d\vartheta^* + h\dot{\vartheta}^*\right)\left[\exp(-\frac{dt}{h}) - 1\right] dV + \frac{1}{2t}\int_0^t\int_D d\chi^2(s) dV ds \\ & \quad + \frac{1}{4t}\int_0^t\int_D \left[\alpha_{mn}e^0_{mn} + \beta_{mn}\varepsilon^0_{mn} - dT^0 - h\dot{T}^0\right][\chi(t+s) + \chi(s)]dV ds. \end{aligned} \tag{61}$$

In equality (61) the inequality Schwarz - Cauchy is applied and considering the relation (51), it is deduced that:

$$\lim_{t\to\infty}\left\{T_C(t) + \frac{1}{4t}\int_0^t\int_D \left(a - \varrho\eta^0\right)[\vartheta(t+s) + \vartheta(s)]dV ds\right\} = 0. \tag{62}$$

The conclusion (40) can be again obtained by using the relations (37), (57), (59), (62) in (56). It is not difficult to obtain the relation (41) considering the Cesaro means in (22) and considering the relations (37), (38) and (40).

(iv) Let us consider the last possibility: $meas\ \partial D_1 = 0$ and $meas\ \partial D_2 = 0$. In this situation the decomposition (20) is used and are took in consideration the relations (14), (15) and the fact that $(v_m, \phi_m) \in \hat{\mathbf{W}}_1(D)$. Thus, it is deduced that:

$$\begin{aligned} & \frac{1}{4t}\int_D \left[\varrho v^0_m \dot{v}_m(2t) + I_{mn}\phi^0_m\dot{\phi}_n(2t)\right] dV = \frac{1}{4t}\int_D \left[\varrho v^*_m \dot{v}^*_m + I_{mn}\phi^*_m\dot{\phi}^*_n\right] dV \\ & + \frac{1}{2}\int_D \left[\varrho \dot{v}^*_m \dot{v}^*_m + I_{mn}\dot{\phi}^*_m\dot{\phi}^*_n\right] dV + \frac{1}{4t}\int_D \left[\varrho \dot{V}^0_m v_m(2t) + I_{mn}\dot{\Phi}^0_m\psi_j(2t)\right] dV. \end{aligned} \tag{63}$$

Considering that $(w_m, \psi_m) \in \hat{\mathbf{W}}_1(D)$, with the help of the Korn inequality (18), the next estimate is obtained:

$$\begin{aligned} & \frac{1}{4t}\int_D [\varrho w_m(\tau)w_m(\tau) + I_{mn}\psi_m(\tau)\psi_n(\tau)]dV \le \frac{k}{m_1}\int_D [A_{mnkl}\bar{e}_{mn}(\tau)\bar{e}_{kl}(\tau) \\ & \qquad + 2B_{mnkl}\bar{e}_{mn}(\tau)\bar{\varepsilon}_{kl}(\tau) + C_{mnkl}\bar{\varepsilon}_{mn}(\tau)\bar{\varepsilon}_{kl}(\tau)]\, dV \\ & \qquad = \frac{k}{m_1}\int_D [A_{mnkl}e_{mn}(\tau)e_{kl}(\tau) + 2B_{mnkl}e_{mn}(\tau)\varepsilon_{kl}(\tau) \\ & \qquad\quad + C_{mnkl}\varepsilon_{mn}(\tau)\varepsilon_{kl}(\tau)]\, dV \le \frac{2k}{m_1}E(0),\ \tau \in [0, \infty), \end{aligned} \tag{64}$$

in which were used the notations: $\bar{e}_{mn} = w_{n,m} + \varepsilon_{nmk}\psi_k$, $\bar{\varepsilon}_{mn} = \psi_{n,m}$.

If the limit in (57) is used, with $t \to \infty$, and are considered the limits (37), the estimates (57), (60), (64) and the equality (63) it is arrived to the relation (42).

At the end, to obtain the relation (43) there are used the Cesaro means in the law (22) and there are considered the limits (37), (38), (42) and the estimate (57). In this way, theorem 4 is completely proven. □

6. Conclusions

If we consider the initial data for which $\dot{v}^*_m = \dot{\phi}^*_m = 0$, then it can be established, from the relations (40) and (42), that it is insured the equipartition of asymptotic type, in mean, of the strain and kinetic energies.

References

[1] Adams, R.S., *Sobolev Spaces*, Academic Press, New York, 1975.

[2] Day, W.A., *Means and autocorrections in elastodynamics*, Arch. Rational Mech. Anal., **73** (1980), 243-256.

[3] Gurtin, M.E., *The dynamics of solid-solid phase transitions*, Arch. Rational Mech. Anal., **4** (1994), 305-335.

[4] Hlavacek, I. and Necas, J., *On inequalities of Korn's type*, Arch. Rational Mech. Anal., **36** (1970), 305-334.

[5] Iesan, D., *Thermoelastic deformation of porous Cosserat beams*, Journal Thermal Stresses, **31** (2008), 823-847.

[6] Levine, H.A., *An equipartition of energy theorem for weak solutions of evolutionary equations in Hilbert spaces*, J. Diff. Eqns., **24** (1977), 197-210.

[7] Marin, M., *An uniqueness result for body with voids in linear thermoelasticity*, Rendiconti di Matematica Applicata, Roma, **17** (1997), no. 1, 103-113.

[8] Marin, M., *On the domain of influence in thermoelasticity of bodies with voids*, Archivum Mathematicum, **33** (1997), no. 4, 301-308.

[9] Marin, M., *A temporally evolutionary equation in elasticity of micropolar bodies with voids*, U.P.B. Sci. Bull., Series A-Applied Mathematics Physics, **60** (1998), 3-12 .

[10] Marin, M., *On the minimum principle for dipolar materials with stretch*, Nonlinear Analysis: RWA, **10** (2009), no. 3, 1572-1578.

[11] Marin, M., *Lagrange identity method for microstretch thermoelastic materials*, J. Math. Anal. Appl., **363** (2010), no. 1, 275-286.

[12] Rionero, S. and Chirita, S., *The Lagrange identity method in linear thermo-elasticity*, Int. J. Engng. Sci., **25** (1987), 935-946.

[13] Vlase, S., Danasel, C., Scutaru, M.L. and Mihalcica, M., *Finite element analysis of a two-dimensional linear elastic systems with a plane rigid motion.* Romanian Journal of Physics, **59** (2014), 476-487.

[14] Vlase, S., Negrean, I., Marin, M. and Scutaru, M.L., *Energy of accelerations used to obtain the motion equations of a three-dimensional finite element.* Symmetry, **12** (2020), Art. No. 321.

[15] Vlase, S., Purcarea, R., Teodorescu-Draghicescu, H., Calin, M.R. and Mihalcica, M., *Behavior of a new Heliopol/Stratimat 300 composite laminate.* Optoelectronics and Advanced Materials, Rapid Communications, **7** (2013), 569-572.