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A SURVEY ON PERTURBATION INVARIANCE OF QUATERNIONIC EXPONENTIALLY DICHOTOMOUS OPERATORS

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Dedicated to Professor Marin Marin on the occasion of his 70th anniversary

Abstract

In this review paper, we present some basic notions and properties of quaternionic exponentially dichotomous operators. Some perturbation results of quaternionic exponentially dichotomous operators are illustrated which will help to consider the exponential dichotomous solutions to quaternionic evolution equations through semigroup theory.

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1 Slice quaternionic Banach algebra and multiplicative quaternionic linear functionals

The notion of quaternions that is a noncommutative extension of complex numbers is a mathematical concept introduced by Irish mathematician Hamilton in 1843 and it has been widely applied to both pure and applied mathematics and physics (see [1, 18]). Quaternionic algebra has been widely applied to dynamic

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equations on time scales (see [16, 25, 27]), differential and difference equations (see [9, 24, 29]) and fuzzy dynamic equations (see [26]), etc.

The further development of spectral theory has deeply promoted the development of the theory of operators and dynamic equations (see [4]).As a core notion in dichotomy spectrum theory, exponentially dichotomous operators are the natural evolution operators of first order linear homogeneous differential equations in an arbitrary Banach space in which causal effects will have influence on both future and past events (see [5, 10, 11, 17, 19, 20, 21, 22]).

Based on the theory of quaternionic operators and spectral theory on S-spectrum (see [3, 6, 7, 8, 12, 13, 14, 15, 23]), in this paper, we will present some perturbation results of quaternionic exponentially dichotomous operators. For more details, one may refer to [28].

Firstly, we begin with the fundamental knowledge of quaternionic space $\mathbb{H}$.

Definition 1 ([8]). *The algebra of quaternions space $\mathbb{H}$ is given by the elements $1, i, j, k$ satisfying the following relations*

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Then a quaternion q is denoted by $q = q_0 + q_1 i + q_2 j + q_3 k$, $q_l \in \mathbb{R}$, $l = 0, 1, 2, 3$, while the conjugate and the norm of q are given by

$$\bar{q} = q_0 - q_1 i - q_2 j - q_3 k, \quad |q| = \sqrt{q\bar{q}} = \sqrt{\bar{q}q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

The real part and the imaginary part of q are denoted by $\mathrm{Re}(q)$, $\mathrm{Im}(q)$, respectively and $q^{-1} = \bar{q}/|q|^2$. For the convenience of later discussion, we let $\mathbb{S}$ be the 2-dimensional sphere of purely imaginary unit quaternions, i.e.,

$$\mathbb{S} = \left\{ q = q_1 i + q_2 j + q_3 k \in \mathbb{H} : q_1^2 + q_2^2 + q_3^2 = 1 \right\}.$$

To each quaternion q it is possible to associate an element in $\mathbb{S}$:

$$I_q = \begin{cases} \frac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|} & \text{if } \mathrm{Im}(q) \neq 0, \\ \text{any element of } \mathbb{S} & \text{otherwise.} \end{cases}$$

Given $I \in \mathbb{S}$ we denote by $\mathbb{C}_I$ the complex plane $\mathbb{R} + I\mathbb{R}$ containing elements of the form $x + Iy$, $x, y \in \mathbb{R}$. Obviously, the imaginary unit I_p determines the complex plane $\mathbb{C}_{I_p}$ containing p.

Based on the notion of two-sided (i.e., bilateral) vector space, the quaternionic linear operator can be defined and classified as follows.

Definition 2 ([23]). *Let V be a two-sided vector space on $\mathbb{H}$. A map $T : V \to V$ is said to be a right linear operator if*

$$T(u + v) = T(u) + T(v), \quad T(us) = T(u)s, \quad \text{for all } s \in \mathbb{H}, \text{ and for all } u, v \in V.$$

The set of right linear operators on V is both a left and a right vector space on $\mathbb{H}$ with respect to the operations

$$(sT)(v) = sT(v), \quad (Ts)(v) = T(sv), \quad \text{for all } s \in \mathbb{H}, \text{ and for all } v \in V.$$

Similarly, a map $T : V \to V$ is said to be a left linear operator if

$$T(u + v) = T(u) + T(v), \quad T(su) = sT(u), \quad \text{for all } s \in \mathbb{H}, \text{ and for all } u, v \in V.$$

The set of left linear operators on V is both a left and a right vector space on $\mathbb{H}$ with respect to the operations

$$(Ts)(v) = T(v)s, \quad (sT)(v) = T(vs), \quad \text{for all } s \in \mathbb{H}, \text{ and for all } v \in V.$$

Now we state the quaternionic version of Cauchy kernel formula.

Definition 3 ([7])**.** *Let $s, q \in \mathbb{H}$, the so-called left slice Cauchy kernel and right slice Cauchy kernel are defined as follows*

$$S_L^{-1}(s, q) = -(q^2 - 2q\mathrm{Re}(s) + |s|^2)^{-1}(q - \overline{s}),$$

and

$$S_R^{-1}(s, q) = -(q - \overline{s})(q^2 - 2q\mathrm{Re}(s) + |s|^2)^{-1}.$$

From the literature [14], for $I \in \mathbb{S}$, there exists a real *-algebra isomorphism $\tilde{\varphi}_I : \mathbb{C} \to \mathbb{C}_I$ as follows:

$$\tilde{\varphi}_I(x + iy) := x + Iy, \quad x, y \in \mathbb{R}.$$

Now we introduce some notations. Let $\mathcal{M}$ be a complex Banach algebra, then define $\mathcal{M}_L := \big\{m = m_0 + m_1 I : I \in \mathbb{S}, m_\xi \in \mathcal{M}, \xi = 0, 1\big\}$ and $\mathcal{M}_I^c = \big\{m \in \mathcal{M}_L : mI = Im, I \in \mathbb{S}\big\}$, we can expand the complex Banach algebra $\mathcal{M}$ to a quaternionic Banach algebra $\mathcal{M}_L$ and introduce a slice quaternionic Banach algebra as follows.

Definition 4 ([28])**.** *Let $D \subset \mathbb{C}$ and $\mathcal{M}$ be a complex Banach algebra with unite element e. If $\mathcal{M}_I^c$ over the field $\tilde{\varphi}_I(D)$ is a quaternionic Banach space, then $\mathcal{M}_I^c$ is called a slice quaternionic Banach algebra* (*or s-quaternionic Banach algebra*) *for $I \in \mathbb{S}$. Generally, if $\mathcal{M}_I^c$ over the field $\tilde{\varphi}_I(D)$ is a quaternionic Hilbert space, then $\mathcal{M}_I^c$ is called a slice quaternionic Hilbert space* (*or s-quaternionic Hilbert space*)*.*

Let $\mathcal{M}_I^c$ be a s-quaternionic Banach algebra with unit element e and $\mathcal{Z}$ be a commutative Banach subalgebra of $\mathcal{M}_I^c$ satisfying $e \in \mathcal{Z}$ and $mz = zm$ for $m \in \mathcal{M}_I^c$ and $z \in \mathcal{Z}$. Let $\mathcal{Y}$ be a closed subalgebra of $\mathcal{M}_I^c$. Then by $\mathcal{Z} \otimes \mathcal{Y}$ we denote the algebraic tensor product of $\mathcal{Z}$ and $\mathcal{Y}$, i.e.,

$$\mathcal{Z} \otimes \mathcal{Y} = \left\{ \sum_{\xi=1}^{n} z_\xi y_\xi : z_\xi \in \mathcal{Z}, y_\xi \in \mathcal{Y}, n \in \mathbb{Z}^+ \right\}.$$

For the convenience of discussion, some following basic definitions will be introduced and we assume that $\mathcal{Z} \otimes \mathcal{Y}$ is a dense linear subspace of $\mathcal{M}_I^c$.

Definition 5 ([28])**.** *Let $\mathcal{Z} \subset \mathcal{M}_I^c$ ($I \in \mathbb{S}$) be a commutative s-quaternionic Banach algebra, φ_I is a bounded linear functional on $\mathcal{Z}$, if for any $z_1, z_2 \in \mathcal{Z}$,*

$$\varphi_I(z_1 z_2) = \varphi_I(z_1)\varphi_I(z_2).$$

Then φ_I is called multiplicative linear functionals for $I \in \mathbb{S}$. Let $\widetilde{U}$ be the set of all continuous multiplicative linear functionals on $\mathcal{Z}$.

Next, we will give some basic properties of s-quaternionic Banach algebra which shall be used in the later discussion.

Definition 6 ([28])**.** *Let $\mathcal{M}_I^c$ be a s-quaternionic Banach algebra and $m \in \mathcal{M}_I^c$, if there exists $m^{-1} \in \mathcal{M}_I^c$ such that*

$$mm^{-1} = m^{-1}m = e,$$

then m is called invertible. Define the S-spectrum of a $m \in \mathcal{M}_I^c$ as follows

$$\sigma_I(m) := \big\{ s \in \mathbb{C}_I : m^2 - 2\mathrm{Re}(s)m + |s|^2 e \text{ is not invertible in } \mathcal{M}_I^c, I \in \mathbb{S} \big\}$$

and for $s \notin \sigma_I(m)$, define $R_s(m) = S_R^{-1}(s, m)$.

Theorem 1 ([28])**.** *Let $\mathcal{M}_I^c(I \in \mathbb{S})$ be a s-quaternionic Banach algebra, if $m \in \mathcal{M}_I^c$ satisfies $\|m\|_{\mathcal{M}_I^c} < 1$, then $e - m$ is reversible and $(e-m)^{-1} = \sum_{\xi=0}^{\infty} m^{\xi}$.*

Corollary 1 ([28])**.** *Let $\mathcal{M}_I^c(I \in \mathbb{S})$ be a s-quaternionic Banach algebra, $m \in \mathcal{M}_I^c$.*

(1) *$(e - tm)^{-1} \to e$ as $t \to 0$.*

(2) *For m_0 is invertible in $\mathcal{M}_I^c$ and $m_1 \in \mathcal{M}_I^c$ such that $\|m_1\|_{\mathcal{M}_I^c} < \|m_0^{-1}\|_{\mathcal{M}_I^c}^{-1}$, then $m = x_0 + m_1$ is invertible and*

$$m^{-1} = (e + m_0^{-1}m_1)^{-1}m_0^{-1}.$$

(3) *$R_s(m)$ is continuous function on $\mathbb{C}_I \backslash \sigma_I(m)$ respect to s.*

Theorem 2 ([28])**.** *Let $\mathcal{M}_I^c(I \in \mathbb{S})$ be a s-quaternionic Banach algebra, for any $m \in \mathcal{M}_I^c$, $s, q \in \mathbb{C}_I \backslash \sigma_I(m)$, the following properties holds:*

(1) *$R_s(m)R_q(m) = R_q(m)R_s(m)$;*

(2) *$R_s(m) - R_q(m) = (q - s)R_s(m)R_q(m)$.*

Corollary 2 ([28])**.** *Let $\mathcal{M}_I^c(I \in \mathbb{S})$ be a s-quaternionic Banach algebra, $m \in \mathcal{M}_I^c$ and $s_0 \in \mathbb{C}_I \backslash \sigma_I(m)$,*

$$\frac{d}{ds}\big(R_s(m)\big)\big|_{s=s_0} = -\big(R_{s_0}(m)\big)^2.$$

Theorem 3 ([28])**.** *Let $\mathcal{M}_I^c(I \in \mathbb{S})$ be a s-quaternionic Banach algebra, for $m \in \mathcal{M}_I^c$, then*

(1) *for any* $f \in \big(\mathcal{M}_I^c\big)^*$, *the function* $F(s) = f\big(R_s(m)\big)$ *is s-regular on* $\mathbb{C}_I \backslash \sigma_I(m)$ *and* $F(s) \to 0$ *as* $|s| \to \infty$.

(2) $\sigma_I(m)$ *is a nonempty compact set on* $\mathbb{C}_I$ *and* $r(m) \leqslant \|m\|_{\mathcal{M}_I^c}$, *where* $r(m) = \sup_{s\in\sigma_I(m)} |s|$.

In what follows, a sufficient condition is given to guarantee the s-quaternionic Banach algebra $\mathcal{M}_I^c$ is isometric and isomorphic to $\mathbb{C}_I$.

Lemma 1 ([28])**.** *Let* $\mathcal{M}_I^c\,(I \in \mathbb{S})$ *be a s-quaternionic Banach algebra, if* $\mathcal{M}_I^c$ *is a field, then* $\mathcal{M}_I^c$ *is isometric and isomorphic to* $\mathbb{C}_I$.

A proper and maximal ideal of a s-quaternionic Banach algebra has the following property.

Lemma 2 ([28])**.** *Let* $\mathcal{M}_I^c\,(I \in \mathbb{S})$ *be a s-quaternionic Banach algebra and* $\mathcal{J}$ *be a proper ideal of* $\mathcal{M}_I^c$, *then* $\mathcal{J}$ *is contained in some maximal ideal of* $\mathcal{M}_I^c$ *and any maximal ideal of* $\mathcal{M}_I^c$ *is closed.*

By using a closed proper ideal of the s-quaternionic Banach algebra, one can obtain a new s-quaternionic Banach algebra through a quotient algebra in the following theorem.

Theorem 4 ([28])**.** *Let* $\mathcal{M}_I^c$ *be a s-quaternionic Banach algebra and* $\mathcal{J}$ *is a closed proper ideal in* $\mathcal{M}_I^c$. *Then quotient algebra* $\mathcal{M}_I^c/\mathcal{J}$ *is a s-quaternionic Banach algebra with the unit* $[e]$.

Lemma 3 ([28])**.** *Let* $\mathcal{Z} \subset \mathcal{M}_I^c$ $(I \in \mathbb{S})$ *be a commutative s-quaternionic Banach algebra, for any maximal ideal* $\mathcal{N}$ *in* $\mathcal{Z}$, *there exists a unique continuous multiplicative linear functional* φ_I *such that* $\mathcal{N} = \operatorname{Ker}\,\varphi_I$.

According to Lemma 3, the following theorem will justify the compactness of the maximal ideal space of $\mathcal{Z}$ for $I \in \mathbb{S}$.

Theorem 5 ([28])**.** *Let* $\mathcal{Z} \subset \mathcal{M}_I^c$ $(I \in \mathbb{S})$ *be a commutative s-quaternionic Banach algebra, the set of all maximal ideals of* $\mathcal{Z}$ *is a compact Hausdorff space for any* $I \in \mathbb{S}$.

According to Theorem 5, the set $\widetilde{U}$ of all multiplicative linear functionals on $\mathcal{Z}$ is compact, we say $\widetilde{U}$ the maximal ideal space of $\mathcal{Z}$, then we introduce the following definition.

Definition 7 ([28])**.** *Let* $\mathcal{Z}$ *be a commutative Banach subalgebra of s-quaternionic Banach algebra* $\mathcal{M}_I^c$ *for* $I \in \mathbb{S}$ *and* $\widetilde{U}$ *be the maximal ideal space of* $\mathcal{Z}$, *for every* $\varphi_I \in \widetilde{U}$, *define* $\phi_{\varphi_I} : \mathcal{Z} \otimes \mathcal{Y} \to \mathcal{M}_I^c$ *as follows*

$$\phi_{\varphi_I}\left(\sum_{\xi=1}^{n} z_\xi y_\xi\right) = \sum_{\xi=1}^{n} \varphi_I(z_\xi) y_\xi.$$

Then $\mathcal{M}_I^c$ *is said to be realizable as a tensor product of* $\mathcal{Z}$ *and* $\mathcal{Y}$ *if and only if* ϕ_{φ_I} *extends to a bounded linear operator on* $\mathcal{M}_I^c$ *for each* $\varphi_I \in \widetilde{U}$.

According to Definition 7,

$$\phi_{\varphi_I}(\mathcal{M}_I^c) \subset \mathcal{Y}, \quad (\phi_{\varphi_I})^2 = \phi_{\varphi_I}, \quad \phi_{\varphi_I}(m_1 m_2) = \phi_{\varphi_I}(m_1)\phi_{\varphi_I}(m_2)$$

for $m_1, m_2 \in \mathcal{M}_I^c$. Thus, ϕ_{φ_I} will be called the multiplicative projection associated with the multiplicative linear functional φ_I.

From the properties of the s-quaternionic Banach algebra and the maximal ideals, one can obtain a basic lemma as follows.

Lemma 4 ([28])**.** *Let $\mathcal{Z} \subset \mathcal{M}_I^c\,(I \in \mathbb{S})$ be a closed commutative Banach algebra with $e \in \mathcal{Z}$. If $\mathcal{J}$ is a maximal left (resp. right) ideal in $\mathcal{M}_I^c$, then $\mathcal{J} \cap \mathcal{Z}$ is a maximal ideal in $\mathcal{Z}$.*

Based on Lemma 4, the following Allan-Bochner-Phillips Theorem of the quaternionic version can be established.

Theorem 6 ([28])**.** *Let $\mathcal{M}_I^c\,(I \in \mathbb{S})$ be a s-quaternionic Banach algebra with the unit element e realized as a tensor product of a commutative subalgebra $\mathcal{Z}$ and some subalgebra $\mathcal{Y}$, where $e \in \mathcal{Z}$. Let $\widetilde{U}$ denote the maximal ideal space of $\mathcal{Z}$. Then $m \in \mathcal{M}_I^c$ is left (resp. right) invertible in $\mathcal{M}_I^c$ if and only if $\phi_{\varphi_I}(m)$ is left (resp. right) invertible in $\mathcal{Y}$ for each $\varphi_I \in \widetilde{U}$.*

2 Perturbation results of quaternionic exponentially dichotomous operators

The following concept of the semigroup in quaternionic version was introduced by F. Colombo and I. Sabadini (see [8]), which will be used to analyze the exponential dichotomy of evolution operators in this paper.

Definition 8 ([8])**.** *Let X be a two-sided quaternionic Banach space and $t \in \mathbb{R}$, then we call $\{E(\cdot)\}$ a quaternionic strongly continuous semigroup on quaternionic Banach space X if the function $E(\cdot) : [0, \infty) \to \mathcal{B}(X)$ having the following properties*

(1) $E(t+s) = E(t)E(s)$ *for* $t, s \geqslant 0$,

(2) $E(\cdot)x : [0, \infty) \to X$ *is continuous for* $x \in X$,

(3) $E(0) = \mathcal{I}_X$.

In addition, if the map $t \to E(t)$ is continuous in the uniform operator topology, then we call the family $\{E(\cdot)\}$ a uniformly continuous quaternionic semigroup in $\mathcal{B}(X)$.

We introduce the concept of the quaternionic bisemigroup as follows.

Definition 9 ([28])**.** *Let X be a quaternionic Banach space, by a (strongly continuous) bisemigroup we mean a function $E(\cdot) : \mathbb{R}\backslash\{0\} \to \mathcal{B}(X)$ having the following properties:*

(1) *If* $t, s > 0$ *we have* $E(t)E(s) = E(t+s)$ *and for* $t, s < 0$ *we have* $E(t)E(s) = -E(t+s)$.

(2) *For every* $x \in X$ *the function* $E(\cdot)x : \mathbb{R}\backslash\{0\} \to X$ *is continuous, apart from a jump discontinuity in* $t = 0$. *That is,*

$$\lim_{t\to 0^{\pm}} \|E(t)x - E(0^{\pm})x\|_X = 0, \quad x \in X.$$

(3) $E(0^+)x - E(0^-)x = x$ *for every* $x \in X$.

(4) *There exist* $M, \lambda > 0$ *such that* $\|E(t)\|_{\mathcal{B}(X)} \leqslant Me^{-\lambda|t|}$ *for* $t \in \mathbb{R}\backslash\{0\}$.

Definition 10 ([28])**.** *Let* $\{E(t)\}_{t\geqslant 0}$ *be a quaternionic strongly continuous semigroup, the infimum of all* $\lambda \in \mathbb{R}$ *satisfying* $\|E(t)\|_{\mathcal{B}(X)} = O(e^{\lambda t})$ *as* $t \to \infty$, *is called the exponential growth bound of* $\{E(t)\}_{t\geqslant 0}$ *and denoted by* $\lambda(E)$.

Proposition 3.1 in [28] implies that $E(0^+)$ and $-E(0^-)$ are bounded complementary, we may introduce the concept of the constituent semigroup of a bisemigroup $\{E(t)\}_{t\in\mathbb{R}\backslash\{0\}}$ as follows.

Definition 11. *Let* $\{E(t)\}_{t\in\mathbb{R}\backslash\{0\}}$ *be a strongly continuous bisemigroup, then we call the operator* $P = -E(0^-)$ *the separating projection of the bisemigroup* $\{E(t)\}_{t\in\mathbb{R}\backslash\{0\}}$. *The restriction of* $E(t)$ *to* Ker P *is a quaternionic strongly continuous semigroup on* Ker P, *while the restriction of* $-E(-t)$ *to* Im P *is a strongly continuous semigroup on* Im P. *These two semigroups are called the constituent semigroups of the bisemigroup* $\{E(t)\}_{t\in\mathbb{R}\backslash\{0\}}$.

Definition 11 indicates that we can describe the exponential growth bounds of $\{E(t)\}_{t\in\mathbb{R}\backslash\{0\}}$ through the exponential growth bounds of its corresponding constituent semigroups, hence we introduce the following notion.

Definition 12 ([28])**.** *Let* $E_j : [0, \infty) \to X_j (j = 1, 2)$ *be the quaternionic strongly continuous semigroups, and both have a negative exponential growth bound, we define the strongly continuous bisemigroup* $\{E(t)\}_{t\in\mathbb{R}\backslash\{0\}}$ *on* $X = X_1 \oplus X_2$ *by*

$$E(t) = \begin{cases} E_1(t) \oplus 0_{X_2}, & t > 0, \\ 0_{X_1} \oplus \big(- E_2(-t)\big), & t < 0, \end{cases}$$

which has $\{E_1(t)\}_{t\geqslant 0}$ *and* $\{E_2(t)\}_{t\geqslant 0}$ *as its constituent semigroups. For the pair of exponential growth bounds of a bisemigroup* $\{E(t)\}_{t\in\mathbb{R}\backslash\{0\}}$, *we denote the pair of (necessarily negative) exponential growth bounds of its constituent semigroups by:*

$$\big\{\lambda_+(E), \lambda_-(E)\big\}.$$

For the exponential growth bound $\lambda(E)$ *of a bisemigroup* $\{E(t)\}_{t\in\mathbb{R}\backslash\{0\}}$, *we denote it by*

$$\lambda(E) \overset{\text{def}}{=} \max\big\{\lambda_-(E), \lambda_+(E)\big\} < 0.$$

Definition 13 ([28])**.** *Let $T_+(\text{Ker } P \to \text{Ker } P)$ and $-T_-(\text{Im } P \to \text{Im } P)$ stand for the infinitesimal generators of the constituent semigroups of the bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$ on X, then the linear quaternionic operator $T(X \to X)$ defined by*

$$\mathcal{D}(T) = \big\{x_+ \oplus x_- : x_+ \in \mathcal{D}(T_+), x_- \in \mathcal{D}(T_-)\big\},$$

$$T(x_+ \oplus x_-) = T_+(x_+) - T_-(x_-)$$

is called the (infinitesimal) generator of the bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$, since $T(X \to X)$ is closed and densely defined, then we define the constituent Laplace transform formulas as follows:

$$S_R^{-1}(s, T_+)x_+ = \int_0^\infty e^{-st}E(t)x_+ \, dt, \quad x_+ \in \text{Ker } P, \quad \text{Re}(s) > \lambda_+(E),$$
$$S_R^{-1}(-s, -T_-)x_- = -\int_0^\infty e^{st}E(-t)x_- \, dt, \quad x_- \in \text{Im } P, \quad \text{Re}(-s) > \lambda_-(E),$$

where both of $\lambda_\pm(E) < 0$, which imply the Laplace transform formula

$$S_R^{-1}(s, T)x = \int_{-\infty}^\infty e^{-st}E(t)x \, dt, \quad \lambda_+(E) < \text{Re}(s) < -\lambda_-(E), \tag{1}$$

where the (Bochner) integral converges absolutely in the norm of X. From now on we will write $E(t, T)$ for the strongly continuous bisemigroup with infinitesimal generator T.

In what follows, we introduce the concept of a quaternionic exponentially dichotomous operator.

Definition 14 ([28])**.** *A closed and densely defined linear quaternionic operator $T(X \to X)$ on a quaternionic Banach space X is called exponentially dichotomous if it is the infinitesimal generator of a strongly continuous bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$ on X.*

Definition 15 ([8])**.** *Let X be a two-sided quaternionic Banach space.*

(i) *We denote by $\mathcal{K}^R(X)$ the set of right linear closed operators $T : \mathcal{D}(T) \subset X \to X$, such that*

  (1) *$\mathcal{D}(T)$ is dense in X,*

  (2) *$\mathcal{D}(T^2) \subset \mathcal{D}(T)$ is dense in X,*

  (3) *$T - \bar{s}\mathcal{I}$ is densely defined in X.*

(ii) *We denote by $\mathcal{K}^L(X)$ the set of left linear closed operators satisfying (1) and (2) and such that $T - \mathcal{I}\bar{s}$ is densely defined in X.*

(iii) *We use the symbol $\mathcal{K}(X)$ when we do not distinguish between $\mathcal{K}^L(X)$ and $\mathcal{K}^R(X)$.*

We can obtain that $T^2 - 2\mathrm{Re}(s)T + |s|^2 I : \mathcal{D}(T^2) \subset X \to X$ *is a closed operator. If* $T \in \mathcal{K}(X)$ *we denote by* $\rho_S(T)$ *the S-resolvent set of* T *as*

$$\rho_S(T) = \{s \in \mathbb{H} : (T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1} \in \mathcal{B}(X)\},$$

and $\sigma_S(T) = \mathbb{H}\backslash\rho_S(T)$*. Now, we denote by* $Q_s(T)$ *the operator*

$$Q_s(T) := (T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1} \quad where \quad Q_s(T) : X \to \mathcal{D}(T^2).$$

Moreover, let A *be an operator containing the term* $Q_s(T)T$ *(resp.* $TQ_s(T)$*) and let* $s \in \rho_S(T)$*, we will denote* $\hat{A}$ *the operator obtained from* A *according to substituting each occurrence of* $Q_s(T)T$ $\big($*resp.* $TQ_s(T)\big)$ *by* $TQ_s(T)$ $\big($*resp.* $Q_s(T)T\big)$.

Definition 16 ([28])**.** *Let* X *be a quaternionic Banach space and* $\mathbb{S}_c^I(X) = \big\{T \in \mathcal{K}(X) : TI = IT, I \in \mathbb{S}\big\}$*, a quaternionic operator* T *is called slice quaternionic operator on* X *if* $T \in \mathbb{C}_I^{\mathbb{S}}(X)$*, where*

$$\mathbb{C}_I^{\mathbb{S}}(X) := \big\{T \in \mathcal{K}(X) : T = T_0 + IT_1, I \in \mathbb{S}, T_l \in \mathbb{S}_c^I(X), l = 0, 1\big\}$$

and $\mathbb{C}_I^{\mathbb{S}}(X)$ *is called a slice operator space with respect to* $I \in \mathbb{S}$*.*

Naturally, we introduce a convolution of functions on s-quaternionic Banach space and it will be used later.

Definition 17 ([28])**.** *Let* $\mathcal{M}_I^c$ $(I \in \mathbb{S})$ *be a s-quaternionic Banach space,* $\varphi, \psi \in \mathbb{C}_I^{\mathbb{S}}(\mathcal{M}_I^c)$ *be Bochner integrable functions on* $\mathbb{R}$ *with values in* $\mathcal{M}_I^c$*, the convolution of* φ *and* ψ *is denoted by* $\varphi * \psi$ *and defined as follows*

$$\big(\varphi * \psi\big)(t) = \int_{-\infty}^{\infty} \varphi(\tau)\psi(t-\tau)\ d\tau = \int_{-\infty}^{\infty} \varphi(t-\tau)\psi(\tau)\ d\tau.$$

Theorem 7 (Convolution Theorem on s-quaternionic Banach space, [28])**.** *Let* $\mathcal{M}_I^c (I \in \mathbb{S})$ *be a s-quaternionic Banach space,* $\varphi, \psi \in \mathbb{C}_I^{\mathbb{S}}(\mathcal{M}_I^c)$ *be Bochner integrable functions on* $\mathbb{R}$ *with values in* $\mathcal{M}_I^c$*, let* $\mathfrak{L}(\varphi), \mathfrak{L}(\psi)$ *be the Laplace transforms of* φ *and* ψ*, respectively. Then*

$$\mathfrak{L}\big(\varphi(t)\big)\mathfrak{L}\big(\psi(t)\big) = \mathfrak{L}\big((\varphi * \psi)(t)\big).$$

Next, an additive compact perturbations of the quaternionic exponentially dichotomous operators will be taken into account.

Theorem 8 ([28])**.** *Let* $T_0 \in \mathbb{C}_I^{\mathbb{S}}(\mathcal{M}_I^c)$ $(I \in \mathbb{S})$ *be a quaternionic exponentially dichotomous operator on s-quaternionic Banach space* $\mathcal{M}_I^c$ *and* $\mathcal{H}$ *be a compact operator on* $\mathcal{M}_I^c$ *such that* $T = T_0 + \mathcal{H} \in \mathbb{C}_I^{\mathbb{S}}(\mathcal{M}_I^c)$*,* $\mathcal{D}(T) = \mathcal{D}(T_0)$*. Suppose that the hyper-imaginary axis* $I \in \mathbb{S}$ *is contained in the S-resolvent set of* T*. Then* T *is a quaternionic exponentially dichotomous operator. Furthermore,* $E(t, T) - E(t, T_0)$ *is a compact operator in the limits as* $t \to 0^{\pm}$*.*

To obtain the perturbation theorem of the quaternionic exponentially dichotomous operator, we need the following two lemmas.

Lemma 5 ([28]). *Let $\mathcal{M}_I^c$ $(I \in \mathbb{S})$ be a s-quaternionic Banach space, suppose a map $K : \mathbb{R} \times \mathcal{M}_I^c \to \mathcal{M}_I^c$ such that*

(1) *$K(\cdot, m) \in L^1(\mathbb{R}, \mathcal{M}_I^c)$ for every $m \in \mathcal{M}_I^c$;*

(2) *$K(t, s_1 m_1 + s_2 m_2) = s_1 K(t, m_1) + s_2 K(t, m_2)$ for $s_1, s_2 \in \mathbb{C}_I$, $I \in \mathbb{S}$, $m_1, m_2 \in \mathcal{M}_I^c$, and a.e. $t \in \mathbb{R}$.*

Then there exists a unique bounded linear operator H on $L^1(\mathbb{R}, \mathcal{M}_I^c)$ such that

$$(H\phi_I)(t) = \int_{-\infty}^{\infty} K\big(t - \tau, \phi_I(\tau)\big)\ d\tau \tag{2}$$

for integrable simple function $\phi_I (I \in \mathbb{S})$, while the norm of H is bounded above by

$$\sup_{\|m\|_{\mathcal{M}_I^c}=1} \int_{-\infty}^{\infty} \|K(t, m)\|_{\mathcal{B}(\mathcal{M}_I^c)} dt.$$

Lemma 6 ([28]). *Let $\mathcal{M}_I^c$ $(I \in \mathbb{S})$ be a s-quaternionic Hilbert space and let $K : \mathbb{R} \times \mathcal{M}_I^c \to \mathcal{M}_I^c$ be a quaternionic map such that the assumptions of Lemma 5 holds, then the linear operator defined by (2) for each integrable simple function ϕ_I extend to a unique bounded linear operator on $L^2(\mathbb{R}, \mathcal{M}_I^c)$ with norm given by*

$$\|H\|_{\mathcal{B}(\mathcal{M}_I^c)} = \sup_{\|x\|_{\mathcal{M}_I^c}=1, \tau \in \mathbb{R}} \big\|\mathfrak{F}\big[K(\tau, m)\big]\big\|_{\mathcal{M}_I^c},$$

where

$$\mathfrak{F}\big[K(\tau, m)\big] = \int_{-\infty}^{\infty} e^{-2\pi I \tau t} K(t, m)\ dt,\ I \in \mathbb{S}.$$

Theorem 9 ([28]). *Let $T \in \mathbb{C}_I^{\mathcal{S}}(\mathcal{M}_I^c)$ be a closed and densely defined linear operator on the quaternion Banach space X satisfying the conditions (a) and (b). Then T is a quaternionic exponentially dichotomous operator if and only if there exist $E : \mathbb{R} \times X \to X$ and constants $r > 0$, $M > 0$ such that for every $x \in X$ we have $E(\cdot, x) \in L^\infty(\mathbb{R}, X)$ and*

$$\|E(\cdot, x)\|_{L^\infty(\mathbb{R}, X)} \leqslant M e^{-r|\cdot|} \|x\|_X,$$

and for some $\eta > 0$, we have the Laplace transform formula

$$S_R^{-1}(s, T)x = \int_{-\infty}^{\infty} e^{-st} E(t, x)\ dt,$$

where $x \in X$, $s \in \{s \in \mathbb{H} : |\mathrm{Re}(s)| \leqslant \eta\} \cap \mathbb{C}_{I_\lambda}$.

Now we will establish a perturbation theorem of a quaternionic exponentially dichotomous operator as follows.

Theorem 10 ([28]). *Let $T_0 \in \mathbb{C}_I^{\mathcal{S}}(\mathcal{M}_I^c)$ $(I \in \mathbb{S})$ be a quaternionic exponentially dichotomous operator on s-quaternionic Banach space $\mathcal{M}_I^c$, and let $\mathcal{H} \in \mathcal{B}(\mathcal{M}_I^c) \cap \mathbb{C}_I^{\mathcal{S}}(\mathcal{M}_I^c)$. Then there exists $\widetilde{\delta} = \widetilde{\delta}(T_0) > 0$ such that $\|\mathcal{H}\|_{\mathcal{B}(\mathcal{M}_I^c)} < \widetilde{\delta}$ implies $T = T_0 + \mathcal{H}$ is a quaternionic exponentially dichotomous operator.*

The following proof is from reference [28].

Proof. Let T_0 be a quaternionic exponentially dichotomous operator on a s-quaternionic Banach space $\mathcal{M}_I^c$, according to Theorem 9, for $M>0$ and a fixed $r>0$,

$$\|E(t,T_0)m\|_{\mathcal{M}_I^c} \leqslant Me^{-r|t|}\|m\|_{\mathcal{M}_I^c}, \quad t\in\mathbb{R}\backslash\{0\},\ m\in\mathcal{M}_I^c.$$

Thus, for $0<\varepsilon<r$, we have

$$\int_{-\infty}^{\infty} e^{\varepsilon|t|}\big\|E(t,T_0)m\big\|_{\mathcal{M}_I^c}\ dt \leqslant \frac{2M\|m\|_{\mathcal{M}_I^c}}{r-\varepsilon}, \quad m\in\mathcal{M}_I^c.$$

Now, for each $\widetilde{\delta}\in(-r,r)$, consider $K(\cdot,m)=e^{\widetilde{\delta}|\cdot|}\mathcal{H}E(\cdot,T_0)m$, then $K(\cdot,m)$ satisfy the conditions (1) and (2) in Lemma 5. Thus, for integrable simple function $\phi_I(I\in\mathbb{S})$, the operator

$$\big(H_{\widetilde{\delta}}\phi_I\big)(t)=\int_{-\infty}^{\infty} e^{\widetilde{\delta}|t-\tau|}\mathcal{H}E(t-\tau,T_0)\phi_I(\tau)\ d\tau \tag{3}$$

is bounded on $L^1(\mathbb{R},\mathcal{M}_I^c)$ with norm satisfying

$$\|H_{\widetilde{\delta}}\|_{\mathcal{B}(\mathcal{M}_I^c)}=\sup_{\|m\|_{\mathcal{M}_I^c}=1}\int_{-\infty}^{\infty}\big\|e^{\widetilde{\delta}|t|}\mathcal{H}E(t,T_0)m\big\|_{\mathcal{M}_I^c}\ dt\leqslant\frac{2Mb\|\mathcal{H}\|_{\mathcal{B}(\mathcal{M}_I^c)}}{r-\widetilde{\delta}}.$$

Therefore, $\|\mathcal{H}\|_{\mathcal{B}(\mathcal{M}_I^c)}<r/2M$ and $0<\widetilde{\delta}<r-2M\|\mathcal{H}\|_{\mathcal{B}(\mathcal{M}_I^c)}$ imply $\frac{2M\|\mathcal{H}\|_{\mathcal{B}(\mathcal{M}_I^c)}}{r-\widetilde{\delta}}<1$. Hence, consider the integral equation as follows

$$\mathfrak{S}F(t,m)=\int_{-\infty}^{\infty}E(t-\tau,T_0)\mathcal{H}F(\tau,m)\ d\tau+E(t,T_0)m. \tag{4}$$

Then, we have

$$\begin{aligned}
&\big\|\mathfrak{S}F_1(t,m)-\mathfrak{S}F_2(t,m)\big\|_{\mathcal{M}_I^c}=\\
&\bigg\|\int_{-\infty}^{\infty}E(t-\tau,T_0)\mathcal{H}F_1(\tau,m)\ d\tau-\int_{-\infty}^{\infty}E(t-\tau,T_0)\mathcal{H}F_2(\tau,m)\ d\tau\bigg\|_{\mathcal{M}_I^c}\\
\leqslant&\int_{-\infty}^{\infty}\big\|E(t-\tau,T_0)\mathcal{H}\big\|_{\mathcal{M}_I^c}\ d\tau\cdot\big\|F_1(t,m)-F_2(t,m)\big\|_{\mathcal{M}_I^c}\\
\leqslant&\frac{2M\|\mathcal{H}\|_{\mathcal{B}(\mathcal{M}_I^c)}}{r-\widetilde{\delta}}\big\|F_1(t,m)-F_2(t,m)\big\|_{\mathcal{M}_I^c}.
\end{aligned}$$

According to the contraction mapping principle (see [2]), (4) has a unique solution $F(\cdot,m)\in L^1(\mathbb{R},\mathcal{M}_I^c)$ for every $m\in\mathcal{M}_I^c$ and note that $F(\cdot,m)$ is strongly measurable for each $m\in\mathcal{M}_I^c$. Thus, we obtain

$$F(t,m)=\int_{-\infty}^{\infty}E(t-\tau,T_0)\mathcal{H}F(\tau,m)\ d\tau+E(t,T_0)m. \tag{5}$$

Then for $\|\mathcal{H}\|_{\mathcal{B}(\mathcal{M}_I^c)} < r/2M$,

$$\begin{aligned}\big\|F(t,m)\big\|_{\mathcal{M}_I^c} =&\Big\|\int_{-\infty}^{\infty} E(t-\tau,T_0)\mathcal{H}F(\tau,m)\ d\tau + E(t,T_0)m\Big\|_{\mathcal{M}_I^c}\\ \leqslant&\Big\|\int_{-\infty}^{\infty} E(t-\tau,T_0)\mathcal{H}F(\tau,m)\ d\tau\Big\|_{\mathcal{M}_I^c} + \big\|E(t,T_0)m\big\|_{\mathcal{M}_I^c}\\ \leqslant&M\|\mathcal{H}\|_{\mathcal{B}(\mathcal{M}_I^c)}\int_{-\infty}^{\infty}\|F(\tau,m)\|_{\mathcal{M}_I^c}e^{-r|t-\tau|}\ d\tau + Me^{-r|t|}\|m\|_{\mathcal{M}_I^c}.\end{aligned}$$

Thus, there exists $\widetilde{\varphi}:\mathbb{R}\to\mathbb{R}$ such that the norms of (5) are dominated by

$$\widetilde{\varphi}(t) - M\|\mathcal{H}\|_{\mathcal{B}(\mathcal{M}_I^c)}\int_{-\infty}^{\infty} e^{-r|t-\tau|}\widetilde{\varphi}(\tau)\ d\tau = Me^{-r|t|}\|m\|_{\mathcal{M}_I^c},$$

where

$$\widetilde{\varphi}(t) = \frac{M\|m\|_{\mathcal{M}_I^c}}{r-2M\|\mathcal{H}\|_{\mathcal{B}(\mathcal{M}_I^c)}}e^{-|t|\left(r-2M\|\mathcal{H}\|_{\mathcal{B}(\mathcal{M}_I^c)}\right)}$$

is its unique solution. Then

$$\|F(t,m)\|_{\mathcal{B}(\mathcal{M}_I^c)} \leqslant \frac{M\|m\|_{\mathcal{M}_I^c}}{r-2M\|\mathcal{H}\|_{\mathcal{B}(\mathcal{M}_I^c)}}e^{-|t|(r-2M\|\mathcal{H}\|_{\mathcal{B}(\mathcal{M}_I^c)})}$$

for a.e. $t\in\mathbb{R}$ and each $m\in\mathcal{M}_I^c$. Moreover, similar to the proof of Theorem 8, we have

$$S_R^{-1}(s,T)m = \int_{-\infty}^{\infty} e^{-st}\left[E(t,T_0)m + \int_{-\infty}^{\infty} E(\tau,T_0)\widetilde{\phi}_{\varphi_I}(t-\tau)\ d\tau\right]\ dt,$$

where $\widetilde{\phi}_{\varphi_I}(t) = \mathcal{H}F(t,m)$, noting that

$$\int_{-\infty}^{\infty} E(\tau,T_0)\widetilde{\phi}_{\varphi_I}(t-\tau)\ d\tau = \int_{-\infty}^{\infty} E(t-\tau,T_0)\mathcal{H}F(\tau,m)\ d\tau$$

which implies that

$$S_R^{-1}(s,T)m = \int_{-\infty}^{\infty} e^{-st}F(t,m)\ dt,\quad |\mathrm{Re}(s)| < r-2M\|\mathcal{H}\|_{\mathcal{B}(\mathcal{M}_I^c)}.$$

By Theorem 9, it follows that T is a quaternionic exponentially dichotomous operator. This completes the proof. □

To obtain the most general result for perturbation theorem on s-quaternionic Hilbert space, we need the following Parseval's Theorem in quaternionic version.

Theorem 11 ([28]). *Let $\mathcal{M}_I^c(I\in\mathbb{S})$ be a s-quaternionic Banach space, $f,g\in L^2\big(\mathbb{R},\mathcal{M}_I^c\big)$, then*

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}\ dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathfrak{F}\big[f(t)\big]\overline{\mathfrak{F}\big[g(t)\big]}\ dt, \tag{6}$$

where $\overline{g(t)}$ is the conjugate of $g(t)$, $\mathfrak{F}\big[f(\tau)\big] = \int_{-\infty}^{\infty} e^{-2\pi I\tau t}f(t)\ dt$, $I\in\mathbb{S}$ is the Fourier transform on s-quaternionic Banach space $\mathcal{M}_I^c$.

The following result is the most general perturbation theorem on s - quaternionic Hilbert space.

Theorem 12 ([28]). *Let $T_0 \in \mathbb{C}_I^{\mathbb{S}}(\mathcal{M}_I^c)$ $(I \in \mathbb{S})$ be a quaternionic exponentially dichotomous operator on a s-quaternionic Hilbert space $\mathcal{M}_I^c$, $\mathcal{H} \in \mathcal{B}(\mathcal{M}_I^c) \cap \mathbb{C}_I^{\mathbb{S}}(\mathcal{M}_I^c)$, and $T = T_0 + \mathcal{H}$. If*

$$Q_\eta^I = \{s \in \mathbb{C}_I : |\mathrm{Re}(s)| \leqslant \eta\} \subset \rho_{S_I}(T)$$

for some $\eta > 0$ and $S_R^{-1}(s,T)$ is bounded on Q_η^I, then T is a quaternionic exponentially dichotomous operator.

The proof progress presented in the following is from reference [28].

Proof. Let $\varepsilon \in (0,\eta]$, $m \in \mathcal{M}_I^c$, then $e^{\varepsilon|\cdot|}E(\cdot,T_0)m : \mathbb{R} \to \mathcal{M}_I^c$ is Pettis integrable. Therefore, according to Lemma 6, for any $\widetilde{\delta} \in (-\varepsilon,\varepsilon)$ the operator given by (3) is bounded on $L^2(\mathbb{R},\mathcal{M}_I^c)$ with norm bounded above by

$$\sup_{s \in I\mathbb{R}} \left\| W(s-\widetilde{\delta}) - \mathcal{I} \right\|_{\mathcal{B}(\mathcal{M}_I^c)}, \quad I \in \mathbb{S}, \tag{7}$$

where $W(s) = \mathcal{I} - S_R^{-1}(s,T_0)\mathcal{H}$, $|\mathrm{Re}(s)| < \varepsilon$. Thus, for each $\widetilde{\delta} \in (-\varepsilon,\varepsilon)$ the operator $H_{\widetilde{\delta}}$ is bounded on $L^2(\mathbb{R},\mathcal{M}_I^c)$ with norm bounded above by (7). From the proof of Theorem 8, we have $W^{-1}(s) = \mathcal{I} + S_R^{-1}(s,T)\mathcal{H}$ for $|\mathrm{Re}(s)|$ sufficiently small, then

$$M_{\widetilde{\delta}} \overset{def}{=} \sup_{\mathrm{Re}(s) = -\widetilde{\delta}} \left\| W^{-1}(s) \right\|_{\mathcal{B}(\mathcal{M}_I^c)} < \infty, \quad \widetilde{\delta} \in [-\eta,\eta]. \tag{8}$$

Next, recall the integral equation (5). Then for $\widetilde{\delta} \in (-\varepsilon,\varepsilon)$, $e^{\widetilde{\delta}|t|}F(\cdot,m) \in L^2(\mathbb{R},\mathcal{M}_I^c)$ and $F(\cdot,m)$ satisfies

$$\int_{-\infty}^{\infty} e^{-st}F(t,m)\,dt = W^{-1}(s)S_R^{-1}(s,T_0)m = S_R^{-1}(s,T)m, \tag{9}$$

where $m \in \mathcal{M}_I^c$, $|\mathrm{Re}(s)| < \varepsilon$. By the hypothesis (7) and Theorem 11, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \|F(t,m)\|^2 e^{2\widetilde{\delta}|t|}\,dt =& \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\| \mathfrak{F}[e^{\widetilde{\delta}|t|}F(t,m)] \right\|_{\mathcal{M}_I^c}^2 dt \\
=& \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\| \int_{-\infty}^{\infty} e^{-2\pi It\tau} e^{\widetilde{\delta}|\tau|}F(\tau,m)\,d\tau \right\|_{\mathcal{M}_I^c}^2 dt \\
=& \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \|S_R^{-1}(-\widetilde{\delta}+Iu,T)m\|_{\mathcal{M}_I^c}^2\,du \\
\leqslant& \frac{(M_{\widetilde{\delta}})^2}{(2\pi)^2}\int_{-\infty}^{\infty} \|S_R^{-1}(-\widetilde{\delta}+Iu,T_0)m\|_{\mathcal{M}_I^c}^2\,du \\
\leqslant& \frac{(M_{\widetilde{\delta}})^2}{2\pi}\int_{-\infty}^{\infty} \|E(t,T_0)m\|_{\mathcal{M}_I^c}^2 e^{2\widetilde{\delta}|t|}\,dt < \infty,
\end{aligned}$$

where $m \in \mathcal{M}_I^c$, $I \in \mathbb{S}$.

Note that for $e^{\widetilde{\delta}|t|}F(\cdot, m) \in L^2(\mathbb{R}, \mathcal{M}_I^c)$, any $\theta \in (-\widetilde{\delta}, \widetilde{\delta})$ and each $m \in \mathcal{M}_I^c$, by Hölder inequality, we have

$$\begin{aligned}\int_{-\infty}^{\infty} e^{\theta|t|}\|F(t,m)\|_{\mathcal{M}_I^c}\, dt &\leqslant \left[\int_{-\infty}^{\infty} e^{-2(\widetilde{\delta}-\theta)|t|}\, dt\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} e^{2\widetilde{\delta}|t|}\|F(\cdot,m)\|_{\mathcal{M}_I^c}^2\, dt\right]^{\frac{1}{2}} \\ &\leqslant \frac{1}{\sqrt{2(\widetilde{\delta}-\theta)}}\left[\int_{-\infty}^{\infty} e^{2\widetilde{\delta}|t|}\|E(t,T_0)m\|_{\mathcal{M}_I^c}^2\, dt\right]^{\frac{1}{2}} \\ &\leqslant \frac{1}{\sqrt{2(\widetilde{\delta}-\theta)}}\frac{M}{\sqrt{r-\widetilde{\delta}}}\|m\|_{\mathcal{M}_I^c} \overset{def}{=} \widehat{u}\|m\|_{\mathcal{M}_I^c}.\end{aligned}$$

Since

$$F(t,m) = E(t,T_0)m - \int_{-\infty}^{\infty} E(\tau,T_0)\mathcal{H}F(t-\tau,m)\, d\tau,$$

then, from the proof of Theorem 10 we obtain

$$\|F(t,m)\|_{\mathcal{M}_I^c} \leqslant \frac{M\|m\|_{\mathcal{M}_I^c}}{r-2\widehat{u}\|\mathcal{H}\|_{\mathcal{B}(\mathcal{M}_I^c)}} e^{-|t|(r-2\widehat{u}\|\mathcal{H}\|_{\mathcal{B}(\mathcal{M}_I^c)})}, \quad a.e.\ t \in \mathbb{R}.$$

Then by Theorem 9, we have T is a quaternionic exponentially dichotomous operator. This completes the proof. □

Since the significance of perturbation invariance of quaternionic exponentially dichotomous operators and their comprehensive applications, the further investigation of their related properties will be our main future work.

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