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MINIMAL NUMBER OF SENSORS FOR 3D COVERAGE

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Abstract

This paper presents some theoretical results on the smaller number $N_k(a,b,c)$ of sensors to achieve k coverage for the 3D rectangular area $[0,a]\times[0,b]\times[0,c]$. The first properties outline some theoretical results for the numbers $N_k(a,b,c)$, including symmetry, sub-additivity and monotony on each variable. We use then these results to establish some lower and upper bounds for $N_k(a,b,c)$. The main contribution proposes a result concerning the minimal density of sensors to achieve k-coverage.

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Key words: sensor coverage, sensor density.

1 Problem statement

We start from $S=\{s_1,s_2,s_3,...,s_n\}$, a set of sensors in the 3D plane with the same sensing range r, which means that each sensor can cover a spherical volume or radius r. The position of each sensor s_i is known given by (x_i,y_i,z_i). The target area A to monitor is defined as a 3D rectangle $A=[0,w]\times[0,l]\times[0,h]$ of width w, length l and height h.

Definition 1. *A point $(x,y,z)\in A$ is covered by a sensor s_i if*

$$\sqrt{(x_i-x)^2+(y_i-y)^2+(z_i-z)^2}\leq r.$$

The target area A is k-covered by the sensors S if each point $(x,y,z)\in A$ is covered by at least k different sensors.

Most covering problems present huge difficulties to solve or to derive a polynomial algorithm even in particular cases like regular or simple shapes and lower dimensional space. The 2D and 3D problems of covering a bounded domain with arbitrary shaped objects were proven to be exponential on the size of the packing

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space [9]. The particular case of covering any 2D or 3D polygon with n similar disks is known to be NP-hard [3]. Consequently, the problem of finding the least number of disks to k-cover a 3D rectangle is NP-hard. All these works have shown that calculating the minimal number of circles to pack a rectangle is a hard problem and there is no pattern associated with this covering. Moreover, the results concerning the minimal number of circles for k-coverage are all either asymptotical based on some limits or approximative based on some inequalities.

The $3D$ coverage problem can have multiple practical applications when sensors have to be deployed in a $3D$ environments. The first scenario that can be considered is the deployment of the sensors in buildings. This problem can investigate how to deploy the sensors to have a good coverage, however one may need to know how many sensors are needed. Because, building environments are accessible easily, these sensors can be deployed manually at some predefined locations. The second scenario can consider the distribution of sensors in $3D$ terrain environments, like forests, in order to monitor fire hazards. Usually, a number of sensors are thrown from airplanes to cover the areas that are not reachable. Some sensors can fall on trees and some on the ground however those landing on trees would have a $3D$ distribution. For this problem, one should know a minimal number of sensors that can thrown in order to have good connectivity of the WSN network.

2 Related works

The literature review has been carried out based on the PRISMA methodology outlined in [11]. Firstly, a scholar google search has been performed using the filtering words like 'coverage in WSN' or '3D coverage' or variants of that. The period of the search has been set between 2000 to date however only few search results have been generated in recent years. The search results have been then processed in 2 steps. Firstly, the abstract has been analysed to see if the contribution matches the literature review context. Secondly, the articles of interest have been read to identify the main important results. To report on the PRISMA outputs, more than 50 articles have been 'abstract' screened, from which 15 have been retrieved for further reading. Finally, the following articles are assessed as eligible and important for the context of optimal coverage.

2.1 Coverage in 2D

Investigating 1-coverage or circle packing has been researched as geometrical combinatorial since 1939. Researchers have tried to find mathematical equations for the optimal 1-covering or even to prove that some configurations are optimal. For example Kershner [6] investigated the problem of covering any 2D set of points with similar circles based on some geometrical combinatorial techniques. This early work proved that the minimal number of circles $N(\varepsilon)$ of radius ε to cover a close set of point M satisfies

$$\lim_{\varepsilon \to 0} N(\varepsilon) = \frac{2\sqrt{3}}{9 \cdot \varepsilon^2} \cdot |M|,$$

where $|M|$ denotes the area of closed by M. The result was proven by using a double inequality for the quantity $\pi\varepsilon^2 N(\varepsilon)$ representing the total area covered by the circles. An important consequence of this result is that the proportion of unavoidable overlapping can be approximated by $\frac{2\pi\sqrt{3}}{9} \simeq 1.209$. We can also mention the early work of Verblunsky [15] who proved that the minimum number $N(l)$ of circles of radius 1 to cover a square of length l should satisfy

$$N(l) \geq \frac{2l^2 + l}{3\sqrt{3}}.$$

These two results come to suggest that the sensor density for 1-coverage can be estimated by $\frac{2\sqrt{3}}{9} \approx 0.384$.

However, these early works [6], [15] do not provide any information about the pattern of circles used to achieve the minimal coverage. Recently, several articles on circle packing problems investigated efficient ways to cover a rectangle with similar circles (see [8], [10] or [14] amongst others). These geometrical combinatorics researches confirmed that optimal packing is difficult to be achieve even for small number of circles. Furthermore, no pattern was detected for the packing configuration that gives optimality.

Lately, several papers have dealt with the k-coverage problems in the context of sensor networks studying conditions when this is achieved or algorithms to detected when this happens. Some of these contributions have made marginal reference to the minimum number or equivalently to the minimum density of sensors that assures k-covering of a given area. Generally, all these works have considered that the number of sensors to use is big enough to k-cover the target area. Adlakha and Srivasyava [1] developed an exposure-based model to find the sensor density required to achieve full coverage of a given area. They proved that the number of sensors to achieve 1-covering is in the order of $O(A/r^2)$, where A is the area to cover however they did not provide any constant for the magnitude of A/r^2.

Ammari and Das [2] investigated the problem of k-coverage proposing a condition to achieve it. They considered the target area divided in "Reuleaux" triangles which are formed by the intersection of 3 circles. The main result of their work states that the target area is k-covered if and only if each "Reuleaux" triangle contains at least k-active sensors. Another important results proposed by Ammari and Das gives that the minimal density of sensors to guarantee k-coverage is $\lambda(r, k) = \frac{2k}{(\pi - \sqrt{3}) \cdot r^2} = \frac{1.4188 \cdot k}{r^2}$, where r is the radius of the sensing disks.

Tabirca et.al. [13] proposed a study for the minimal number of sensors for the 2D coverage, where some mathematical equations are provided and some lower and upper bounds are given for these numbers. The authors also table the numbers for small values of a, b.

2.2 Coverage in 3D

The research into the 3D coverage has been driven by the necessity to have good coverage in WSN networks deployed in buildings or even in cellular networks. These works focussed mainly in finding efficient deployments based on 3D geometrical shapes including spheres, octahedrons or dodecahedrons. Another research direction has dealt with theoretical results on the coverage analysing properties like coverage density, uncovered areas etc. However, the transition from 2D coverage to 3D coverage is hugely more difficult posing multiple challenges.

Zhang et.al. [17] investigated the full-coverage deployment problem using the concept of lattice mostly inspired by the some results from both in discrete computational geometry and WSNs. They made a connection between optimal patterns under certain regularity constraints and some important natural constructs that show strong regularity in their components. Because the coverage exposed properties of periodicity and homogeneity, the authors decided to model the coverage using lattices. The main contribution of this research work ca be summarised as follows. Firstly, a new deployment coverage patterns was proposed to obtain optimality for connectivity under some constraints of the lattice. Secondly, the authors showed how their optimal patterns based on the sphere sensing and communication can be used with some practical models.

Huang et. al. [5] work proposed an efficient solution for the coverage based on a reduction of the geometric problem from a 3D space to a 2D space and then to a 1D space. Their approach used spheres and spheres' intersections (spheres' caps) to propose a polynomial algorithm for coverage. They defined the notions of spheres' coverage and field sensing and proposed some theoretical results for them. To determined whether each sensor's sphere is sufficiently covered, the authors looked at how each spherical cap is covered. By projecting each circle on a 1-dimensional line, they proposed a method to determine the level of coverage.

Yun et. al. [16] proposed an optimal patterns for sensors deployment that achieve full coverage as well as k-connectivity for WSNs. The authors stated that there exists a hexagon-based universally elemental pattern that can be used to generate all other patterns. They then proved the optimality of designed pattern coverage and proposed a new deployment polygon based methodology. For this type of deployment, Yun et. al. studied the k-connectivity problem showing that it is optimal for $k < 6$ based on a theoretical result that connects the coverage with connectivity graph topology.

Priyadarshi et.al. [12] developed recently a method in which deterministic sensor deployment to cover the entire space is generate by using some non linear optimisation. Several geometrical properties have been proposed between the sphere radius and the sizes of the area to cover. Then these expressions are integrated into a multivariable function that is then maximised using Lagrange multipliers to generates an optimal value for coverage volume based on the radius. The authors proposed a three-dimensional coverage pattern and deployment structure based on cuboid, where the monitor region is partitioned into three-dimensional grid. Some experiments showed that the number of sensors decreases to a large extent compared with traditional deployed method such as cube and regular tetrahedron deployment schemes.

3 Minimal number of sensors

Consider the following problem *"Find the smallest number of sensors $N_k(a,b,c)$ that should be used to achieve k-coverage for a $3D$ rectangular area $[0,a]\times[0,b]\times[0,c]$ with sensors of the same radius"*. We can suppose that all the sensors have the coverage radius of 1 unit. By convention $N_k(a,b,c)=0$ when $a\le 0$ or $b\le 0$ or $c\le 0$. It is clear that a k-coverage with n sensors satisfies

$$n \ge N_k(a,b,c). \tag{1}$$

The following results can be directly obtained based on Equation 1 and on the definition of $N_k(a,b,c)$.

Lemma 1. *The function $N_k(a,b,c)$ is symmetrical on a,b,c*

$$N_k(a,b,c)=N_k(p(a,b,c)),\quad \forall a,b,c>0 \textit{ and } \forall p \textit{ permutation of } a,b,c.$$

Lemma 2. *The function $N_k(a,b,c)$ is monotonically on each variable a,b,c as well as in k:*

$$a_1 \le a_2 \Rightarrow N_k(a_1,b,c)\le N_k(a_2,b,c).$$
$$b_1 \le b_2 \Rightarrow N_k(a,b_1,c)\le N_k(a,b_2,c).$$
$$c_1 \le c_2 \Rightarrow N_k(a,b,c_1)\le N_k(a,b,c_1).$$
$$k_1 \le k_2 \Rightarrow N_{k_1}(a,b,c)\le N_{k_2}(a,b,c).$$

Lemma 3. *The function $N_k(a,b,c)$ is sub-additive on each variable a,b,c as well as in k:*

$$N_k(a_1+a_2,b,c)\le N_k(a_1,b,c)+N_k(a_2,b,c).$$
$$N_k(a,b_1+b_2,c)\le N_k(a,b_1,c)+N_k(a,b_2,c).$$
$$N_k(a,b,c_1+c_2)\le N_k(a,b,c_1)+N_k(a,b,c_2).$$
$$N_{k_1+k_2}(a,b,c)\le N_{k_1}(a,b,c)+N_{k_2}(a,b,c).$$

It is simple to observe that a cube of size s has the diagonal of size $d=\sqrt{3}\cdot s$, hence a cube o size $s=\frac{2}{\sqrt{3}}$ can be 1-covered by a sphere of radius 1, because the sphere circumscribe the cube (see Figure 1).

Proposition 1. $N_1(\frac{2}{\sqrt{3}}\cdot n, \frac{2}{\sqrt{3}}\cdot m, \frac{2}{\sqrt{3}}\cdot p)\le n\cdot m\cdot p$ *when* $n,m,p\in N$.

Proof. Consider that the $3D$ rectangular area of sizes $a=\frac{2}{\sqrt{3}}\cdot n,\ b=\frac{2}{\sqrt{3}}\cdot m$ *and* $c=\frac{2}{\sqrt{3}}\cdot p$ is divided into a grid of $n\times m\times p$ cubes of size $\frac{2}{\sqrt{3}}$ (see Figure 2). Each such cube can be 1-covered by its circumscribed sphere of radius 1. Hence, $N_1(\frac{2}{\sqrt{3}}\cdot n, \frac{2}{\sqrt{3}}\cdot m, \frac{2}{\sqrt{3}}\cdot p)\le n\cdot m\cdot p$ since there is a 1-coverage with $n\cdot m\cdot p$ spheres of the whole volume. □

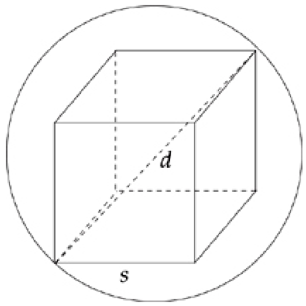

Figure 1: Covering a Cube.

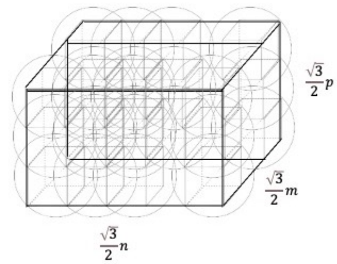

Figure 2: Covering the cuboid

Proposition 2. *The numbers $N_1(a,b,c)$ satisfy the following inequality*

$$N_1(a,b,c) \leq \lceil \frac{\sqrt{3}a}{2} \rceil \cdot \lceil \frac{\sqrt{3}b}{2} \rceil \cdot \lceil \frac{\sqrt{3}c}{2} \rceil, \quad \forall a,b \in R \tag{2}$$

where $\lceil x \rceil$ is the ceiling function.

Proof. Consider $n = \lceil \frac{\sqrt{3}a}{2} \rceil \in N$ so that we have $\frac{\sqrt{3}a}{2} \leq n$ or $a \leq n \cdot \frac{2}{\sqrt{3}}$. Similarly, we have $b \leq m \cdot \frac{2}{\sqrt{3}}$ and $c \leq k \cdot \frac{2}{\sqrt{3}}$. Now, the following inequality can be derived based on Lemma 1

$$N_1(a,b,c) \leq N_1\left(n \cdot \frac{2}{\sqrt{3}}, m \cdot \frac{2}{\sqrt{3}}, n \cdot \frac{2}{\sqrt{3}}\right) \Rightarrow$$

$$N_1(a,b,c) \leq n \cdot m \cdot k \Rightarrow N_1(a,b,c) \leq \lceil \frac{\sqrt{3}a}{2} \rceil \cdot \lceil \frac{\sqrt{3}b}{2} \rceil \cdot \lceil \frac{\sqrt{3}c}{2} \rceil,$$

which it proves the theorem. □

The result above gives only an upper bound of values in which the number $N_1(a,b,c)$ can be located.

Theorem 1. *For k-coverage problem, the numbers $N_k(a,b,c)$ satisfy the following inequality*

$$\frac{3 \cdot k \cdot a \cdot b \cdot c}{4 \cdot \pi} \leq N_k(a,b,c) \leq k \cdot \lceil \frac{\sqrt{3}a}{2} \rceil \cdot \lceil \frac{\sqrt{3}b}{2} \rceil \cdot \lceil \frac{\sqrt{3}c}{2} \rceil, \quad \forall a,b \in R \tag{3}$$

Proof. The sub-additivity property is used as follows

$$N_k(a,b,c) = N_{1+...+1}(a,b,c) \leq N_1(a,b,c) + ... + N_1(a,b,c) =$$

$$= k \cdot N_1(a,b,c) \leq k \cdot \lceil \frac{\sqrt{3}a}{2} \rceil \cdot \lceil \frac{\sqrt{3}b}{2} \rceil \cdot \lceil \frac{\sqrt{3}c}{2} \rceil,$$

which proves the right hand side inequality. For the left hand side we considered that each point of the $3D$ rectangle is covered by at least k spheres. Hence, the

$N_k(a,b,c)$ spheres cover the whole $3D$ rectangle volume by k times. Hence, the volume of the spheres is greater than k times the volume of the $3D$ rectangle.

$$\frac{4}{3} \cdot N_k(a,b,c) \cdot \pi \cdot 1^3 \geq k \cdot a \cdot b \cdot c \Rightarrow N_k(a,b,c) \geq \frac{3 \cdot k \cdot a \cdot b \cdot c}{4 \cdot \pi}.$$

□

Now we can look into the density of sensors to cover an area are the number of sensors used over the volume of the target area. It make sense to investigate the minimal density of sensors to cover the rectangular area $[0,a] \times [0,b] \times [0,c]$ as given by $\lambda_{a,b,c}(k) = \frac{N_k(a,b,c)}{a \cdot b \cdot c}$. We can assume that the minimal density of sensors $\lambda_{a,b,c}(k)$ does not depend on a,b,c for large values of a,b,c, so that we can write $\lambda(k) \simeq \frac{N_k(a,b,c)}{a \cdot b \cdot c}, \forall a,b,c$. In this case the minimal density of sensors can be evaluated by the following result.

Theorem 2. *The minimum density of sensors to achieve k-covering for rectangular areas satisfies*

$$\frac{3}{4 \cdot \pi} \cdot k \leq \lambda(k) \leq \frac{3 \cdot \sqrt{3}}{8} \cdot k. \tag{4}$$

Proof. Each member of Equation 3 is divided by $a \cdot b \cdot c$ to obtain

$$\frac{3 \cdot k}{4 \cdot \pi} \leq \frac{N_k(a,b,c)}{a \cdot b \cdot c} \leq k \cdot \frac{\lceil \frac{\sqrt{3}a}{2} \rceil}{a} \cdot \frac{\lceil \frac{\sqrt{3}b}{2} \rceil}{b} \cdot \frac{\lceil \frac{\sqrt{3}c}{2} \rceil}{c}.$$

Based on the above assumption that the minimum density to achieve k-covering is independent of the area to cover when a,b,c are big, we can take $a,b,c \to \infty$. Hence, the fractions become $\lim_{a \to \infty} \frac{\lceil \frac{\sqrt{3}a}{2} \rceil}{a} = \frac{\sqrt{3}a}{2}$ and similar for b and c so that $\lambda(k)$ sadisfies

$$\frac{3}{4 \cdot \pi} \cdot k \leq \lambda(k) \leq \frac{3 \cdot \sqrt{3}}{8} \cdot k, \quad \forall a,b,c > 0.$$

□

Theorem 2 shows that the minimal density to achieve k-coverage with sensors of radius 1 is between $0.238732 \cdot k$ and $0.64951905 \cdot k$.

On the other hand, the minimal density of sensors $\frac{N_k(a,b,c)}{a \cdot b \cdot c}$ can also have the following upper bound for any $a,b,c \geq 0$.

$$\frac{N_k(a,b,c)}{a \cdot b \cdot c} \leq k \cdot \frac{\lceil \frac{\sqrt{3}a}{2} \rceil}{a} \cdot \frac{\lceil \frac{\sqrt{3}b}{2} \rceil}{b} \cdot \frac{\lceil \frac{\sqrt{3}c}{2} \rceil}{c} \leq k \cdot \frac{\frac{\sqrt{3}a}{2} + 1}{a} \cdot \frac{\frac{\sqrt{3}b}{2} + 1}{b} \cdot \frac{\frac{\sqrt{3}c}{2} + 1}{c}$$

$$\frac{N_k(a,b,c)}{a \cdot b \cdot c} \leq k \cdot \left(\frac{\sqrt{3}}{2} + \frac{1}{a}\right) \cdot \left(\frac{\sqrt{3}}{2} + \frac{1}{b}\right) \cdot \left(\frac{\sqrt{3}}{2} + \frac{1}{c}\right) =$$

$$= k \cdot \left[\frac{3\sqrt{3}}{8} + \frac{3}{4} \cdot \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + \frac{\sqrt{3}}{2} \cdot \left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \right) + \frac{1}{abc} \right] =$$

$$= k \cdot \left[\frac{3\sqrt{3}}{8} + \frac{3}{8} \cdot \frac{2 \cdot a \cdot b + 2 \cdot b \cdot c + 2 \cdot c \cdot a}{a \cdot b \cdot c} + \frac{\sqrt{3}}{8} \cdot \frac{4 \cdot a + 4 \cdot b + 4 \cdot c}{a \cdot b \cdot c} + \frac{1}{abc} \right] =$$

$$\frac{k}{8} \cdot \left[3\sqrt{3} + \frac{3 \cdot A + \sqrt{3} \cdot P + 8}{V} \right] \Longrightarrow$$

$$\frac{N_k(a,b,c)}{a \cdot b \cdot c} \leq \frac{k}{8} \cdot \left[3\sqrt{3} + \frac{3 \cdot A + \sqrt{3} \cdot P + 8}{V} \right],$$

where $P = 4 \cdot a + 4 \cdot b + 4 \cdot c$, $A = 2 \cdot a \cdot b + 2 \cdot b \cdot c + 2 \cdot c \cdot a$ and $V = a \cdot b \cdot c$ are the perimeter, the area and the volume respectively of the $3D$ rectangle. This provides an upper bound for the density based on the geometrical sizes of the target area.

4 Conclusion

The article presented some theoretical results related to the minimal number of sensors $N_k(a,b,c)$. It was proven that these numbers are sub-additive and increasing. Then some lower and upper bounds were proposed for $N_k(a,b,c)$ to show that the minimal density for k-coverage satisfies $\frac{3}{4 \cdot \pi} \cdot k \leq \lambda(k) \leq \frac{3 \cdot \sqrt{3}}{8} \cdot k$.

The work can be extended by applying these theoretical results to some concrete $3D$ deployments, the most obvious being building deployments. Considering the blueprints or a building plus the sensors' coverage radius, some algorithms for deployments can be developed.

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