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RICCI SOLITONS ON α-SASAKIAN MANIFOLDS WITH QUARTER SYMMETRIC METRIC CONNECTION

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Abstract

The main object of the present paper is to discuss about the quarter symmetric metric connection on α-Sasakian Manifold with respect to Ricci Soliton. In this paper, firstly, we discuss the quarter symmetric metric connection on α-Sasakian manifold. Secondly, we elaborate the results of quarter symmetric metric connection on α-Sasakian manifold which admits Ricci Soliton and also flourish a non-trivial example of α-sasakian manifold and validate some of our results. Thirdly, we classify certain curvature properties of Ricci α-Sasakian manifold in regard to Quarter symmetric metric connection. Finally, we show Ricci Soliton on submanifold of α-Sasakian manifold in term of Levi-Civita and quarter symmetric metric connection.

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1 Introduction

In differential geometry, R. S. Hamilton [9] was the first one who introduced the concept of Ricci flow in 1982 and if it moves only by a one parameter family of diffeomorphism and scaling then self-similar solution to the Ricci flow is said to be Ricci soliton (shortly, RS) which is represented below:

$$(\pounds_V g)(Y,Z) + 2S(Y,Z) + 2\lambda g(Y,Z) = 0, \tag{1}$$

where $\pounds_V$, S, g and λ are respectively denoted by the Lie derivative in the direction of vector field V, Ricci tensor (shortly, RT), Riemannian metric and a scalar

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or constant.

Also, RS is said to be shrinking, steady and expanding according to λ's nature, that is, negative, zero and positive respectively. A RS on a Riemannian manifold (M, g) is known as a generalization of an Einstein metric. During the last two decades, in contact metric manifold the geometry of Ricci solitons has been studied by various authors such as Chen and Deshmukh [5], Deshmukh et. al [6], Tripathi [20], Hui et. al ([12, 16]) and many more (see [4]). Recently, Vandana et al. [21] investigated significant findings concerning generic contact CR-submanifolds embedded in Sasakian manifolds and accompanied by concurrent vector fields. Subsequently, they explored the practical applications of solitons, specifically Ricci and Ricci-Yamabe solitons, on such submanifolds possessing concurrent vector fields within the same overarching manifold.

On the other hand, H. A. Hayden [10] introduced the idea of a metric connection with non-zero torsion on a Riemannian manifold in 1932. The connection ∇ on M is said to be a metric connection if there is g on M such that $\nabla g = 0$, otherwise it is non-metric. In 1975, quarter symmetric (shortly, QS) linear connection on a differentiable manifold was studied by S. Golab in [8]. The QS connection is said to be quarter symmetric metric (shortly, QSM) connection if $\overline{\nabla} g = 0$ otherwise quarter symmetric non-metric connection [3]. The relation between Levi-Civita connection ∇ and QSM connection $\overline{\nabla}$ of a contact metric manifold is given by

$$\overline{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{2}$$

Note that QS connection can be considered as a generalization of semi-symmetric connection because if we take $\phi X = X$ and $\phi Y = Y$, then QS becomes semi-symmetric connection [7]. Golab studied semi-symmetric and quarter symmetric linear connections in [8]. QSM-connection on Riemannian, Kaehlerian and Sasakian manifolds were studied by Mishra and Pandey [14] in 1980. In 1982, QSM connections and their curvature tensors defined by Yano and Imai [22] on Hermitian and Kaehlerian manifolds. Moreover, M.D. Siddiqi studied semi-symmetric metric connection on δ-Lorentzian trans-Sasakian manifolds [18] and on (ε)-Kenmotsu manifolds with a semi-symmetric metric connection [19]. In [17], M.D. Siddiqi and O. Bahadır came up with new study of η-Ricci soliton with the generalized symmetric metric connection on Kenmotsu manifold, in which they discussed Ricci and η-Ricci solitons with generalized symmetric metric connection of type (α, β) satisfying the conditions $\bar{R}.\bar{S} = 0$, $\bar{S}.\bar{R} = 0$, $\bar{W_2}.\bar{S} = 0$ and $\bar{S}.\bar{W_2} = 0$.

QS connection on submanifolds firstly defined by S. Ali and R. Nivas in [1]. Further, some curvature properties [15] on Riemannian manifold are represented as

$$C^*(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y], \tag{3}$$

and

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2}\left[S(Y, Z)X - S(X, Z)Y\right], \tag{4}$$

where (3) and (4) are respectively known as a concircular and projective curvature [23].

Based on the preceding discourse, the organization of this paper is structured as follows: Following the introductory section, Section 2 is dedicated to presenting fundamental definitions and notations, which will serve as the foundation for subsequent discussions. In Section 3, a comprehensive examination of the QSM connection applied to α-Sasakian manifolds is conducted. The analysis of RS on α-Sasakian manifolds concerning the QSM connection is pursued in Section 4. This section establishes a pivotal result: if an α-Sasakian manifold accommodates RS in conjunction with the QSM connection, it inherently exhibits both Einstein characteristics and consistent contraction. Section 5 encompasses the construction of a pertinent example, illustrating an α-Sasakian manifold denoted as M that aligns with the outcomes derived in this study. Furthermore, the investigation of curvature properties on α-Sasakian manifolds that admit RS under the QSM connection is undertaken in Section 6. Section 7 introduces the delineation of RS on a specific submanifold of α-Sasakian manifolds, employing the framework of the Levi-Civita connection. Transitioning to Section 8, the establishment of the QSM connection on a distinct submanifold of α-Sasakian manifolds is addressed, alongside the proof of RS existence on such a submanifold within the purview of the QSM connection. The conclusive Section 9 encapsulates the paper with final remarks and conclusions drawn from the undertaken study.

2 Preliminaries

This section related to some basic definitions and formulas on para-contact metric manifolds and α-Sasakian manifolds. Also, all the manifolds are assumed to be connected and smooth. An $n(= 2m + 1)$-dimensional connected almost contact metric manifold M with an almost contact metric structure (ϕ, ξ, η, g), where ϕ denotes a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and g is the compatible Riemannian metric satisfies the following:

$$\phi^2 X = X - \eta(X)\xi, \tag{5}$$

$$\eta(\xi) = 1,\ \phi(\xi) = 0,\ \eta(\phi\xi) = 0, \tag{6}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{7}$$

$$g(X, \xi) = \eta(X),\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{8}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ denotes the collection of smooth vector fields.

An almost contact metric manifold M is said to be α-Sasakian manifold [2] if it satisfies the following conditions:

$$(\nabla_X \phi) = \alpha \left(g\left(X, Y \right) \xi - \eta \left(Y \right) X \right), \tag{9}$$

$$\nabla_X \xi = -\alpha \phi X, \tag{10}$$

$$\left(\nabla_X \eta \right) Y = \alpha g \left(X, \phi Y \right), \tag{11}$$

for a non-zero real constant α on M. Here ∇ denotes Levi-Civita on α-Sasakian manifold.

For α-Sasakian manifold M, we have some relations which are defined below:

$$R(X, Y)\xi = \alpha^2 \left[\eta(Y)X - \eta(X)Y \right], \tag{12}$$

$$R(\xi, X)Y = \alpha^2 \left[g(X, Y)\xi - \eta(Y)X \right], \tag{13}$$

$$\eta(R(X, Y)Z) = \alpha^2 \left[\eta(X)g(Y, Z) \right] - \eta(Y)g(X, Z), \tag{14}$$

$$S(X, \xi) = \alpha^2 (n-1)\eta(X), \tag{15}$$

$$S(\xi, \xi) = \alpha^2 (n-1), \tag{16}$$

$$Q\xi = \alpha^2 (n-1)\xi, \tag{17}$$

for all vector fields X, Y, Z on M. Here R represents the Riemannian curvature tensor and S is the Ricci curvature tensor of M.

On the other hand, α-sasakian manifold M is said to be an η-Einstein if the Ricci curvature tensor has the following form:

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{18}$$

for any vector fields X, Y on M, where a and b being constants.

3 Quarter symmetric metric connection on α-Sasakian manifolds

A QS connection on a differentiable manifold with affine connection introduced by S. Golab in [8]. If torsion tensor τ of the linear connection $\overline{\nabla}$ on an n-dimensional differentiable manifold M is the form of

$$\begin{aligned} \tau(X, Y) &= \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y] \\ &= \eta(Y)\phi X - \eta(X)\phi Y, \end{aligned} \tag{19}$$

where η is a 1-form and ϕ is a tensor of type $(1, 1)$.

Again, $\overline{\nabla}$ is said to be a QSM connection if the QS connection $\overline{\nabla}$ follows the condition defined below:

$$(\overline{\nabla}_X g)(Y, Z) = 0, \tag{20}$$

for all $X, Y, Z \in \chi(M)$.

Let $\overline{\nabla}$ be a QSM connection on α-Sasakian manifolds such that

$$\overline{\nabla}_X Y = \nabla_X Y + H(X, Y), \tag{21}$$

where H is a tensor of type $(1, 1)$ such that

$$H(X, Y) = \frac{1}{2}\left[\tau(X, Y) + \tau^{'}(X, Y) + \tau^{'}(Y, X)\right], \tag{22}$$

where τ and $\tau^{'}$ are related by

$$g(\tau^{'}(X, Y), Z) = g(\tau(Z, X), Y). \tag{23}$$

From (19) and (23), we get

$$\tau^{'}(X, Y) = -\eta(X)\phi Y - g(\phi X, Y)\xi. \tag{24}$$

Using (19) and (24), we have

$$H(X, Y) = -\eta(X)\phi Y. \tag{25}$$

From (21) and (25), we obtain

$$\overline{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{26}$$

Let R and $\overline{R}$ are the curvature tensor of ∇ and $\overline{\nabla}$ on α-Sasakian manifold, which are shown below:

$$\overline{R}(X, Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z. \tag{27}$$

Using (26), (6) and $[X, Y] = \nabla_X Y - \nabla_Y X$ in (27), we get

$$\begin{aligned}\overline{R}(X, Y)Z = R(X, Y)Z &+ \eta(X)(\nabla_Y\phi)Z - \eta(Y)(\nabla_X\phi)Z\\ &- (\nabla_X\eta)(Y)\phi Z + (\nabla_Y\eta)(X)\phi Z.\end{aligned} \tag{28}$$

By putting (9) and (11) into (28), we arrive at

$$\begin{aligned}\overline{R}(X, Y)Z = R(X, Y)Z &- 2\alpha g(X, \phi Y)\phi Z\\ &+ \alpha\left[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\right]\xi\\ &+ \alpha\left[\eta(Y)X - \eta(X)Y\right]\eta(Z).\end{aligned} \tag{29}$$

Taking inner product of (29) with W, we get

$$\begin{aligned}\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) &- 2\alpha g(X,\phi Y)g(\phi Z,W)\\ &+ \alpha[\eta(X)g(Y,Z)-\eta(Y)g(X,Z)]g(\xi,W)\\ &+ \alpha[\eta(Y)g(X,W)-\eta(X)g(Y,W)]\eta(Z).\end{aligned} \tag{30}$$

Take $X = W = v_i$ in (30), we obtain

$$\overline{S}(Y,Z) = S(Y,Z) - \alpha g(Y,Z) + \alpha n\eta(Y)\eta(Z). \tag{31}$$

Contracting (32), we have

$$\overline{r} = r. \tag{32}$$

Again, by (29) and (12), we get

$$\overline{R}(X,Y)\xi = (\alpha^2+\alpha)(\eta(Y)X - \eta(X)Y). \tag{33}$$

From (31), we have

$$\overline{S}(Y,\xi) = S(Y,\xi) - \alpha g(Y,\xi) + \alpha n\eta(Y)\eta(\xi). \tag{34}$$

Using (6), (8) and (15), we obtain

$$\overline{S}(Y,\xi) = (\alpha^2+\alpha)(n-1)\eta(Y). \tag{35}$$

Again, by first Bianchi identity and (29), we arrive at

$$\begin{aligned}&\overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y\\ = \;& -2\alpha\left[g(X,\phi Y)\phi Z + g(Y,\phi Z)\phi X + g(Z,\phi X)\phi Y\right].\end{aligned} \tag{36}$$

So, we conclude the following result:

Theorem 1. *If α-Sasakian manifold M admits QSM connection $\overline{\nabla}$, then*

1. *the expression for $\overline{R}$ is provided by equation (29).*
2. *the definition of $\overline{S}$ is stipulated in equation (31).*
3. *$\overline{r}$ is determined through equation (32).*
4. *If M satisfies the first Bianchi identity, then*

$$\begin{aligned}\overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y &= -2\alpha[g(X,\phi Y)\phi Z + g(Y,\phi Z)\phi X\\ &\quad + g(Z,\phi X)\phi Y].\end{aligned}$$

5. *M is an η-Einstein with respect to $\overline{\nabla}$ if M is Einstein or η-Einstein with respect to ∇.*

4 Ricci solitons on α-Sasakian manifolds with respect to quarter symmetric metric connection

A RS (g, V, λ) on α-Sasakian manifolds admitting QSM connection defined by

$$(\overline{\pounds}_V g)(Y, Z) + 2\overline{S}(Y, Z) + 2\lambda g(Y, Z) = 0. \tag{37}$$

As we know that

$$(\overline{\pounds}_V g)(Y, Z) = g(\overline{\nabla}_Y V, Z) + g(Y, \overline{\nabla}_Z V). \tag{38}$$

Using (26), we get

$$(\overline{\pounds}_V g)(Y, Z) = (\pounds_V g)(Y, Z) - \eta(Y)g(\phi V, Z) - \eta(Z)g(Y, \phi V). \tag{39}$$

We put (31) and (39) in (37), we obtain

$$-\eta(Y)g(\phi V, Z) - \eta(Z)g(Y, \phi V) - 2\alpha g(Y, Z) + 2n\alpha\eta(Y)\eta(Z) = 0. \tag{40}$$

So, we can state that

Theorem 2. *A QSM connection on α-Sasakian manifold admitting RS (g, V, λ) is invariant if and only if its satisfies (40).*

Next, if we put $V = \xi$ in (37), we deduce that

$$(\overline{\pounds}_\xi g)(Y, Z) + 2\overline{S}(Y, Z) + 2\lambda g(Y, Z) = 0, \tag{41}$$

which can be rewritten as

$$(\overline{\pounds}_\xi g)(Y, Z) = g(\overline{\nabla}_Y \xi, Z) + g(Y, \overline{\nabla}_Z \xi).$$

Using (26), we have

$$(\overline{\pounds}_\xi g)(Y, Z) = (\pounds_\xi g)(Y, Z). \tag{42}$$

Substituting (10) into (42), we get

$$(\overline{\pounds}_\xi g)(Y, Z) = 0. \tag{43}$$

By (41) and (43), we obtain

$$\overline{S}(Y, Z) = -\lambda g(Y, Z). \tag{44}$$

Relations (31) and (44) together give the following:

$$S(Y, Z) = -\alpha n\eta(Y)\eta(Z) + (\alpha - \lambda)g(Y, Z). \tag{45}$$

From (44), we conclude the following:

Theorem 3. *A RS (g, ξ, λ) on α-Sasakian manifold M with QSM connection is Einstein manifold.*

Now, we put $Z = \xi$ into (44), we have

$$\overline{S}(Y, \xi) = -\lambda \eta(Y).$$

By (35), we get

$$\lambda = -(n-1)(\alpha^2 + \alpha).$$

Since, we know $\alpha^2 + \alpha \geq 0$ always, then either

$$\lambda = -(n-1)(\alpha^2 + \alpha) < 0,$$

or

$$\lambda = 0, \text{ if } \alpha = -1.$$

Thus, we have

Theorem 4. *A RS (g, ξ, λ) on α-Sasakian manifold M with QSM connection is steady if $\alpha = -1$ otherwise always shrinking.*

5 A non-trivial example

Let $M = \mathbb{R}^3$ be a 3-dimensional manifold. We choose three linear independent vector fields as

$$v_1 = e^x \frac{\partial}{\partial y}, \ v_2 = e^x \left(\frac{\partial}{\partial y} + 2y \frac{\partial}{\partial z} \right), \ v_3 = \frac{\partial}{\partial z}.$$

Consider the Riemannian metric g on M as

$$g(v_i, v_j) = 0, i \neq j, \ i, j = 1, 2, 3,$$
$$g(v_1, v_1) = g(v_2, v_2) = g(v_3, v_3) = 1.$$

Let η be the 1-form defined by $\eta(W) = g(W, v_3)$, for any $W \in \chi(M)$ and ϕ be the (1, 1) tensor field which is defined by $\phi(v_1) = v_2, \phi(v_2) = -v_1$ and $\phi(v_3) = 0$ with $\xi = v_3$. Thus, (ϕ, ξ, η, g) defines a Sasakian structure on M.

Next, we assume ∇ as the Levi-Civita connection with respect to g. Then by the definition of Lie bracket we have

$$[v_1, v_2] = -e^x v_1 + 2e^{2x} v_3, \quad [v_1, v_3] = 0,$$
$$[v_2, v_1] = e^x v_1 - 2e^{2x} v_3, \quad [v_2, v_3] = 0.$$

Taking $v_3 = \xi$ and using Koszul's formula, we derive

$$\nabla_{v_1} v_2 = -e^x v_1 + e^{2x} v_3, \ \nabla_{v_1} v_3 = -e^{2x} v_2, \ \nabla_{v_1} v_1 = e^x v_2,$$
$$\nabla_{v_2} v_3 = e^{2x} v_1, \ \nabla_{v_2} v_2 = 0, \ \nabla_{v_2} v_1 = -e^{2x} v_3,$$
$$\nabla_{v_3} v_3 = 0, \ \nabla_{v_3} v_2 = e^{2x} v_1, \ \nabla_{v_3} v_1 = -e^{2x} v_2.$$

Clearly, (ϕ, ξ, η, g) is a structure on α-Sasakian manifold M with $\alpha = e^{2x}$ and we define the curvature tensor R and the Ricci tensor S respectively as follows:

$$\begin{aligned} &R(v_1, v_2)v_2 = -3e^{4x}v_1 - e^{2x}v_1,\ R(v_1, v_3)v_3 = e^{4x}v_1,\\ &R(v_2, v_1)v_1 = -v_2(3e^{2x} + 1)v_2, R(v_2, v_3)v_3 = e^{4x}v_2,\\ &R(v_3,\ v_1)v_1 = e^{4x}v_3,\ R(v_3, v_2)v_2 = e^{4x}v_3,\\ &R(v_1, v_2)v_3 = 0,\ R(v_3, v_2)v_3 = -e^{4x}v_2,\ R(v_3, v_1)v_2 = 0, \end{aligned}$$

and

$$S(v_1, v_1) = -e^{2x}(1 + 2e^{2x}), \quad S(v_2, v_2) = -e^{2x}(1 + 2e^{2x}), \quad S(v_3, v_3) = 2e^{4x}.$$

Hence, the scalar curvature $r = -2e^{2x}(1 + e^{2x})$.

Now, let $\overline{\nabla}$ is a QSM connection on M. Then from (26), we obtain

$$\begin{aligned} \overline{\nabla}_{v_1}v_2 = -e^{x}v_1 + e^{2x}v_3, \quad \overline{\nabla}_{v_1}v_3 = -e^{2x}v_2, \quad \overline{\nabla}_{v_1}v_1 = e^{x}v_2,\\ \overline{\nabla}_{v_2}v_3 = e^{2x}v_1, \quad \overline{\nabla}_{v_2}v_2 = 0, \quad \overline{\nabla}_{v_2}v_1 = -e^{2x}v_3,\\ \overline{\nabla}_{v_3}v_3 = 0, \quad \overline{\nabla}_{v_3}v_2 = (e^{2x} + 1)v_1, \quad \overline{\nabla}_{v_3}v_1 = -(e^{2x} + 1)v_2. \end{aligned}$$

From the above relation, $\overline{R}$ is defined as

$$\begin{aligned} &\overline{R}(v_1, v_2)v_2 = -3e^{2x}(e^{2x} + 1)v_1, \ \ \overline{R}(v_1, v_3)v_3 = e^{2x}(e^{2x} + 1)v_1,\\ &\overline{R}(v_2, v_3)v_3 = e^{2x}(e^{2x} + 1)v_2 \ \ \overline{R}(v_2, v_1)v_1 = -3e^{2x}(e^{2x} + 1)v_2,\\ &\overline{R}(v_3, v_1)v_1 = e^{2x}(e^{2x} + 1)v_3, \ \ \overline{R}(v_3, v_2)v_2 = e^{2x}(e^{2x} + 1)v_3\\ &\overline{R}(v_1, v_2)v_3 = 0, \ \ \overline{R}(v_3, v_2)v_3 = -e^{2x}(e^{2x} + 1)v_2, \ \ \overline{R}(v_3, v_1)v_2 = 0, \end{aligned}$$

which satisfies (29) of Theorem 1 and also satisfies (36) of Theorem 1, that is,

$$\overline{R}(v_1, v_2)v_3 + \overline{R}(v_2, v_3)v_1 + \overline{R}(v_3, v_1)v_2 = 0.$$

Next, Ricci tensor of M with QSM connection is defined by

$$\overline{S}(v_1, v_1) = -2e^{2x}(1 + e^{2x}), \overline{S}(v_2, v_2) = -2e^{2x}(e^{2x} + 1), \overline{S}(v_3, v_3) = 2e^{2x}(e^{2x} + 1) \tag{46}$$

which again satisfies Theorem 1. Hence, we compute the scalar curvature with QSM connection as $\overline{r} = -2e^{2x}(1 + e^{2x}) = r$, which satisfies (32) of Theorem 1.

Now, putting $Y = Z = v_3$ in (42) and (41) and using (10), we find

$$(\overline{\pounds}_{\xi}g)(v_3, v_3) = 0, \tag{47}$$

and

$$\overline{S}(v_3, v_3) = \lambda\eta(v_3). \tag{48}$$

On using (46), (47) and (48) in (41), we easily find

$$\lambda = -2e^{2x}(e^{2x} + 1) < 0,$$

where $e^{2x} \neq 0$ and satisfies Theorem 4.

6 Curvature properties on α-Sasakian manifolds admit Ricci soliton with respect to quarter symmetric connection

Concircular curvature with respect to QSM connection is defined by

$$\overline{C}^*(X,Y)Z = \overline{R}(X,Y)Z - \frac{\overline{r}}{n(n-1)}\left[g(Y,Z)X - g(X,Z)Y\right]. \tag{49}$$

On putting (29) and (32) into (49), we have

$$\begin{aligned}\overline{C}^*(X,Y)Z = C^*(X,Y)Z - 2\alpha g(X,\phi Y)\phi Z + \alpha[\eta(X)g(Y,Z) \\ - \eta(Y)g(X,Z)]\xi + \alpha\left[\eta(Y)X - \eta(X)Y\right]\eta(Z),\end{aligned} \tag{50}$$

where C^* denotes concircular curvature with respect to Levi-Civita connection.

For $Z = \xi$ in (50), we find a relation as follows:

$$\overline{C}^*(X,Y)\xi = C^*(X,Y)\xi + \alpha[\eta(Y)X - \eta(X)Y]. \tag{51}$$

From above, we state the following:

Theorem 5. *A concircular curvature on α-Sasakian manifold admits RS with QSM connection is given by (51).*

Again, we can write (51) as

$$\begin{aligned}\overline{C}^*(X,Y)\xi = R(X,Y)\xi - \frac{r}{n(n-1)}\left[\eta(Y)X - \eta(X)Y\right] \\ + \alpha[\eta(Y)X - \eta(X)Y].\end{aligned} \tag{52}$$

The use of (12) in (52) gives

$$\overline{C}^*(X,Y)\xi = \left(\alpha^2 + \alpha - \frac{r}{n(n-1)}\right)\left(\eta(Y)X - \eta(X)Y\right). \tag{53}$$

Now, we put $Y = Z = v_i$ in (45) and contracting it, then we get

$$r = -\lambda n. \tag{54}$$

By making the use of relations (53) and (54), we find

$$\overline{C}^*(X,Y)\xi = \left(\alpha^2 + \alpha + \frac{\lambda}{(n-1)}\right)\left(\eta(Y)X - \eta(X)Y\right),$$

which further gives us the following result:

Theorem 6. *A α-Sasakian manifold M admits RS with QSM connection is ξ-concircular flat if and only if $\lambda = -(n-1)(\alpha^2+\alpha)$.*

Furthermore, the projective curvature with respect to QSM connection is defined by

$$\overline{P}(X,Y)Z = \overline{R}(X,Y)Z - \frac{1}{2}\left[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y\right]. \tag{55}$$

Taking into account of $Z = \xi$ in (55), we derive

$$\overline{P}(X,Y)\xi = \overline{R}(X,Y)\xi - \frac{1}{2}\left[\overline{S}(Y,\xi)X - \overline{S}(X,\xi)Y\right]. \tag{56}$$

By adopting (44), (33) and (56), we get

$$\overline{P}(X,Y)\xi = \left(\alpha^2 + \alpha + \frac{\lambda}{2}\right)(\eta(Y)X - \eta(X)Y). \tag{57}$$

This relation gives us the following result:

Theorem 7. *A α-Sasakian manifold M admits RS with QSM connection is ξ-Projective flat if and only if $\lambda = -2\alpha(\alpha+1)$.*

7 Ricci solitons on submanifolds with respect to Levi-Civita connection

Let m-dimensional $\tilde{M}$ be a submanifold of α-Sasakian manifold M, where $m < n$, with induced metric g. Also, let $\tilde{\nabla}$ and $\tilde{\nabla}^{\perp}$ be the levi-civita connection on the tangent bundle $T\tilde{M}$ and the normal bundle $T^{\perp}\tilde{M}$ of $\tilde{M}$, respectively. Then the Gauss and Weingarten formulae are represented by [23]:

$$\nabla_X Y = \tilde{\nabla}_X Y + \tilde{\sigma}(X,Y), \tag{58}$$

and

$$\nabla_X N = -A_N X + \nabla^{\perp}_X N, \tag{59}$$

for all $X, Y \in T\tilde{M}$ and $N \in T^{\perp}\tilde{M}$, where $\tilde{\sigma}$ and A_N are second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of $\tilde{M}$ into M. The second fundamental form $\tilde{\sigma}$ and the shape operator A_N are related by

$$g(\tilde{\sigma}(X,Y),N) = g(A_N X, Y), \tag{60}$$

for any $X, Y \in T\tilde{M}$ and $N \in T^{\perp}\tilde{M}$.

A submanifold $\tilde{M}$ of an almost contact metric manifold M is said to be invariant if the structure vector field ξ is tangent to $\tilde{M}$ at every point of $\tilde{M}$ and ϕX is tangent to $\tilde{M}$ for every vector field X tangent to $\tilde{M}$ at every point of $\tilde{M}$.

Let us take RS (g, ξ, λ) on $\tilde{M}$ as

$$(\tilde{\mathcal{L}}_\xi g)(Y, Z) + 2\tilde{S}(Y, Z) + 2\lambda g(Y, Z) = 0. \tag{61}$$

Put $Y = \xi$ in (58) and use (10), we have

$$-\alpha\phi X = \tilde{\nabla}_X \xi + \tilde{\sigma}(X, \xi). \tag{62}$$

Since $\tilde{M}$ is invariant therefore $\phi X, \xi \in T\tilde{M}$. Then, on comparing tangential and normal components of (62), we get

$$-\alpha\phi X = \tilde{\nabla}_X \xi \text{ and } \tilde{\sigma}(X, \xi) = 0. \tag{63}$$

From (63), we have

$$\begin{aligned}(\tilde{\mathcal{L}}_\xi g) = g(\tilde{\nabla}_Y \xi, Z) + g(Y, \tilde{\nabla}_Z \xi) \\ = 0.\end{aligned} \tag{64}$$

Using (61) and (64), we get

$$\tilde{S}(Y, Z) = -\lambda g(Y, Z). \tag{65}$$

Again, using (63) and the curvature formula becomes

$$\tilde{R}(X, Y)\xi = \tilde{\nabla}_X \tilde{\nabla}_Y \xi - \tilde{\nabla}_Y \tilde{\nabla}_X \xi - \tilde{\nabla}_{[X,Y]}\xi. \tag{66}$$

We obtain

$$\tilde{R}(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y]. \tag{67}$$

Now, we have

$$\tilde{S}(X, \xi) = \alpha^2(m-1)\eta(X). \tag{68}$$

Put $Z = \xi$ into (65) and use (68), then we arrive at

$$\lambda = -\alpha^2(m-1) < 0. \tag{69}$$

From (65) and (69), we yield the following results:

Theorem 8. *If invariant submanifold $\tilde{M}$ of α-Sasakian manifold M admits a RS (g, ξ, λ), then $\tilde{M}$ is*

1. *Einstein manifold.*

2. *always shrinking.*

8 Ricci solitons on submanifolds with respect to QSM connection

Assuming that Levi-Civita connection $\tilde{\nabla}$ and QSM connection $\overline{\tilde{\nabla}}$ on submanifold $\tilde{M}$ of α-Sasakian manifold M with Levi-Civita connection ∇ and QSM connection $\overline{\nabla}$ [13]. Here we denote the second fundamental form with respect to QSM connection by $\overline{\tilde{\sigma}}$. Then Gauss formula with respect to QSM connection can be represented as

$$\overline{\nabla}_X Y = \overline{\tilde{\nabla}}_X Y + \overline{\tilde{\sigma}}(X, Y). \tag{70}$$

Now, use (26) in (70), we get

$$\overline{\tilde{\nabla}}_X Y + \overline{\tilde{\sigma}}(X, Y) = \nabla_X Y - \eta(X)\phi Y. \tag{71}$$

Using (58) and (71), we obtain

$$\overline{\tilde{\nabla}}_X Y + \overline{\tilde{\sigma}}(X, Y) = \tilde{\nabla}_X Y + \tilde{\sigma}(X, Y) - \eta(X)\phi Y \tag{72}$$

Since $\tilde{M}$ is a invariant submanifold, therefore tangential and normal parts are given by

$$\overline{\tilde{\nabla}}_X = \tilde{\nabla}_X Y - \eta(X)\phi Y, \tag{73}$$

and

$$\overline{\tilde{\sigma}}(X, Y) = \tilde{\sigma}(X, Y). \tag{74}$$

Thus, we conclude the following:

Theorem 9. *Let $\tilde{M}$ be an invariant submanifold of α-Sasakian manifold M endowed with two connections namely Levi-civita and QSM connections ∇, $\overline{\nabla}$ and $\tilde{\nabla}$, $\overline{\tilde{\nabla}}$ be the induced Levi-civita and QSM connections on $\tilde{M}$ from ∇, $\overline{\nabla}$, respectively. Then we have*

1. *$\tilde{M}$ admits QSM connection.*
2. *The second fundamental form are same with respect to ∇ and $\overline{\nabla}$.*

Now, we define the curvature tensor, Ricci tensor (shortly, RT) and scalar curvature tensor on $\tilde{M}$ of M:

Theorem 10. *If $\tilde{M}$ be a submanifold of M with QSM connection $\overline{\tilde{\nabla}}$ then*

1. *The curvature tensor $\overline{\tilde{R}}$ is given as*

$$\begin{aligned}\overline{\tilde{R}}(X, Y)Z = \tilde{R}(X, Y)Z &- 2\alpha g(X, \phi Y)\phi Z \\ &+ \alpha\left[\eta(X)g(X, Y) - \eta(Y)g(X, Z)\right]\xi \\ &+ \alpha\left[\eta(Y)X - \eta(X)Y\right]\eta(Z).\end{aligned} \tag{75}$$

2. *Ricci tensor* $\overline{\tilde{S}}$ *is represented as*

$$\overline{\tilde{S}}(Y,Z) = \tilde{S}(Y,Z) - \alpha g(Y,Z) + \alpha m\eta(Y)\eta(Z). \tag{76}$$

3. *The scalar curvature is given by* $\overline{\tilde{r}} = \tilde{r}$.

Again, let RS (g, ξ, λ) on an invariant submanifold $\tilde{M}$ with QSM connection, which is defined as

$$(\overline{\tilde{\mathcal{L}}}_\xi g)(Y,Z) + 2\overline{\tilde{S}}(Y,Z) + 2\lambda g(Y,Z) = 0. \tag{77}$$

By adopting similar steps done in proving Theorems 3 and 4, we have the following results:

Theorem 11. *Let* (g, ξ, λ) *be a RS on an invariant submanifold* $\tilde{M}$ *of* α*-Sasakian manifold* M *with QSM connection then* $\tilde{M}$ *is*

1. *Einstein manifold.*
2. *always shrinking.*
3. *steady when* $\alpha = -1$.

9 Conclusions and remarks

In relation to the QSM connection, our investigation in this study has centered around α-Sasakian manifolds denoted as M, coupled with RS (Ricci solitons). Within this context, we have both examined the characterization of submanifolds in M that admit RS and established the definition of RS on a submanifold of M, utilizing the framework of the QSM connection. The outcomes of our discourse can be summarized as follows:

1. RS on M endowed with the QMS connection, exhibits the properties of being Einstein and consistently contracting, except in the case of $\alpha = -1$ where it remains steady.
2. RS on an invariant submanifold of M possessing either the Levi-Civita connection or the QSM connection, demonstrates Einstein characteristics.

It is pertinent to highlight that these findings generate certain inquiries that offer avenues for future exploration:

1. Does the entire array of outcomes established in this paper extend to scenarios involving torqued vector fields?
2. If connections other than the QSM connection are employed, will the conclusions remain unaltered or undergo modifications?
3. Can the assertions in Theorems 6 and 7 be generalized to encompass any submanifold of α-Sasakian manifolds?
4. In cases where a submanifold is anti-invariant, what implications can be drawn regarding Theorems 8 and 11?

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