Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 4(66), No. 1 - 2024, 163-174
https://doi.org/10.31926/but.mif.2024.4.66.1.12

CLASSIFICATIONS OF $\mathcal{THA}$-SURFACES IN $\mathbb{I}^3$

Bendehiba SENOUSSI[*,1]

Abstract

In classical differential geometry, the problem of obtaining Gaussian and mean curvatures of a surface is one of the most important problems. A surface $\mathcal{M}^2$ in $\mathbb{I}^3$ is a $\mathcal{THA}$-surface of first type if it can be parameterized by

$$r(s,\ t) = (s,\ t,\ Af(s+at)g(t) + B(f(s+at) + g(t))).$$

A surface $\mathcal{M}^2$ in $\mathbb{I}^3$ is a $\mathcal{THA}$- surface of second type if it can be parameterized by

$$r(s,\ t) = (s,\ Af(s+at)g(t) + B(f(s+at) + g(t)),\ t),$$

where A and B are non-zero real numbers [16, 17, 18]. In this paper, we classify two types $\mathcal{THA}$-surfaces in the 3-dimensional isotropic space $\mathbb{I}^3$ and study $\mathcal{THA}$-surfaces with zero curvature in $\mathbb{I}^3$.

2000 *Mathematics Subject Classification:* 53A10, 53C42.
Key words: $\mathcal{THA}$- surfaces, minimal surfaces, Isotropic space, mean curvature.

1 Introduction

M.K. Karacan, D.W.Yoon, B. Bukcu [6], M.E.Aydin [1, 2] have studied some classes of surfaces in $\mathbb{I}^3$. R. López [9] studied translation surfaces in the 3-dimensional hyperbolic space $\mathbb{H}^3$ and classified minimal translation surfaces. R. López and M. I. Munteanu [10] constructed translation surfaces in Sol_3 and investigated properties of minimal one. In a different aspect, H. Liu [7] considered the translation surfaces with constant mean curvature in 3-dimensional Euclidean space and Lorentz-Minkowski space.

Recently, K. Seo [20] gave a classification of the translation hypersurfaces with constant mean curvature or constant Gauss-Kronecker curvature in space forms.

Related works on minimal translation surfaces of $\mathbb{E}^3$ are ([7], [14], [22]). B. Senoussi et al. [19] studied the translation surfaces in Lorentz-Heisenberg 3-space Nil_1^3. In this paper, we classify two types $\mathcal{THA}$-surfaces in the 3-dimensional isotropic space $\mathbb{I}^3$ and study $\mathcal{THA}$- surfaces with zero curvature in $\mathbb{I}^3$.

[1*] *Corresponding author*, Department of Mathematics, *Ecole Normale Supérieure* Mostaganem, Algeria, e-mail: snoussi.bendehiba19@gmail.com

Theorem 1 ([11]). *i) The only translation surfaces with constant Gauss curvature $K_G = 0$ are cylindrical surfaces.*

ii) There are no translation surfaces with constant Gauss curvature $K_G \neq 0$ if one of the generating curves is planar.

Definition 1. *A homothetical (factorable) surface $\mathcal{M}^2$ in 3-dimensional Euclidean space $\mathbb{E}^3$ is a surface that is a graph of a function*

$$z(u,\ v) = f(u)g(v),$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ and $g : J \subset \mathbb{R} \to \mathbb{R}$ are two smooth functions.

Theorem 2 ([11]). *Planes and helicoids are the only minimal homothetical surfaces in Euclidean space.*

Theorem 3 ([8]). *Let $r(x, y) = (x,\ y,\ z(x, y) = f(x) + g(ax + y))$ be a minimal affine translation surface. Then either $z(x, y)$ is linear or can be written as*

$$z(x,y) = \frac{1}{c} \log \frac{\cos(c\sqrt{1+a^2}x)}{\cos[c(ax+y)]}. \tag{1}$$

Remark 1. *If $a = 0$, the minimal affine translation surface given by (1) is the classical Scherk surface.*

Definition 2 ([8]). *The minimal affine translation surface (1) is called generalized Scherk surface or affine Scherk surface in Euclidean 3 - space.*

2 Preliminaries

The 3-dimensional isotropic space $\mathbb{I}^3$ was introduced by Strubecker. The group G_6 of motions of $\mathbb{I}^3$ is a 6 parameter group, defined by (see [1], [12], [13]).

$$\psi :\ (x_1, x_2, x_3) \mapsto (x_1', x_2', x_3') : \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ \lambda & \mu & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

where (x_1, x_2, x_3) denote the affine coordinates and $\phi,\ a, b, c, \lambda, \mu \in \mathbb{R}$.
The isotropic metric induced by the absolute figure is given by

$$g_{\mathbb{I}^3} = \mathrm{d}s^2_{\mathbb{I}^3} = \mathrm{d}x_1^2 + \mathrm{d}x_2^2.$$

Consider the points $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$. The isotropic distance of two points X and Y is defined by

$$\mathrm{d}_{\mathbb{I}^3}(X,\ Y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}.$$

Two points (x_1, x_2, x_3) and (x_1, x_2, y_3) with the same top view are called parallel points. The lines in x_3 direction are called isotropic lines. The planes containing an isotropic line are called isotropic planes. Non-isotropic planes are planes non-parallel to the z - direction.

Let $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ be vectors in $\mathbb{I}^3$. The isotropic inner product of X and Y is defined by

$$g_{\mathbb{I}^3}(X,Y) = \begin{cases} x_3y_3, & \text{if } \ x_j = 0 \ \text{ and } \ y_j = 0, \ (j=1,2) \\ x_1y_1 + x_2y_2, & \text{if otherwise.} \end{cases}$$

We call the surface $\mathcal{M}^2$ admissible if it has no isotropic tangent planes.

If some admissible surface is locally parameterized by

$$r: \ \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{I}^3, \quad (u,v) \longmapsto (u, v, z(u,v)).$$

The coefficients of the first fundamental form and the second fundamental form are

$$\begin{aligned} E &= g_{\mathbb{I}^3}(r_u, r_u), \ F = g_{\mathbb{I}^3}(r_u, r_v), \ G = g_{\mathbb{I}^3}(r_v, r_v), \\ L &= \frac{\det(r_{uu}, r_u, r_v)}{\sqrt{EG - F^2}}, \ M = \frac{\det(r_{uv}, r_u, r_v)}{\sqrt{EG - F^2}}, \ N = \frac{\det(r_{vv}, r_u, r_v)}{\sqrt{EG - F^2}}, \end{aligned}$$

where $r_u = \frac{\partial r}{\partial u}$, $r_v = \frac{\partial r}{\partial v}$.

The isotropic mean curvature H and the isotropic Gaussian curvature K_G are, respectively, defined by

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)} \quad \text{and} \quad K_G = \frac{LN - M^2}{EG - F^2}.$$

The surface $\mathcal{M}^2$ is said to be isotropic minimal (resp. isotropic flat) if H (resp. K_G) vanishes ([1], [2], [6]).

The main purpose of this paper is to complete classification of $\mathcal{THA}$-surfaces in the 3-dimensional isotropic space $\mathbb{I}^3$.

3 $\mathcal{THA}$-surfaces in $\mathbb{I}^3$

Let $\mathcal{M}^2$ be a 2-dimensional surface, of the isotropic 3- space $\mathbb{I}^3$. Using the standard coordinate system of $\mathbb{E}^3$ we denote the parametric representation of the surface $r(u,v)$ by

$$r(u,v) = (r_1(u,v), \ r_2(u,v), \ r_3(u,v)).$$

In $\mathbb{I}^3$, a surface is called a translation surface if it is given by an immersion

$$r : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3 : (u,v) \mapsto (u, v, f(u) + g(v)),$$

where f and g are smooth functions on opens of $\mathbb{R}$.

Definition 3. *i) A surface $\mathcal{M}^2$ in $\mathbb{I}^3$ is a $\mathcal{THA}$-surface of first type if it can be parameterized by*

$$r(s,\ t) = (s,\ t,\ Af(s+at)g(t) + B(f(s+at)+g(t))). \tag{2}$$

ii) A surface $\mathcal{M}^2$ in $\mathbb{I}^3$ is a $\mathcal{THA}$- surface of second type if it can be parameterized by

$$r(s,\ t) = (s,\ Af(s+at)g(t) + B(f(s+at)+g(t)),\ t), \tag{3}$$

where A and B are non-zero real numbers [16, 17, 18].

Remark 2. *i) If $A \neq 0$ and $B = 0$ in (2), then $\mathcal{M}^2$ is a affine factorable (homothetical) surface.*
ii) If $A = 0$ and $B \neq 0$ in (2), then $\mathcal{M}^2$ is a affine translation surface.

4 $\mathcal{THA}$- surfaces of first type with zero Gaussian curvature in $\mathbb{I}^3$

We classify the $\mathcal{THA}$- surfaces of first type with zero Gaussian curvature in $\mathbb{I}^3$.

Let $\mathcal{M}^2$ be a $\mathcal{THA}$-surface in $\mathbb{I}^3$ parameterized by (2). By a transformation

$$\begin{cases} x = s + at \\ y = t, \end{cases} \tag{4}$$

and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$.
From (4) and (2) we have

$$r(x,\ y) = (x - ay,\ y,\ z(x,\ y) = Af(x)g(y) + B(f(x)+g(y))), \tag{5}$$

The coefficients of the first fundamental form of $\mathcal{M}^2$ are given by

$$E = 1, \quad F = -a, \ G = 1 + a^2.$$

The coefficients of the second fundamental form are given by

$$L = \frac{\gamma\alpha''}{A}, \quad M = \frac{\alpha'\gamma'}{A}, \quad N = \frac{\alpha\gamma''}{A},$$

where $\alpha = (Af + B)$ and $\gamma = (Ag + B)$.

A $\mathcal{THA}$-surfaces of first type in $\mathbb{I}^3$ parameterized by (5) has Gaussian curvature

$$K = \frac{\alpha\gamma\alpha''\gamma'' - \gamma'^2\alpha'^2}{A^2}.$$

Hence that if $K = 0$, then

$$\alpha\gamma\alpha''\gamma'' - \gamma'^2\alpha'^2 = 0. \tag{6}$$

We discuss the different cases according the functions α and γ.

The proof given in [2, 15]. We can obtain the following:

Theorem 4. *Let a $\mathcal{THA}$-surface of first type in $\mathbb{I}^3$ have constant Gaussian curvature K_0 Then, for λ, $c_0, c_1, c_2 \in \mathbb{R}$, we have*
(1) *if $K_0 = 0$, then*
(*a*) $z(x,y) = c_0\alpha(x)$ *or* $z(x,y) = c_0\gamma(y)$
(*b*) $\alpha(x) = \lambda_3 e^{k_1 x} + \lambda_4$ *and* $\gamma(y) = \lambda_5 e^{k_2 y} + \lambda_6$
c) $\alpha(x) = c_3((1-\lambda)k_1 x + c_1)^{\frac{1}{1-\lambda}} + c_4$ *and* $\gamma(y) = c_5((\frac{\lambda-1}{\lambda})k_2 y + c_2)^{\frac{\lambda}{\lambda-1}} + c_6$.
(2) *Otherwise, i.e. $K_0 \neq 0$ then K_0 is negative and*
(*a*) $z(x,y) = c_0(\sqrt{-K_0}y + c_1)(x - ay + c_2)$
(*b*) $z(x,y) = c_0(\sqrt{-K_0}(x - ay)y + c_1)(y + c_2)$.

5 Minimal $\mathcal{THA}$-surfaces of first type in $\mathbb{I}^3$

The expression of H is

$$H = \frac{\alpha\gamma'' + 2a\alpha'\gamma' + \gamma\alpha''(1+a^2)}{2A}. \tag{7}$$

Then $\mathcal{M}^2$ is a minimal surface if and only if

$$\alpha\gamma'' + 2a\alpha'\gamma' + \gamma\alpha''(1+a^2) = 0. \tag{8}$$

Theorem 5. *Let a $\mathcal{THA}$-surface of first type in $\mathbb{I}^3$ be minimal.*
Then, for $\lambda, \mu_0, \mu_1, \mu_2 \in \mathbb{R}$, either
i) *it is a non-isotropic plane; or*
ii) $z(x,\ y) = \mu_0 e^{\psi(x,\ y)}[\mu_1 \cos(\varphi(x,\ y)) + \mu_2 \sin(\varphi(x,\ y))]$,
where $\psi(x,\ y) = \frac{b(x-ay)}{1+a^2}$, $\varphi(x,\ y) = \frac{b(ax+y)}{1+a^2}$; *or*
iii) $z(x,\ y) = \mu_0 e^{by}[\mu_1 \cos(bx) + \mu_2 \sin(bx)]$.

Proof. The proof given in [2]. □

6 Minimal $\mathcal{THA}$-surfaces of second type in $\mathbb{I}^3$

Let $\mathcal{M}^2$ be a $\mathcal{THA}$-surface of second type in $\mathbb{I}^3$ parameterized by (3). Let us put

$$x = s + at, \quad y = t, \tag{9}$$

and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$. From (9) and (3) we have

$$E = \frac{\alpha'^2\gamma^2 + A^2}{A^2}, \quad F = \frac{-aA^2 + \alpha\gamma\alpha'\gamma'}{A^2}, \; G = \frac{\alpha^2\gamma'^2 + A^2a^2}{A^2},$$

where $\alpha = Af + B$ and $\gamma = Ag + B$.
The coefficients of the second fundamental form are given by

$$L = -\frac{\gamma\alpha''}{AW}, \quad M = -\frac{\alpha'\gamma'}{AW}, \quad N = -\frac{\alpha\gamma''}{AW},$$

where $W = A^{-1}(\alpha\gamma' + a\gamma\alpha')$.

$\mathcal{M}^2$ is a minimal surface if and only if

$$\alpha\gamma''(\alpha'^2\gamma^2 + A^2) - 2\alpha'\gamma'(-aA^2 + \alpha\gamma\alpha'\gamma') + \gamma\alpha''(\alpha^2\gamma'^2 + a^2A^2) = 0. \qquad (10)$$

We distinguish two cases for (10):
Case 1. $a = 0$. Equation (10) writes as

$$\gamma''(\alpha'^2\gamma^2 + A^2) + \gamma\gamma'^2(\alpha\alpha'' - 2\alpha'^2) = 0. \qquad (11)$$

We distinguish several cases.
(i). Assume $\gamma' = 0$, then $\gamma(y) = b_1 \in \mathbb{R} - \{0\}$. In such case, $H = 0$ is satisfied for any function α.
(ii). Assume $\alpha' = 0$ ($\gamma' \neq 0$), then $\alpha(x) = b_2 \in \mathbb{R} - \{0\}$. From (11), we have $\gamma'' = 0$.
(iii). Assume $\alpha'' = 0$. Then ($\alpha' = b_2 \neq 0$), and (11) gives

$$\gamma''(b_2^2\gamma^2 + A^2) = 2b_2^2\gamma\gamma'^2. \qquad (12)$$

A direct integration implies that there exist c_1, c_2 such that

$$\gamma = \frac{A}{b_2}\tan(c_1 y + c_2).$$

(iv). Assume $\alpha'' \neq 0$. Equation (11) writes as

$$\frac{\gamma''}{\gamma\gamma'^2}(\alpha'^2\gamma^2 + A^2) = -(\alpha\alpha'' - 2\alpha'^2). \qquad (13)$$

Differentiating (13) with respect to x, we have an identity of two functions, one depending only on y and the other one depending only on x. Then both functions are equal to a same constant

$$-\frac{(\alpha\alpha'' - 2\alpha'^2)_{,x}}{2\alpha'\alpha''} = c = \frac{\gamma\gamma''}{\gamma'^2}. \qquad (14)$$

If $c = 0$, then the second equation of (14) implies $\gamma'' = 0$.
Then (11) gives

$$\alpha\alpha'' - 2\alpha'^2 = 0.$$

Integration with respect to x leads to

$$\alpha = -\frac{1}{c_3 x + c_4},\ c_3, c_4 \in \mathbb{R}.$$

If $c \neq 0$. Substituting $c = \dfrac{\gamma\gamma''}{\gamma'^2}$ in (13), one obtain

$$A^2\frac{\gamma''}{\gamma\gamma'^2} = -(\alpha\alpha'' - 2\alpha'^2) - \alpha'^2 c. \tag{15}$$

Since α and γ are functions of two independent variables, the above equation can be written as

$$A^2\frac{\gamma''}{\gamma\gamma'^2} = c_4, \;\; (\alpha\alpha'' - 2\alpha'^2) - \alpha'^2 c = -c_4.$$

Hence from $A^2\frac{\gamma''}{\gamma\gamma'^2} = c_4$ and $\frac{\gamma\gamma''}{\gamma'^2} = c$ we can write

$$A^2 c = c_4\gamma^2. \tag{16}$$

From (16), $\gamma = constant$ leads to a contradiction.

Case 2. $a \neq 0$. We distinguish several cases.

(i). Assume $\gamma' = 0$. Then $\gamma(y) = b_1 \in \mathbb{R} - \{0\}$ and (10) implies $\alpha'' = 0$.

(ii). Assume $\gamma'' = 0$. Then $\gamma' = b_2 \in \mathbb{R} - \{0\}$ and (10) implies

$$\alpha''(\alpha^2 b_2^2 + a^2A^2) = \frac{2aA^2\alpha' b_2}{\gamma} - 2\alpha\alpha'^2 b_2^2. \tag{17}$$

Differentiating (17) with respect to y, we get $\alpha' = 0$.

(iii). Assume $\alpha'' = 0$. Then $\alpha' = b_3 \in \mathbb{R} - \{0\}$ and (10) implies

$$\gamma''(\gamma^2 b_3^2 + A^2) = \frac{2ab_3A^2\gamma'}{\alpha} - 2\gamma\gamma'^2 b_3^2. \tag{18}$$

Differentiating (18) with respect to x, we get $\gamma' = 0$.

(iv). Assume $\alpha\alpha'\gamma\gamma' \neq 0$ $\alpha''\gamma'' \neq 0$. Then (10) implies

$$\frac{A^2\gamma''}{\gamma\alpha'^2\gamma'^2} + \frac{\gamma\gamma''}{\gamma'^2} + \frac{a^2A^2\alpha''}{\alpha\alpha'^2\gamma'^2} + \frac{\alpha\alpha''}{\alpha'^2} + \frac{2aA^2}{\alpha\gamma\alpha'\gamma'} - 2 = 0. \tag{19}$$

Let us differentiate with respect to x and then with respect to y, to see

$$A^2\left(\frac{\gamma''}{\gamma\gamma'^2}\right)_{,y}\left(\frac{1}{\alpha'^2}\right)_{,x} + a^2A^2\left(\frac{\alpha''}{\alpha\alpha'^2}\right)_{,x}\left(\frac{1}{\gamma'^2}\right)_{,y} + 2aA^2\left(\frac{1}{\alpha\alpha'}\right)_{,x}\left(\frac{1}{\gamma\gamma'}\right)_{,y} = 0. \tag{20}$$

If we divide (20) by $\left(\frac{1}{\alpha'^2}\right)_{,x}\left(\frac{1}{\gamma'^2}\right)_{,y}$, we have

$$A^2\frac{\left(\frac{\gamma''}{\gamma\gamma'^2}\right)_{,y}}{\left(\frac{1}{\gamma'^2}\right)_{,y}} + a^2A^2\frac{\left(\frac{\alpha''}{\alpha\alpha'^2}\right)_{,x}}{\left(\frac{1}{\alpha'^2}\right)_{,x}} + 2aA^2\frac{\left(\frac{1}{\alpha\alpha'}\right)_{,x}\left(\frac{1}{\gamma\gamma'}\right)_{,y}}{\left(\frac{1}{\alpha'^2}\right)_{,x}\left(\frac{1}{\gamma'^2}\right)_{,y}} = 0. \tag{21}$$

Differentiating now with respect to x and next with respect to y, we get

$$\begin{cases} \left(\dfrac{1}{\gamma\gamma'}\right)_{,y} = c_3\left(\dfrac{1}{\gamma'^2}\right)_{,y} \\ \left(\dfrac{1}{\alpha\alpha'}\right)_{,x} = c_7\left(\dfrac{1}{\alpha'^2}\right)_{,x}. \end{cases} \tag{22}$$

Substituting this into (21), we get

$$\begin{cases} \left(\dfrac{\gamma''}{\gamma\gamma'^2}\right)_{,y} = c_1\left(\dfrac{1}{\gamma'^2}\right)_{,y} \\ \left(\dfrac{\alpha''}{\alpha\alpha'^2}\right)_{,x} = c_5\left(\dfrac{1}{\alpha'^2}\right)_{,x}. \end{cases} \tag{23}$$

From (22) and (23) we get

$$\begin{cases} \dfrac{\gamma''}{\gamma\gamma'^2} = c_1\left(\dfrac{1}{\gamma'^2}\right) + c_2 \\ \dfrac{1}{\gamma\gamma'} = c_3\left(\dfrac{1}{\gamma'^2}\right) + c_4 \end{cases} \tag{24}$$

$$\begin{cases} \dfrac{\alpha''}{\alpha\alpha'^2} = c_5\left(\dfrac{1}{\alpha'^2}\right) + c_6 \\ \dfrac{1}{\alpha\alpha'} = c_7\left(\dfrac{1}{\alpha'^2}\right) + c_8. \end{cases} \tag{25}$$

Differentiating (19) with respect to y, we have

$$A^2\left(\frac{\gamma''}{\gamma\gamma'^2}\right)_{,y}\frac{1}{\alpha'^2} + \left(\frac{\gamma\gamma''}{\gamma'^2}\right)_{,y} + a^2A^2\frac{\alpha''}{\alpha\alpha'^2}\left(\frac{1}{\gamma'^2}\right)_{,y} + 2aA^2\left(\frac{1}{\gamma\gamma'}\right)_{,y}\frac{1}{\alpha\alpha'} = 0. \tag{26}$$

Substituting (25) in (26) gives

$$A^2\left(\frac{\gamma''}{\gamma\gamma'^2} + a^2c_5\frac{1}{\gamma'^2} + 2ac_7\frac{1}{\gamma\gamma'}\right)_{,y} + \alpha'^2\left(\frac{\gamma\gamma''}{\gamma'^2} + a^2A^2c_6\left(\frac{1}{\gamma'^2}\right)_{,y} + 2aA^2c_8\left(\frac{1}{\gamma\gamma'}\right)_{,y}\right) = 0.$$

For each fixed y, we can view this expression as a polynomial equation on α' and thus, all coefficients vanish. Then

$$A^2\frac{\gamma''}{\gamma\gamma'^2} + a^2A^2c_5\frac{1}{\gamma'^2} + 2aA^2c_7\frac{1}{\gamma\gamma'} = \lambda_1, \quad \lambda_1 \in \mathbb{R}, \tag{27}$$

$$\frac{\gamma\gamma''}{\gamma'^2} + a^2A^2c_6\frac{1}{\gamma'^2} + 2aA^2c_8\frac{1}{\gamma\gamma'} = \lambda_2, \quad \lambda_2 \in \mathbb{R}. \tag{28}$$

Substituting (25) in (19) gives

$$\frac{A^2\gamma''}{\gamma\gamma'^2}\frac{1}{\alpha'^2}+\frac{\gamma\gamma''}{\gamma'^2}+\frac{a^2A^2}{\gamma'^2}\left(c_5\left(\frac{1}{\alpha'^2}\right)+c_6\right)+\frac{\alpha\alpha''}{\alpha'^2}+\frac{2aA^2}{\gamma\gamma'}\left(c_7\left(\frac{1}{\alpha'^2}\right)+c_8\right)-2=0. \tag{29}$$

Then (29) can be written as

$$\frac{\lambda_1}{\alpha'^2}+\lambda_2+\frac{\alpha\alpha''}{\alpha'^2}-2=0. \tag{30}$$

Differentiating (30) with respect to x, we have

$$\lambda_1\left(\frac{1}{\alpha'^2}\right)_{,x}+\left(\frac{\alpha\alpha''}{\alpha'^2}\right)_{,x}=0. \tag{31}$$

Using (25), we have

$$\alpha'^2-\alpha^2c_5-\lambda_1=0.$$

Differentiating this equation with respect to x, we obtain $\alpha''=c_5\alpha$. From (25), we get $c_6=0$.

Differentiating (19) with respect to x, we have

$$\frac{A^2\gamma''}{\gamma\gamma'^2}\left(\frac{1}{\alpha'^2}\right)_{,x}+a^2A^2\left(\frac{\alpha''}{\alpha\alpha'^2}\right)_{,x}\left(\frac{1}{\gamma'^2}\right)+\left(\frac{\alpha\alpha''}{\alpha'^2}\right)_{,x}+2aA^2\left(\frac{1}{\gamma\gamma'}\right)\left(\frac{1}{\alpha\alpha'}\right)_{,x}=0. \tag{32}$$

Using (24), we have

$$A^2c_1\left(\frac{1}{\alpha'^2}\right)_{,x}+a^2A^2\left(\frac{\alpha''}{\alpha\alpha'^2}\right)_{,x}+2aA^2c_3\left(\frac{1}{\alpha\alpha'}\right)_{,x}+$$
$$\gamma'^2\left(A^2c_2\left(\frac{1}{\alpha'^2}\right)_{,x}+\left(\frac{\alpha\alpha''}{\alpha'^2}\right)_{,x}+2aA^2c_4\left(\frac{1}{\alpha\alpha'}\right)_{,x}\right)=0. \tag{33}$$

For each fixed x, we can view this expression as a polynomial equation on γ' and thus, all coefficients vanish. Then

$$A^2c_1\left(\frac{1}{\alpha'^2}\right)+a^2A^2\left(\frac{\alpha''}{\alpha\alpha'^2}\right)+2aA^2c_3\left(\frac{1}{\alpha\alpha'}\right)=\lambda_3,\quad \lambda_3\in\mathbb{R}, \tag{34}$$

$$A^2c_2\left(\frac{1}{\alpha'^2}\right)+\left(\frac{\alpha\alpha''}{\alpha'^2}\right)+2aA^2c_4\left(\frac{1}{\alpha\alpha'}\right)=\lambda_4,\quad \lambda_4\in\mathbb{R}. \tag{35}$$

Using (24) in (19), we obtain

$$\frac{\lambda_3}{\gamma'^2}+\lambda_4+\frac{\gamma\gamma''}{\gamma'^2}-2=0. \tag{36}$$

If we differentiate this equation with respect to y, we get

$$\lambda_3 \left(\frac{1}{\gamma'^2}\right)_{,y} + \left(\frac{\gamma\gamma''}{\gamma'^2}\right)_{,y} = 0. \tag{37}$$

Substituting (24) in (37) gives

$$\gamma'^2 - c_1\gamma^2 - \lambda_3 = 0.$$

Differentiating this equation with respect to y, we obtain $\gamma'' = c_1\gamma$. From (24) , we get $c_2 = 0$.
Then

$$\alpha'' = c_5\alpha, \;\; \gamma'' = c_1\gamma. \tag{38}$$

By substituting (38) into (10) and differentiating with respect to x and next with respect to y, we get

$$\left(\frac{\alpha'}{\alpha}\right)_{,x}\left(\frac{\gamma'}{\gamma}\right)_{,y} = 0.$$

Hence there are constants $\delta_1, \; \delta_2 \in \mathbb{R} - \{0\}$ such that

$$\gamma' = \delta_1\gamma, \;\; \alpha' = \delta_2\alpha. \tag{39}$$

From (39) and (38), we obtain

$$\delta_2^2 = c_5, \;\; \delta_1^2 = c_1.$$

By using of relations (30) and (36) we find

$$\lambda_1 + c_5(\lambda_2 - 1)\alpha^2 = 0, \;\; \lambda_3 + c_1(\lambda_4 - 1)\gamma^2 = 0.$$

Then $\lambda_1 = 0, \; \lambda_2 = 1$ and $\lambda_3 = 0, \; \lambda_4 = 1$. Since $\lambda_2 = 1, \;\; c_6 = 0$ and $\delta_1^2 = c_1$, we must have $c_8 = 0$.
Now, from the equation (25) we obtain $\alpha' = c_7\alpha$.
Then $c_7^2 = c_5$, and (27) implies $\sqrt{c_1} = -a\sqrt{c_5} = -ac_7$.
Substituting $\lambda_4 = 1$ into (35) we obtain $c_4 = 0$. This implies $c_3 = \sqrt{c_1}$. From (39), we obtain

$$\gamma = b_1 e^{-ac_7 y}, \;\; \alpha = b_2 e^{c_7 x},$$

where $b_1, b_2, c_7 \in \mathbb{R} - \{0\}$.

Thus, we can state the following theorem:

Theorem 6. *Let $\mathcal{M}^2$ be a $\mathcal{THA}$-surface in $\mathbb{I}^3$. If $\mathcal{M}^2$ is minimal surface, then $\mathcal{M}^2$ parameterized as (3), where*
(1). *if $a = 0$, then*
(i) $g(y) = y_0 \in \mathbb{R} - \{0\}$ and f is any arbitrary function.
(ii) $f(x) = x_0 \in \mathbb{R} - \{0\}$ and $g(y) = b_1 y + b_2$; $b_1, b_2 \in \mathbb{R}$.

(iii) $f(x) = b_3x + b_4$, $b_3, b_4 \in \mathbb{R}$ *and* $g(y) = \dfrac{1}{Ab_3}\tan(c_1y + c_2) + c_3$; $b_3, c_1, c_2, c_3 \in \mathbb{R} - \{0\}$.
(iv) $f(x) = -\dfrac{1}{c_5x + c_6} - \dfrac{B}{A}$ *and* $g(y) = c_7y + c_8$; $c_i \in \mathbb{R} - \{0\}$.
(2). *if* $a \neq 0$, *then*
(i) $g(y) = y_0 \in \mathbb{R} - \{0\}$ *and* $f(x) = d_1x + d_2$, $d_1, d_2 \in \mathbb{R} - \{0\}$.
(ii) $f(x) = b_0 \in \mathbb{R} - \{0\}$ *and* $g(y) = b_1y + b_2$, $b_1, b_2 \in \mathbb{R} - \{0\}$.
(iii) $f(x) = \lambda_3x + \lambda_4$; $\lambda_3, \lambda_4 \in \mathbb{R}$ *and* $g(y) = \lambda_0 \in \mathbb{R} - \{0\}$.
(iv) $f(x) = \dfrac{\lambda_5e^{cx}}{A} - \dfrac{B}{A}$ *and* $g(y) = \dfrac{\lambda_6e^{-acy}}{A} - \dfrac{B}{A}$; $\lambda_i \in \mathbb{R} - \{0\}$.

References

[1] Aydin, M.E., *A generalization of translation surfaces with constant curvature in the isotropic space*, J.Geom. **107** (2016), no. 3, 603-615.

[2] Aydin, M.E., *Affine factorable surfaces in isotropic space*, TWMS J. Pure Appl. Math. **11** (2020), no. 1, 72-88.

[3] Chen, B.Y., *Total mean curvature and submanifolds of finite type*, World Scientific, Singapore, 1984.

[4] Dillen, F., Pas, J. and Verstraelen, L., *On surfaces of finite type in Euclidean 3-space*, Kodai Math. J. **13** (1990), 10-21.

[5] Ferrandez, A., Garay, O.J. and Lucas, P., *On a certain class of conformally flat Euclidean hypersurfaces*, Proc. of the Conf, in Global Analysis and Global Differential Geometry, Berlin. (1990).

[6] Karacan, M.K., Yoon, D.W. and Bukcu, B., *Translation surfaces in the three dimensional simply isotropic space* $\mathbb{I}_3^1$, Int. J. Geom. Methods Mod. Phys. **13** (2016), 1650088.

[7] Liu, H., *Translation surfaces with constant mean curvature in 3-dimensional spaces*, J. Geom. **64** (1999), 141-149.

[8] Liu, H. and Yu. Y., *Affine translation surfaces in Euclidean* 3*-space*, Proc. Japan Acad. **89** (2013), 111-113.

[9] López, R., *Minimal translation surfaces in hyperbolic space*, Beitr. Algebra Geom. **52** (2011), 105-112.

[10] López, R. and Munteanu, M.I., *Minimal translation surfaces in* Sol_3, J. Math. Soc. Japan, **64** (2012), 985-1003.

[11] López, R. and Moruz, M., *Translation and homothetical surfaces in Euclidean space with constant curvature*, J. Korean Math. Soc. **52** (2015), 523-535.

[12] Milin - Sipus, Z., *Translation surfaces of constant curvatures in a simply isotropic space*, Period. Math. Hung. **68** (2014), 160-175.

[13] Milin - Sipus, Z. and Divjak, B., *Curves in n - dimensional k - isotropic space*, Glasnik Matematicki, (1998), 267-286.

[14] Munteanu, M.I. and Nistor, A.I., *Polynomial translation Weingarten surfaces in 3-dimensional Euclidean space*, Proceedings of the VIII International Colloquium on Differential Geometry, World Scientific, (2009), 316-320.

[15] Senoussi, B. and Bekkar, M., *Translation and homothetical TH-surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$ and Lorentzian-Minkowski space $\mathbb{E}^3_1$*, Open J. Math. Sci. (2019), 74-83.

[16] Senoussi, B., Bennour, A. and Beddani, K., *THA-surfaces in the Galilean space $\mathbb{G}^3$*, J. Adv. Math. Stud. **14** (2021), 187-196.

[17] Senoussi, B., Bennour, A. and Beddani, K., *THA-surfaces in 3-dimensional Euclidean space*, Asia Pac. J. Math. **14** (2021), 1-15.

[18] Senoussi, B., Beddani, K. and Bennour, A., *THA-surfaces of finite type in the Galilean space $\mathbb{G}^3$*, Annals of West University of Timisoara Mathematics and Computer Science. **58** (2022), No. 1, 85-99.

[19] Senoussi, B., Beddani, K. and Bennour, A., *Translation surfaces in Lorentz-Heisenberg 3-space Nil^3_1*, International Journal of Geometry (2022), No. 4, 34-46.

[20] Seo, K., *Translation hypersurfaces with constant curvature in space forms*, Osaka J. Math. **50** (2013), 631-641.

[21] Takahashi, T., *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan. **18** (1966), 380-385.

[22] Verstraelen, L., Walrave, J. and Yaprak, S., *The minimal translation surfaces in Euclidean space*, Soochow J. Math. **20** (1994), 77-82.