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A NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH q-CALCULUS

Saurabh PORWAL [*,1] **and Manoj Kumar SINGH** [2]

Abstract

The purpose of the present paper is to introduce a new subclass of harmonic univalent functions by applying q-calculus. Coefficient inequalities, extreme points, distortion bounds, covering results, convolution condition and convex combination are determined for this class. Finally, we discuss a class preserving integral operator for this class.

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1 Introduction

A continuous complex valued function $f = u + iv$ defined in a simply-connected domain $\mathbb{D}$ is said to be harmonic in $\mathbb{D}$ if both u and v are real harmonic in $\mathbb{D}$. In any simply-connected domain we can express$f = h+\bar{g}$, where h and g are analytic in $\mathbb{D}$, called the analytic and co-analytic part of the function f, respectively. The jacobian of the function $f = h + \bar{g}$ is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. According to Lewy's theorem every harmonic function $f = h + \bar{g}$ is locally univalent and sense preserving in $\mathbb{D}$ is that $|h'(z)|^2 > |g'(z)|^2$, $z \in \mathbb{D}$. For detail study one may refer to Clunie and Sheil-Small [5], Duren [7],(see also [1, 2, 8, 15, 18]).

Further, we denote S_H the class of function $f = h + \bar{g}$ which are harmonic, univalent and sense-preserving in the open unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$.

If $f = h + \bar{g} \in S_H$ then h and g are of the form

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \tag{1.1}$$

[1*] *Corresponding author*, Department of Mathematics, Ram Sahai Government Degree College, Bairi-Shivrajpur, Kanpur-209205, (U.P.), India, e-mail: saurabhjcb@rediffmail.com
[2]Department of Mathematics, Government Engineering College-Dahod, Gujarat-389151, India, e-mail: ms84ddu@gmail.com

It is worthy to note that for $g(z) = 0$ the class S_H reduced to the class S of analytic univalent functions. For this class $f(z)$ can be written as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

The following definitions of fractional derivatives are given by Owa [12] and Srivastwa and Owa [21]

Definition 1.1. *The fractional derivative of order λ is defined for a function $f(z)$ by*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi, \tag{1.3}$$

where $0 \leq \lambda < 1$, $f(z)$ is an analytic function in a simply-connected region of the z plane containing the origin and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $log(z-\xi)$ to be real when $(z-\xi) > 0$.

Definition 1.2. *Under the hypothesis of Definition 1.1, the fractional derivative of order $n+\lambda$ is defined for a function $f(z)$ by*

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z),$$

where $0 \leq \lambda < 1$ and $n \in N_0 = \{0, 1, 2, 3, ...\}$.

Using the Definition 1.1 and 1.2, Owa and Srivastava [13] introduced the following fractional calculus operator
$\Omega^{\lambda} : A \longrightarrow A$, which is defined as $\Omega^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z), (\lambda \neq 2, 3, ...)$, where A denotes the class of functions f of the form (1.2) which are analytic in U.

Recently, it has come to know that the concept of q-calculus is widely used in Geometric function theory. By using the definition of q- calculus various new subclasses of analytic and harmonic univalent functions were investigated by several researchers. In this direction noteworthy contribution may be found in [3], [11] and [16].

The concept of q-calculus were initially introduced by Jackson [9] (see also [4]). They defined the q-number for $k \in N$ in the following way

$$[k]_q = \frac{1-q^k}{1-q}, 0 \leq q < 1. \tag{1.4}$$

It is easy to see that $[k]_q$ can be represented as a geometric series in the following way $[k]_q = \sum_{i=0}^{k-1} q^i$.

Obviously, $\lim_{k \to -\infty} [k]_q = \frac{1}{1-q}$ and $\lim_{q \to -1} [k]_q = k$.

The q-derivative for a function f is defined as

$$D_q(f(z)) = \frac{f(qz) - f(z)}{(q-1)z}, q \neq 1, z \neq 0$$

and $D_q(f(0)) = f'(0)$, provided $f'(0)$ exists.

If we take the function $h(z) = z^k$ then the q- derivative of $h(z)$ is defined as $D_q(h(z)) = D_q(z^k) = \frac{1-q^k}{1-q} z^{k-1} = [k]_q z^{k-1}$.

Then $\lim_{q\to 1} D_q(h(z)) = \lim_{q\to 1}[k]_q(z^{k-1}) = kz^{k-1} = h'(z)$

where h' is the ordinary derivative

Now, we define the subclass $S_H(\alpha, \lambda, \mu, q)$ of S_H consisting of functions f of the form (1.1) satisfying the following condition

$$\Re\left\{\frac{z[D_q(\Omega^\lambda h(z))] - \overline{zD_q(\Omega^\lambda g(z))}}{\mu(z[D_q(\Omega^\lambda h(z))] - \overline{z[D_q(\Omega^\lambda g(z))]}) + (1-\mu)\{\Omega^\lambda h(z) + \overline{\Omega^\lambda g(z)}\}}\right\} \geq \alpha \quad (1.5)$$

where $0 \leq \alpha < 1$, $0 \leq \mu < 1$, $0 \leq \lambda < 1$, $0 < q < 1$.

Next, we define $TS_H(\alpha, \lambda, \mu, q)$ be the subclass of $S_H(\alpha, \lambda, \mu, q)$ for which $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k|z^k, g(z) = \sum_{k=1}^{\infty} |b_k|z^k, |b_1| < 1. \quad (1.6)$$

It should be worthy to note that for specific values of α, μ, λ, q on the subclasses $S_H(\alpha, \lambda, \mu, q)$ and $TS_H(\alpha, \lambda, \mu, q)$, we obtain the following known subclasses of S_H studied earlier by various researchers.

1. $S_H(\alpha, \lambda, \mu, 1) \equiv S_H(\alpha, \lambda, \mu)$ and $TS_H(\alpha, \lambda, \mu, 1) \equiv TS_H(\alpha, \lambda, \mu)$ studied by Porwal and Kanaujia [17].

2. $S_H(\alpha, \lambda, 0, 1) \equiv S_H(\alpha, \lambda)$ and $TS_H(\alpha, \lambda, 0, 1) \equiv TS_H(\alpha, \lambda)$ studied by Dixit and Porwal [6].

3. $S_H(\alpha, 0, \mu, 1) \equiv S_H(\alpha, \mu)$ and $TS_H(\alpha, 0, \mu, 1) \equiv TS_H(\alpha, \mu)$ studied by Öztürk et.al. [14].

4. $S_H(\alpha, 0, 0, 1) \equiv S_H^*(\alpha,)$ and $TS_H(\alpha, 0, 0, 1) \equiv TS_H^*(\alpha)$ studied by Jahangiri [10].

5. $S_H(0, 0, 0, 1) \equiv S_H^*$ and $TS_H(0, 0, 0, 1) \equiv TS_H^*$ studied by Slverman [19], Silverman and Silvia [20].

In the present paper, we obtain coefficient inequality, extreme points, distortion bounds, covering results, convolution condition and convex combination for the class $TS_H(\alpha, \lambda, \mu, q)$. Finally, we discuss a class preserving integral operator and q- Jackson type integral operator for this class.

2 Main results

In our first theorem, we give a sufficient coefficient bound for function in the class $S_H(\alpha, \lambda, \mu, q)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be such that h and g are given by (1.1). Furthermore let*

$$\sum_{k=2}^{\infty} \frac{[k]_q(1-\alpha\mu)-\alpha(1-\mu)}{1-\alpha}\phi(k,\lambda)|a_k| + \sum_{k=1}^{\infty} \frac{[k]_q(1-\alpha\mu)+\alpha(1-\mu)}{1-\alpha}\phi(k,\lambda)|b_k| \leq 1. \tag{2.1}$$

where $0 \leq \alpha < 1$, $0 \leq \mu < 1$, $0 \leq \lambda < 1$, $0 < q < 1$ and

$$\phi(k,\lambda) = \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)}$$

Then f is sense-preserving, harmonic univalent in U and $f \in S_H(\alpha, \lambda, \mu, q)$.

Proof. First we note that f is locally univalent and sense-preserving in U. This is because

$$\begin{aligned}
|h'(z)| &\geq 1 - \sum_{k=2}^{\infty} k|a_k|r^{k-1} \\
&> 1 - \sum_{k=2}^{\infty} k|a_k| \\
&\geq 1 - \sum_{k=2}^{\infty} \frac{[k]_q(1-\alpha\mu)-\alpha(1-\mu)}{1-\alpha}\phi(k,\lambda)|a_k| \\
&\geq \sum_{k=1}^{\infty} \frac{[k]_q(1-\alpha\mu)+\alpha(1-\mu)}{1-\alpha}\phi(k,\lambda)|b_k| \\
&\geq \sum_{k=1}^{\infty} k|b_k| \\
&> \sum_{k=1}^{\infty} k|b_k|r^{k-1} \\
&\geq |g'(z)|.
\end{aligned}$$

To Show that f is univalent in U, suppose that $z_1, z_2 \in U$ such that $z_1 \neq z_2$ then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = \left|\frac{1-\sum_{k=1}^{k=\infty} b_k(z_1^k - z_2^k)}{z_1 - z_2 + \sum_{k=2}^{k=\infty} a_k(z_1^k - z_2^k)}\right|$$

$$> 1 - \frac{\sum_{k=1}^{k=\infty} k|b_k|}{1-\sum_{k=2}^{k=\infty} k|a_k|} \geq \frac{1-\sum_{k=1}^{k=\infty} \frac{[k]_q(1-\alpha\mu)+\alpha(1-\mu)}{1-\alpha}\phi(k,\lambda)|b_k|}{1-\sum_{k=2}^{k=\infty} \frac{[k]_q(1-\alpha\mu)-\alpha(1-\mu)}{1-\alpha}\phi(k,\lambda)|a_k|} \geq 0.$$

Now, we show that $f \in S_H(\alpha, \lambda, \mu, q)$, using the fact that $\Re\{w\} \geq \alpha$, if and only if, $|1-\alpha+w| \geq |1+\alpha-w|$ it suffices to show that

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \geq 0, \tag{2.2}$$

where $A(z) = z[D_q(\Omega^\lambda h(z))] - z\overline{[D_q(\Omega^\lambda g(z))]}$ and

$$B(z) = \mu z[D_q(\Omega^\lambda h(z))] - z\overline{[D_q(\Omega^\lambda g(z))]} + (1-\mu)\{\Omega^\lambda h(z) + \overline{\Omega^\lambda g(z)}\}.$$

Substituting the values of $A(z)$ and $B(z)$ in L.H.S. of (2.2) and performing the simple calculation, we obtain

$$\begin{aligned}
&= |(2-\alpha)z + \sum_{k=2}^{\infty} [k]_q + (1-\alpha)\mu[k]_q + (1-\alpha)(1-\mu)\phi(k,\lambda)a_k z^k \\
&| - \sum_{k=1}^{\infty} [k]_q - (1-\alpha)\mu[k]_q - (1-\alpha)(1-\mu)\phi(k,\lambda)b_k z^k| \\
&| - \alpha z + \sum_{k=2}^{\infty} [k]_q - (1-\alpha)\mu[k]_q - (1-\alpha)(1-\mu)\phi(k,\lambda)a_k z^k| \\
&| - \sum_{k=1}^{\infty} [k]_q - (1-\alpha)\mu[k]_q + (1-\alpha)(1-\mu)\phi(k,\lambda)b_k z^k| \\
&\geq 2(1-\alpha)|z|\Bigg[1 - \sum_{k=2}^{k=\infty}\{\frac{[k]_q(1-\alpha\mu) - \alpha(1-\mu)}{1-\alpha}\}\phi(k,\lambda)|a_k||z|^{k-1} \\
&- \sum_{k=1}^{\infty}\{\frac{[k]_q(1-\alpha\mu) + \alpha(1-\mu)}{1-\alpha}\}\phi(k,\lambda)|b_k||z|^{k-1}\Bigg] \\
&> 2(1-\alpha)|z|\Bigg[1 - \sum_{k=2}^{k=\infty}\{\frac{[k]_q(1-\alpha\mu) - \alpha(1-\mu)}{1-\alpha}\}\phi(k,\lambda)|a_k| \\
&- \sum_{k=1}^{k=\infty}\{\frac{[k]_q(1-\alpha\mu) + \alpha(1-\mu)}{1-\alpha}\}\phi(k,\lambda)|b_k|\Bigg] \geq 0, \qquad \text{(Using (2.1))}.
\end{aligned}$$

The coefficient bound given by (2.1) is sharp because equality holds for the following functions

$$\begin{aligned}
f(z) = z &+ \sum_{k=2}^{\infty}\{\frac{1-\alpha}{[k]_q(1-\alpha\mu) - \alpha(1-\mu)\}\phi(k,\lambda)} x_k z^k \\
&+ \sum_{k=1}^{\infty} \frac{1-\alpha}{\{[k]_q(1-\alpha\mu) + \alpha(1-\mu)\}\phi(k,\lambda)} \overline{y_k z^k},
\end{aligned}$$

where

$$\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1.$$

This completes the proof of Theorem 2.1. □

In our next theorem, we prove that the condition (2.1) is also necessary for the function $f = h + \overline{g}$, where h and g are of the form (1.6).

Theorem 2.2. *Let the function $f = h + \overline{g}$ be such that h and g are given by (1.6). Then $f \in S_H(\alpha, \lambda, \mu, q)$, if and only if*

$$\sum_{k=2}^{\infty}\{[k]_q(1-\alpha\mu)-\alpha(1-\mu)\}\phi(k,\lambda)|a_k| + \sum_{k=1}^{\infty}\{[k]_q(1-\alpha\mu)+\alpha(1-\mu)\}\phi(k,\lambda)|b_k| \leq 1-\alpha, \tag{2.3}$$

where $0 \leq \alpha < 1$, $0 \leq \mu < 1$, $0 \leq \lambda < 1$, $0 < q < 1$ and

$$\phi(k,\lambda) = \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)}.$$

Proof. Since $TS_H(\alpha, \lambda, \mu, q) \subset S_H(\alpha, \lambda, \mu, q)$ this gives the if part of the theorem. To this end, for function f of the form (1.6), we notice that the condition

$$\Re\left\{\frac{z[D_q(\Omega^\lambda h(z))] - \overline{zD_q(\Omega^\lambda g(z))}}{\mu(z[D_q(\Omega^\lambda h(z))] - \overline{z[D_q(\Omega^\lambda g(z))]}) + (1-\mu)\{\Omega^\lambda h(z) + \overline{\Omega^\lambda g(z)}\}}\right\} \geq \alpha$$

is equivalent to

$$\Re\left\{\frac{(1-\alpha)z - \sum_{k=2}^{\infty}\{[k]_q + (1-\alpha\mu) - \alpha(1-\mu)\}\phi(k,\lambda)|a_k|z^k - \sum_{k=1}^{\infty}\{[k]_q(1-\alpha\mu)+\alpha(1-\mu)\}\phi(k,\lambda)|b_k|\bar{z}^k}{z - \sum_{k=2}^{\infty}\{[k]_q\mu + (1-\mu)\}\phi(k,\lambda)|a_k|z^k - \sum_{k=1}^{\infty}\{[k]_q\mu - (1-\mu)\}\phi(k,\lambda)|b_k|\bar{z}^k}\right\} \geq \alpha.$$

The above condition must holds for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$\left\{\frac{(1-\alpha) - \sum_{k=2}^{\infty}\{[k]_q + (1-\alpha\mu) - \alpha(1-\mu)\}\phi(k,\lambda)|a_k|r^{k-1} - \sum_{k=1}^{\infty}\{[k]_q(1-\alpha\mu)+\alpha(1-\mu)\}\phi(k,\lambda)|b_k|r^{k-1}}{1 - \sum_{k=2}^{\infty}\{[k]_q\mu + (1-\mu)\}\phi(k,\lambda)|a_k|r^{k-1} - \sum_{k=1}^{\infty}\{[k]_q\mu - (1-\mu)\}\phi(k,\lambda)|b_k|r^{k-1}}\right\} \geq 0 \tag{2.4}$$

If the condition (2.3) does not hold then the numerator in (2.4) is negative for r sufficiently close to 1.Thus there exists a $z_0 = r_0$ in (0, 1) for which the quotient in (2.4) is negative. This contradicts the required condition for $f \in TS_H(\alpha, \lambda, \mu, q)$ and so the proof is complete.

□

Next, we determine the extreme points of closed convex hulls of $TS_H(\alpha, \lambda, \mu, q)$ denoted by $clcoTS_H(\alpha, \lambda, \mu, q)$.

Theorem 2.3. *If $f \in clcoTS_H(\alpha, \lambda, \mu, q)$, if and only if*

$$f(z) = \sum_{k=1}^{\infty}\{x_k h_k(z)) + y_k g_k(z)\} \tag{2.5}$$

where $h_1(z) = z$

$$h_k(z) = z - \frac{1-\alpha}{\{[k]_q(1-\alpha\mu) - \alpha(1-\mu)\}\phi(k,\lambda)} z^k, (k = 2, 3, 4, ...)$$

$$g_k(z) = z + \frac{1-\alpha}{\{[k]_q(1-\alpha\mu) + \alpha(1-\mu)\}\phi(k,\lambda)} \bar{z}^k, (k = 1, 2, 3, ...),$$

$$x_k \geq 0, y_k \geq 0, \sum_{k=1}^{\infty}(x_k + y_k) = 1.$$

In particular, the extreme points of $TS_H(\alpha, \lambda, \mu, q)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For function f of the form (2.5), we have
$f(z) = \sum_{k=1}^{\infty}\{x_k h_k(z)) + y_k g_k(z)\}$

$$= \sum_{k=1}^{\infty}(x_k + y_k)z - \sum_{k=2}^{\infty} \frac{1-\alpha}{\{[k]_q(1-\alpha\mu) - \alpha(1-\mu)\}\phi(k,\lambda)} x_k z^k$$

$$+ \sum_{k=1}^{\infty} \frac{1-\alpha}{\{[k]_q(1-\alpha\mu) + \alpha(1-\mu)\}\phi(k,\lambda)} y_k \bar{z}^k,$$

Then

$$\sum_{k=2}^{\infty} \frac{[k]_q(1-\alpha\mu) - \alpha(1-\mu)}{1-\alpha}\phi(k,\lambda) \left\{\frac{1-\alpha}{[[k]_q(1-\alpha\mu) - \alpha(1-\mu)]\,\phi(k,\lambda)}\right\} x_k$$

$$+ \sum_{k=1}^{\infty} \frac{[k]_q(1-\alpha\mu) + \alpha(1-\mu)}{1-\alpha}\phi(k,\lambda) \left\{\frac{1-\alpha}{[[k]_q(1-\alpha\mu) + \alpha(1-\mu)]\,\phi(k,\lambda)}\right\} y_k$$

$$= \sum_{k=1}^{\infty} x_k + \sum_{k=2}^{\infty} y_k = 1 - x_1 \leq 1$$

and so $f \in clcoTS_H(\alpha, \lambda, \mu, q)$.
Conversely, suppose that $f \in clcoTS_H(\alpha, \lambda, \mu, q)$. Set

$$x_k = \frac{k(1-\alpha\mu) - \alpha(1-\mu)}{1-\alpha}\phi(k,\lambda)|a_k|, (k = 2, 3, 4...)$$

$$y_k = \frac{\{k(1-\alpha\mu) + \alpha(1-\mu)\}}{1-\alpha}\phi(k,\lambda)|b_k|, (k = 1, 2, 3, ...).$$

then from Theorem 2.2, we have $0 \leq x_k \leq 1 (k = 2, 3, 4, ...)$ and $0 \leq y_k \leq 1 (k = 2, 3, 4, ...)$, we define

$$x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k$$

and note that by Theorem 2.2, $x_1 \geq 0$. Consequently, we obtain

$$f(z) = \sum_{k=1}^{\infty} \{x_k h_k(z)) + y_k g_k(z)\}$$

as required.

□

In our next theorem, we obtain the bounds for function in $TS_H(\alpha, \lambda, \mu, q)$ which yields a covering results for this class.

Theorem 2.4. *Let* $f \in TS_H(\alpha, \lambda, \mu, q)$ *then*

$$|f(z)| \leq (1 + |b_1|)r + \left(\frac{1-\alpha}{1+q-\alpha(1+q\mu)} - \frac{1+\alpha(1-2\mu)}{1+q-\alpha(1+q\mu)}|b_1|\right)\frac{2-\lambda}{2}r^2$$

and

$$|f(z)| \geq (1 - |b_1|)r - \left(\frac{1-\alpha}{1+q-\alpha(1+q\mu)} - \frac{1+\alpha(1-2\mu)}{1+q-\alpha(1+q\mu)}|b_1|\right)\frac{2-\lambda}{2}r^2 |z| = r < 1.$$

Proof. We only prove the right hand inequality. The proof for left hand inequality is similar and will be omitted. Let $f \in TS_h(\alpha, \lambda, \mu, q)$. Then taking the absolute value of f we have

$$\begin{aligned}
|f(z)| &\leq (1 + |b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^2 \\
&= (1 + |b_1|)r + \frac{1-\alpha}{\{1+q-\alpha(1+q\mu)\}\phi(2,\lambda)} \times \\
&\qquad \times \sum_{k=2}^{\infty}\left(\frac{\{1+q-\alpha(1+q\mu)\}\phi(2,\lambda)}{1-\alpha}(|a_k| + |b_k|)\right)r^2 \\
&\leq (1 + |b_1|)r + \frac{(1-\alpha)(2-\lambda)}{2\{1+q-\alpha(1+q\mu)\}}\sum_{k=2}^{\infty}\left(\frac{\{[k]_q(1-\alpha\mu) - \alpha(1-\mu)\}}{1-\alpha}|a_k| + \right. \\
&\left. \frac{\{[k]_q(1-\alpha\mu) + \alpha(1-\mu)\}}{1-\alpha}|b_k|\right)\phi(k,\lambda)r^2 \\
&\leq (1 + |b_1|)r + \frac{(1-\alpha)(2-\lambda)}{2\{1+q-\alpha(1+q\mu)\}}\left(1 - \frac{1+\alpha(1-2\mu)}{1-\alpha}|b_1|\right)r^2 \\
&\leq (1 + |b_1|)r + \left(\frac{1-\alpha}{1+q-\alpha(1+q\mu)} - \frac{1+\alpha(1-2\mu)}{1+q-\alpha(1+q\mu)}|b_1|\right)\frac{2-\lambda}{2}r^2.
\end{aligned}$$

□

The following covering result follows from the left hand inequality of Theorem 2.4.

Corollary 2.5. *Let $f \in TS_H(\alpha, \lambda, \mu, q)$. Then*

$$\left\{ w : |w| < \left(1 - \frac{(1-\alpha)(2-\lambda)}{2\,(1+q-\alpha(1+q\mu))} - \left(1 - \frac{[1+\alpha(1-2\mu)]\,(2-\lambda)}{2(1+q-\alpha(1+q\mu))} \right) |b_1| \right) \right\} \subset f(U).$$

3 Convolution and convex combination

In this section, we prove that the class $TS_H(\alpha, \lambda, \mu, q)$ is closed under convolution and convex combination. Now, we need the following definition of convolution of two harmonic functions.

Definition 3.1. *Let the function $f(z)$ and $F(z)$ be defined by*

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \overline{z}^k$$

and

$$F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| \overline{z}^k.$$

Then the convolution of $f(z)$ and $F(z)$ are denoted by $f(z) * F(z)$ and defined by

$$(f * F)(z) = f(z) * F(z)$$

$$(f * F)(z) = z - \sum_{k=2}^{\infty} |a_k||A_k| z^k + \sum_{k=1}^{\infty} |b_k||B_k| \overline{z}^k. \tag{3.1}$$

Using this definition we show that the class $TS_H(\alpha, \lambda, \mu, q)$ is closed under convolution.

Theorem 3.2. *For $0 \le \beta \le \alpha < 1$, let $f \in TS_H(\alpha, \lambda, \mu, q)$ and $F \in TS_H(\beta, \lambda, \mu, q)$. Then $(f * F) \in TS_H(\alpha, \lambda, \mu, q) \subset TS_H(\beta, \lambda, \mu, q)$.*

Proof. Let $f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \overline{z}^k$ be in $TS_H(\alpha, \lambda, \mu, q)$ and $F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| \overline{z}^k$ be in $TS_H(\beta, \lambda, \mu, q)$. Then the convolution $(f * F)(z)$ is given by (3.1). To prove that $(f * F) \in TS_H(\alpha, \lambda, \mu, q)$, from Theorem 2.2 it is sufficient to show that

$$\sum_{k=2}^{\infty} \left\{ \frac{\{[k]_q(1-\alpha\mu) - \alpha(1-\mu)\}}{1-\alpha} \right\} |a_k A_k| \phi(k, \lambda) +$$
$$\sum_{k=1}^{\infty} \left\{ \frac{\{[k]_q(1-\alpha\mu) + \alpha(1-\mu)\}}{1-\alpha} |b_k B_k| \phi(k, \lambda) \right\} \le 1 - \alpha.$$

Since $F \in TS_H(\beta, \lambda, \mu, q)$ then by Theorem 2.2, we obtain $|A_k| \le 1$, and $|B_k| \le 1$

Now

$$\sum_{k=2}^{\infty}\frac{\{[k]_q(1-\alpha\mu)-\alpha(1-\mu)\}}{1-\alpha}|a_kA_k|\phi(k,\lambda)+$$
$$\sum_{k=1}^{\infty}\frac{\{[k]_q(1-\alpha\mu)+\alpha(1-\mu)\}}{1-\alpha}|b_kB_k|\phi(k,\lambda)$$
$$\leq\sum_{k=2}^{\infty}\frac{\{[k]_q(1-\alpha\mu)-\alpha(1-\mu)\}}{1-\alpha}|a_k|\phi(k,\lambda)+$$
$$\sum_{k=1}^{\infty}\frac{\{[k]_q(1-\alpha\mu)+\alpha(1-\mu)\}}{1-\alpha}|b_k|\phi(k,\lambda)$$
$$\leq 1,\quad \text{since } (f)(z)\in TS_H(\alpha,\lambda,\mu,q).$$

Therefore $(f*F)(z)\in TS_H(\alpha,\lambda,\mu,q)$. □

In our next theorem, we prove that the class $TS_H(\alpha,\lambda,\mu,q)$ is closed under convex combination.

Theorem 3.3. *The class $TS_H(\alpha,\lambda,\mu,q)$ is closed under convex combination.*

Proof. For $\alpha=1,2,3,...$ let $f_i(z)\in TS_H(\alpha,\lambda,\mu,q)$ where $f_i(z)$ is of the form

$$f_i(z)=z-\sum_{k=2}^{\infty}|a_{k_i}|z^k+\sum_{k=1}^{\infty}|b_{k_i}|\overline{z}^k.$$

Then from Theorem 2.2, we have

$$\sum_{k=2}^{\infty}\frac{\{[k]_q(1-\alpha\mu)-\alpha(1-\mu)\}}{1-\alpha}|a_{k_i}|\phi(k,\lambda)+$$
$$\sum_{k=1}^{\infty}\frac{\{[k]_q(1-\alpha\mu)+\alpha(1-\mu)\}}{1-\alpha}|b_{k_i}|\phi(k,\lambda)\leq 1. \tag{3.2}$$

For $\sum_{i=1}^{\infty}t_i=1, 0\leq t_i\leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty}t_if_i(z)=z-\sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{k_i}|\right)z^k+\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{k_i}|\right)\overline{z}^k.$$

Then by the condition (3.2), we have

$$\sum_{k=2}^{\infty}\left\{\frac{\{[k]_q(1-\alpha\mu)-\alpha(1-\mu)\}}{1-\alpha}\right\}(\sum_{i=1}^{\infty}t_i|a_{k_i}|)\phi(k,\lambda)+$$
$$\sum_{k=1}^{\infty}\left\{\frac{\{[k]_q(1-\alpha\mu)+\alpha(1-\mu)\}}{1-\alpha}\right\}(\sum_{i=1}^{\infty}t_i|b_{k_i}|)\phi(k,\lambda)$$

$$= \sum_{i=1}^{\infty} t_i \Big\{ \sum_{k=2}^{\infty} \{ \frac{\{[k]_q(1-\alpha\mu)-\alpha(1-\mu)\}}{1-\alpha} \} |a_{k_i}| \phi(k,\lambda) +$$
$$\sum_{k=1}^{\infty} \{ \frac{\{[k]_q(1-\alpha\mu)+\alpha(1-\mu)\}}{1-\alpha} \} |b_{k_i}| \phi(k,\lambda) \Big\}$$
$$\leq \sum_{i=1}^{\infty} t_i \leq 1.$$

Then by Theorem 2.2, we have $\sum_{i=1}^{\infty} t_i f_i(z) \in TS_H(\alpha,\lambda,\mu,q)$. □

4 A family of class preserving integral operators

Definition 4.1. *Let $f(z) = h(z) + \overline{g(z)}$ be defined by (1.1) then $F(z)$ be defined by the relation*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t)dt + \overline{\frac{c+1}{z^c} \int_0^z t^{c-1} g(t)dt}, \quad (c > -1). \tag{4.1}$$

Theorem 4.2. *Let $f(z) = h(z) + \overline{g(z)} \in S_H$ be given by (1.6) and $f(z) \in TS_H(\alpha,\lambda,\mu,q)$ $0 \leq \alpha < 1$, $0 \leq \mu \leq 1$, $0 \leq \lambda \leq 1$ and $0 < q < 1$. Then $F(z)$ defined by (4.1) is also in the class $TS_H(\alpha,\lambda,\mu,q)$.*

Proof. From the representation of (4.1) it follows that

$$F(z) = z - \sum_{k=2}^{\infty} \frac{c+1}{c+k} |a_k| z^k + \sum_{k=1}^{\infty} \frac{c+1}{c+k} |b_k| \overline{z}^k. \tag{4.2}$$

Since $f \in TS_H(\alpha,\lambda,\mu,q)$, we have

$$\sum_{k=2}^{\infty} \{ \frac{\{[k]_q(1-\alpha\mu)-\alpha(1-\mu)\}}{1-\alpha} \} |a_k| \phi(k,\lambda) +$$
$$\sum_{k=1}^{\infty} \{ \frac{\{[k]_q(1-\alpha\mu)+\alpha(1-\mu)\}}{1-\alpha} \} |b_k| \phi(k,\lambda) \leq 1.$$

Now

$$\sum_{k=2}^{\infty} \{ \frac{\{[k]_q(1-\alpha\mu)-\alpha(1-\mu)\}}{1-\alpha} \} \frac{c+1}{c+k} |a_k| \phi(k,\lambda) +$$
$$\sum_{k=1}^{\infty} \{ \frac{\{[k]_q(1-\alpha\mu)+\alpha(1-\mu)\}}{1-\alpha} \} \frac{c+1}{c+k} |b_k| \phi(k,\lambda)$$

$$\leq \sum_{k=2}^{\infty} \{ \frac{\{[k]_q(1-\alpha\mu)-\alpha(1-\mu)\}}{1-\alpha} \} |a_k| \phi(k,\lambda) +$$

$$\sum_{k=1}^{\infty}\{\frac{\{[k]_q(1-\alpha\mu)+\alpha(1-\mu)\}}{1-\alpha}\}|b_k|\phi(k,\lambda)$$
$$\leq 1.$$

Then by Theorem 2.2, we have $F(z) \in TS_H(\alpha,\lambda,\mu,q)$. □

Definition 4.3. *Let $f = h + \bar{g}$ be defined by (1.1). Then the q- Jackson type integral operator $F_q : H \to H$ is defined by the relation*

$$F_q(z) = \frac{[c]_q}{z^{c+1}}\int_0^z t^c h(t)d_q t + \overline{\frac{[c]_q}{z^{c+1}}\int_0^z t^c g(t)d_q t} \tag{4.3}$$

where $[c]_q$ is the q- number defined by (1.4) and H is the class of functions of the form (1.1), which are harmonic in U.

Theorem 4.4. *Let $f(z) = h(z) + \overline{g(z)} \in S_H$ be given by (1.6) and $f(z) \in TS_H(\alpha,\lambda,\mu,q)$ where $0 \leq \alpha < 1$, $0 \leq \mu \leq 1$, $0 \leq \lambda \leq 1$ and $0 < q < 1$. Then $F_q(z)$ defined by (4.3) is in the class $TS_H(\alpha,\lambda,\mu,q)$.*

Proof. Let

$$f(z) = z - \sum_{k=2}^{\infty}|a_k|z^k + \sum_{k=1}^{\infty}|b_k|\bar{z}^k. \tag{4.4}$$

Since $f \in TS_H(\alpha,\lambda,\mu,q)$, then by Theorem 2.2, we have

$$\sum_{k=2}^{\infty}\{\frac{\{[k]_q(1-\alpha\mu)-\alpha(1-\mu)\}}{1-\alpha}\}|a_k|\phi(k,\lambda)+$$
$$\sum_{k=1}^{\infty}\{\frac{\{[k]_q(1-\alpha\mu)+\alpha(1-\mu)\}}{1-\alpha}\}|b_k|\phi(k,\lambda) \leq 1.$$

From the representation of (4.3), we have

$$F_q(z) = z - \sum_{k=2}^{\infty}\frac{[c]_q}{[k+c+1]_q}|a_k|z^k + \sum_{k=1}^{\infty}\frac{[c]_q}{[k+c+1]_q}|b_k|\bar{z}^k.$$

since

$$[k+c+1]_q - [c]_q$$
$$= \sum_{i=0}^{k+c}q^i - \sum_{k=0}^{c-1}q^i$$
$$= \sum_{i=c}^{k+c}q^i > 0$$
$$\Rightarrow [k+c+1]_q > [c]_q \Rightarrow \frac{[c]_q}{[k+c+1]_q} < 1.$$

Now

$$\sum_{k=2}^{\infty} \frac{[k]_q(1-\alpha\mu)-\alpha(1-\mu)}{1-\alpha} \frac{[c]_q}{[c+k+1]_q} |a_k| \phi(k,\lambda) +$$
$$\sum_{k=1}^{\infty} \frac{[k]_q(1-\alpha\mu)+\alpha(1-\mu)}{1-\alpha} \frac{[c]_q}{[c+k+1]_q} |b_k| \phi(k,\lambda)$$

$$\leq \sum_{k=2}^{\infty} \frac{[k]_q(1-\alpha\mu)-\alpha(1-\mu)}{1-\alpha} |a_k| \phi(k,\lambda) +$$
$$\sum_{k=1}^{\infty} \frac{[k]_q(1-\alpha\mu)+\alpha(1-\mu)}{1-\alpha} |b_k| \phi(k,\lambda) \leq 1.$$

Therefore, by Theorem 2.2, we have $F_q(z) \in TS_H(\alpha,\lambda,\mu,q)$. □

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