A NEW SUBCLASS OF HARMONIC UNIVALENT
FUNCTIONS ASSOCIATED WITH q-CALCULUS

Saurabh PORWAL *1 and Manoj Kumar SINGH 2

Abstract

The purpose of the present paper is to introduce a new subclass of harmonic univalent functions by applying q-calculus. Coefficient inequalities, extreme points, distortion bounds, covering results, convolution condition and convex combination are determined for this class. Finally, we discuss a class preserving integral operator for this class.

2000 Mathematics Subject Classification: 30C45.

Key words: harmonic function, univalent function, q-calculus, fractional calculus.

1 Introduction

A continuous complex valued function \( f = u + iv \) defined in a simply-connected domain \( \mathbb{D} \) is said to be harmonic in \( \mathbb{D} \) if both \( u \) and \( v \) are real harmonic in \( \mathbb{D} \). In any simply-connected domain we can express \( f = h + \bar{g}, \) where \( h \) and \( g \) are analytic in \( \mathbb{D} \), called the analytic and co-analytic part of the function \( f \), respectively. The jacobian of the function \( f = h + \bar{g} \) is given by \( J_f(z) = |h'(z)|^2 - |g'(z)|^2 \). According to Lewy’s theorem every harmonic function \( f = h + \bar{g} \) is locally univalent and sense preserving in \( \mathbb{D} \) is that \( |h'(z)|^2 > |g'(z)|^2, \ z \in \mathbb{D} \). For detail study one may refer to Clunie and Sheil-Small [5], Duren [7], (see also [1, 2, 8, 15, 18]).

Further, we denote \( S_H \) the class of function \( f = h + \bar{g} \) which are harmonic, univalent and sense-preserving in the open unit disc \( \mathbb{D} = \{ z : |z| < 1 \} \) for which \( f(0) = f_z(0) - 1 = 0 \).

If \( f = h + \bar{g} \in S_H \) then \( h \) and \( g \) are of the form

\[
h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1.
\]

* Corresponding author, Department of Mathematics, Ram Sahai Government Degree College, Bairi-Shivrajpur, Kanpur-209205, (U.P.), India, e-mail: saurabhjcbr@rediffmail.com
1Department of Mathematics, Government Engineering College-Dahod, Gujarat-389151, India, e-mail: ms84ddu@gmail.com
It is worthy to note that for $g(z) = 0$ the class $S_H$ reduced to the class $S$ of analytic univalent functions. For this class $f(z)$ can be written as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (1.2)$$

The following definitions of fractional derivatives are given by Owa [12] and Srivastava and Owa [21]

**Definition 1.1.** The fractional derivative of order $\lambda$ is defined for a function $f(z)$ by

$$D_{z}^{\lambda}f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_{0}^{z} \frac{f(\xi)}{(z-\xi)^{1-\lambda}}d\xi, \quad (1.3)$$

where $0 \leq \lambda < 1$, $f(z)$ is an analytic function in a simply-connected region of the $z$ plane containing the origin and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring log$(z-\xi)$ to be real when $(z-\xi) > 0$.

**Definition 1.2.** Under the hypothesis of Definition 1.1, the fractional derivative of order $n + \lambda$ is defined for a function $f(z)$ by

$$D_{z}^{n+\lambda}f(z) = \frac{d^n}{dz^n} D_{z}^{\lambda}f(z),$$

where $0 \leq \lambda < 1$ and $n \in N_0 = \{0, 1, 2, 3, ...\}$.

Using the Definition 1.1 and 1.2, Owa and Srivastava [13] introduced the following fractional calculus operator

$\Omega_{\lambda} : A \rightarrow A$, which is defined as $\Omega_{\lambda} f(z) = \Gamma(2 - \lambda) z^{\lambda} D_{z}^{\lambda} f(z), (\lambda \neq 2, 3, ...)$, where $A$ denotes the class of functions $f$ of the form (1.2) which are analytic in $U$.

Recently, it has come to know that the concept of $q$-calculus is widely used in Geometric function theory. By using the definition of $q$-calculus various new subclasses of analytic and harmonic univalent functions were investigated by several researchers. In this direction noteworthy contribution may be found in [3], [11] and [16].

The concept of $q$-calculus were initially introduced by Jackson [9] (see also [4]). They defined the $q$-number for $k \in N$ in the following way

$$[k]q = \frac{1 - q^k}{1 - q}, 0 \leq q < 1. \quad (1.4)$$

It is easy to see that $[k]q$ can be represented as a geometric series in the following way $[k]q = \sum_{i=0}^{k-1} q^i$.

Obviously, $\lim_{k \to \infty} [k]q = \frac{1}{1-q}$ and $\lim_{q \to 1} [k]q = k$.

The $q$-derivative for a function $f$ is defined as

$$D_{q}(f(z)) = \frac{f(qz) - f(z)}{(q-1)z}, q \neq 1, z \neq 0.$$
and $D_q(f(0)) = f'(0)$, provided $f'(0)$ exists.

If we take the function $h(z) = z^k$ then the $q$-derivative of $h(z)$ is defined as

$$D_q(h(z)) = D_q(z^k) = \frac{1-q}{1-q} z^{k-1} = [k]_q z^{k-1}.$$  

Then $\lim_{q \to 1} D_q(h(z)) = \lim_{q \to 1}[k]_q(z^{k-1}) = k z^{k-1} = h'(z)$

where $h'$ is the ordinary derivative.

Now, we define the subclass $S_H(\alpha, \lambda, \mu, q)$ of $S_H$ consisting of functions $f$ of the form (1.1) satisfying the following condition

$$\mathbb{R} \left\{ \frac{z[D_q(\Omega^\lambda h(z))] - zD_q(\Omega^\lambda g(z))}{\mu(z[D_q(\Omega^\lambda h(z))] - z[D_q(\Omega^\lambda g(z))]) + (1-\mu)\{\Omega^\lambda h(z) + \Omega^\lambda g(z)\}} \right\} \geq \alpha \quad (1.5)$$

where $0 \leq \alpha < 1$, $0 \leq \mu < 1$, $0 \leq \lambda < 1$, $0 < q < 1$.

Next, we define $TS_H(\alpha, \lambda, \mu, q)$ be the subclass of $S_H(\alpha, \lambda, \mu, q)$ for which $f(z) = h(z) + g(z)$, where $h(z)$ and $g(z)$ are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k|z^k, g(z) = \sum_{k=1}^{\infty} |b_k|z^k, |b_1| < 1. \quad (1.6)$$

It should be worthy to note that for specific values of $\alpha$, $\mu$, $\lambda$, $q$ on the subclasses $S_H(\alpha, \lambda, \mu, q)$ and $TS_H(\alpha, \lambda, \mu, q)$, we obtain the following known subclasses of $S_H$ studied earlier by various researchers.

1. $S_H(\alpha, \lambda, \mu, 1) \equiv S_H(\alpha, \lambda, \mu)$ and $TS_H(\alpha, \lambda, \mu, 1) \equiv TS_H(\alpha, \lambda, \mu)$ studied by Porwal and Kanaujia [17].

2. $S_H(\alpha, \lambda, 0, 1) \equiv S_H(\alpha, \lambda)$ and $TS_H(\alpha, \lambda, 0, 1) \equiv TS_H(\alpha, \lambda)$ studied by Dixit and Porwal [6].

3. $S_H(\alpha, 0, \mu, 1) \equiv S_H(\alpha, \mu)$ and $TS_H(\alpha, 0, \mu, 1) \equiv TS_H(\alpha, \mu)$ studied by Öztürk et.al. [14].

4. $S_H(\alpha, 0, 0, 1) \equiv S_H^*(\alpha, )$ and $TS_H(\alpha, 0, 0, 1) \equiv TS_H^*(\alpha)$ studied by Jahangiri [10].

5. $S_H(0, 0, 0, 1) \equiv S_H^*$ and $TS_H(0, 0, 0, 1) \equiv TS_H^*$ studied by Silverman [19], Silverman and Silvia [20].

In the present paper, we obtain coefficient inequality, extreme points, distortion bounds, covering results, convolution condition and convex combination for the class $TS_H(\alpha, \lambda, \mu, q)$. Finally, we discuss a class preserving integral operator and $q$-Jackson type integral operator for this class.

## 2 Main results

In our first theorem, we give a sufficient coefficient bound for function in the class $S_H(\alpha, \lambda, \mu, q)$.
Theorem 2.1. Let \( f = h \mp g \) be such that \( h \) and \( g \) are given by (1.1). Furthermore let

\[
\sum_{k=2}^{\infty} \frac{[k]_q(1 - \alpha \mu) - \alpha(1 - \mu)}{1 - \alpha} \phi(k, \lambda)|a_k| + \sum_{k=1}^{\infty} \frac{[k]_q(1 - \alpha \mu) + \alpha(1 - \mu)}{1 - \alpha} \phi(k, \lambda)|b_k| \leq 1.
\]

(2.1)

where \( 0 \leq \alpha < 1 \), \( 0 \leq \mu < 1 \), \( 0 \leq \lambda < 1 \), \( 0 < q < 1 \) and

\[
\phi(k, \lambda) = \frac{\Gamma(k + 1)\Gamma(2 - \lambda)}{\Gamma(k + 1 - \lambda)}
\]

Then \( f \) is sense-preserving, harmonic univalent in \( U \) and \( f \in S_H(\alpha, \lambda, \mu, q) \).

Proof. First we note that \( f \) is locally univalent and sense-preserving in \( U \). This is because

\[
|h'(z)| \geq 1 - \sum_{k=2}^{\infty} k|a_k|r^{k-1}
\]

\[
> 1 - \sum_{k=2}^{\infty} k|a_k|
\]

\[
\geq 1 - \sum_{k=2}^{\infty} \frac{[k]_q(1 - \alpha \mu) - \alpha(1 - \mu)}{1 - \alpha} \phi(k, \lambda)|a_k|
\]

\[
\geq \sum_{k=1}^{\infty} \frac{[k]_q(1 - \alpha \mu) + \alpha(1 - \mu)}{1 - \alpha} \phi(k, \lambda)|b_k|
\]

\[
\geq \sum_{k=1}^{\infty} k|b_k|
\]

\[
\geq \sum_{k=1}^{\infty} k|b_k|r^{k-1}
\]

\[
\geq |g'(z)|.
\]

To Show that \( f \) is univalent in \( U \), suppose that \( z_1, z_2 \in U \) such that \( z_1 \neq z_2 \) then

\[
\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| \geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| = \left| \frac{1 - \sum_{k=1}^{\infty} b_k(z_1^k - z_2^k)}{z_1 - z_2 + \sum_{k=2}^{\infty} a_k(z_1^k - z_2^k)} \right|
\]

\[
> 1 - \frac{\sum_{k=2}^{\infty} k|b_k|}{1 - \sum_{k=2}^{\infty} k|a_k|} \geq 1 - \frac{\sum_{k=1}^{\infty} [k]_q(1 - \alpha \mu) + \alpha(1 - \mu)}{1 - \sum_{k=1}^{\infty} [k]_q(1 - \alpha \mu) - \alpha(1 - \mu)} \phi(k, \lambda)|b_k|\geq 0.
\]

Now, we show that \( f \in S_H(\alpha, \lambda, \mu, q) \), using the fact that \( \Re \{w\} \geq \alpha \), if and only if, \( |1 - \alpha + w| \geq |1 + \alpha - w| \) it suffices to show that

\[
|A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)| \geq 0,
\]

(2.2)
where $A(z) = z[D_q(\Omega^\lambda h(z))] - z[D_q(\Omega^\lambda g(z))]$ and
\[ B(z) = \mu z[D_q(\Omega^\lambda h(z))] - z[D_q(\Omega^\lambda g(z))] + (1 - \mu)\{\Omega^\lambda h(z) + \overline{\Omega^\lambda g(z)}\}. \]

Substituting the values of $A(z)$ and $B(z)$ in L.H.S. of (2.2) and performing the simple calculation, we obtain
\[
= |(2 - \alpha)z + \sum_{k=2}^{\infty} [k]_q + (1 - \alpha)\mu[k]_q + (1 - \alpha)(1 - \mu)\phi(k, \lambda)a_k z^k | - \sum_{k=1}^{\infty} [k]_q - (1 - \alpha)\mu[k]_q - (1 - \alpha)(1 - \mu)\phi(k, \lambda)b_k z^k |
\]
\[
= |\alpha z + \sum_{k=2}^{\infty} [k]_q - (1 - \alpha)\mu[k]_q - (1 - \alpha)(1 - \mu)\phi(k, \lambda)a_k z^k | - \sum_{k=1}^{\infty} [k]_q - (1 - \alpha)\mu[k]_q - (1 - \alpha)(1 - \mu)\phi(k, \lambda)b_k z^k |
\]
\[
\geq 2(1 - \alpha)z \left[ 1 - \sum_{k=2}^{\infty} \left\{ \frac{[k]_q(1 - \alpha \mu) - \alpha(1 - \mu)}{1 - \alpha} \right\} \phi(k, \lambda)|a_k| |z|^k \right] - \sum_{k=1}^{\infty} \left\{ \frac{[k]_q(1 - \alpha \mu) + \alpha(1 - \mu)}{1 - \alpha} \right\} \phi(k, \lambda)|b_k| |z|^k \right] \geq 0, \quad \text{(Using (2.1)).}
\]

The coefficient bound given by (2.1) is sharp because equality holds for the following functions
\[
f(z) = z + \sum_{k=2}^{\infty} \left\{ \frac{1 - \alpha}{[k]_q(1 - \alpha \mu) - \alpha(1 - \mu)} \right\} \phi(k, \lambda)x_k z^k
\]
\[
+ \sum_{k=1}^{\infty} \left\{ \frac{1 - \alpha}{[k]_q(1 - \alpha \mu) + \alpha(1 - \mu)} \right\} \phi(k, \lambda)y_k z^k,
\]
where
\[
\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1.
\]

This completes the proof of Theorem 2.1. \qed

In our next theorem, we prove that the condition (2.1) is also necessary for the function $f = h + \overline{g}$, where $h$ and $g$ are of the form (1.6).
Theorem 2.2. Let the function $f = h + \bar{g}$ be such that $h$ and $g$ are given by (1.6). Then $f \in S_H(\alpha, \lambda, \mu, q)$, if and only if

$$
\sum_{k=2}^{\infty} \{[k]_{q}(1 - \alpha \mu) - \alpha(1 - \mu)\} \phi(k, \lambda) |a_k| + \sum_{k=1}^{\infty} \{[k]_{q}(1 - \alpha \mu) + \alpha(1 - \mu)\} \phi(k, \lambda) |b_k| \leq 1 - \alpha,
$$

(2.3)

where $0 \leq \alpha < 1$, $0 \leq \mu < 1$, $0 \leq \lambda < 1$, $0 < q < 1$ and

$$
\phi(k, \lambda) = \frac{\Gamma(k + 1) \Gamma(2 - \lambda)}{\Gamma(k + 1 - \lambda)}.
$$

Proof. Since $TS_H(\alpha, \lambda, \mu, q) \subset S_H(\alpha, \lambda, \mu, q)$ this gives the if part of the theorem. To this end, for function $f$ of the form (1.6), we notice that the condition

$$
\Re \left\{ \frac{z[D_q(\Omega^\lambda h(z))] - z[D_q(\Omega^\lambda g(z))]}{\mu(z[D_q(\Omega^\lambda h(z))] - z[D_q(\Omega^\lambda g(z))]) + (1 - \mu)\{\Omega^\lambda h(z) + \Omega^\lambda g(z)\}} \right\} \geq \alpha
$$

is equivalent to

$$
\Re \left\{ \begin{array}{l}
(1 - \alpha)z - \sum_{k=2}^{\infty} \{[k]_{q} + (1 - \alpha \mu) - \alpha(1 - \mu)\} \phi(k, \lambda) |a_k| z^k \\
- \sum_{k=1}^{\infty} \{[k]_{q}(1 - \alpha \mu) + \alpha(1 - \mu)\} \phi(k, \lambda) |b_k| z^k \\
z - \sum_{k=2}^{\infty} \{[k]_{q}\mu + (1 - \mu)\} \phi(k, \lambda) |a_k| z^k \\
- \sum_{k=1}^{\infty} \{[k]_{q}\mu - (1 - \mu)\} \phi(k, \lambda) |b_k| z^k
\end{array} \right\} \geq \alpha.
$$

The above condition must holds for all values of $z$, $|z| = r < 1$. Upon choosing the values of $z$ on the positive real axis where $0 \leq z = r < 1$, we must have

$$
\Re \left\{ \begin{array}{l}
(1 - \alpha) - \sum_{k=2}^{\infty} \{[k]_{q} + (1 - \alpha \mu) - \alpha(1 - \mu)\} \phi(k, \lambda) |a_k| r^{k-1} \\
- \sum_{k=1}^{\infty} \{[k]_{q}(1 - \alpha \mu) + \alpha(1 - \mu)\} \phi(k, \lambda) |b_k| r^{k-1} \\
1 - \sum_{k=2}^{\infty} \{[k]_{q}\mu + (1 - \mu)\} \phi(k, \lambda) |a_k| r^{k-1} \\
- \sum_{k=1}^{\infty} \{[k]_{q}\mu - (1 - \mu)\} \phi(k, \lambda) |b_k| r^{k-1}
\end{array} \right\} \geq 0
$$

(2.4)

If the condition (2.3) does not hold then the numerator in (2.4) is negative for $r$ sufficiently close to 1. Thus there exists a $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.4) is negative. This contradicts the required condition for $f \in TS_H(\alpha, \lambda, \mu, q)$ and so the proof is complete.
A new subclass of harmonic univalent functions ... 155

Next, we determine the extreme points of closed convex hulls of $T S_H(\alpha, \lambda, \mu, q)$ denoted by $clcoTS_H(\alpha, \lambda, \mu, q)$.

**Theorem 2.3.** If $f \in clcoTS_H(\alpha, \lambda, \mu, q)$, if and only if

$$f(z) = \sum_{k=1}^{\infty} \{x_k h_k(z) + y_k g_k(z)\}$$  \hspace{1cm} (2.5)

where $h_1(z) = z$

$$h_k(z) = z - \frac{1 - \alpha}{[k]_q (1 - \alpha \mu) - \alpha (1 - \mu)} \phi(k, \lambda) z^k, (k = 2, 3, 4, ...)
$$

$$g_k(z) = z + \frac{1 - \alpha}{[k]_q (1 - \alpha \mu) + \alpha (1 - \mu)} \phi(k, \lambda) z^k, (k = 1, 2, 3, ...),$$

$$x_k \geq 0, y_k \geq 0, \sum_{k=1}^{\infty} (x_k + y_k) = 1.$$

In particular, the extreme points of $T S_H(\alpha, \lambda, \mu, q)$ are $\{h_k\}$ and $\{g_k\}$.

**Proof.** For function $f$ of the form (2.5), we have

$$f(z) = \sum_{k=1}^{\infty} \{x_k h_k(z) + y_k g_k(z)\}$$

$$= \sum_{k=1}^{\infty} (x_k + y_k) z - \sum_{k=2}^{\infty} \frac{1 - \alpha}{[k]_q (1 - \alpha \mu) - \alpha (1 - \mu)} \phi(k, \lambda) x_k z^k$$

$$+ \sum_{k=1}^{\infty} \frac{1 - \alpha}{[k]_q (1 - \alpha \mu) + \alpha (1 - \mu)} \phi(k, \lambda) y_k z^k,$$

Then

$$\sum_{k=2}^{\infty} \frac{[k]_q (1 - \alpha \mu) - \alpha (1 - \mu)}{1 - \alpha} \phi(k, \lambda) \left\{ \frac{1 - \alpha}{[k]_q (1 - \alpha \mu) - \alpha (1 - \mu)} \phi(k, \lambda) \right\} x_k$$

$$+ \sum_{k=1}^{\infty} \frac{[k]_q (1 - \alpha \mu) + \alpha (1 - \mu)}{1 - \alpha} \phi(k, \lambda) \left\{ \frac{1 - \alpha}{[k]_q (1 - \alpha \mu) + \alpha (1 - \mu)} \phi(k, \lambda) \right\} y_k$$

$$= \sum_{k=1}^{\infty} x_k + \sum_{k=2}^{\infty} y_k = 1 - x_1 \leq 1$$

and so $f \in clcoTS_H(\alpha, \lambda, \mu, q)$.

Conversely, suppose that $f \in clcoTS_H(\alpha, \lambda, \mu, q)$. Set

$$x_k = \frac{k (1 - \alpha \mu) - \alpha (1 - \mu)}{1 - \alpha} \phi(k, \lambda) |a_k|, (k = 2, 3, 4, ...)
$$

$$y_k = \frac{k (1 - \alpha \mu) + \alpha (1 - \mu)}{1 - \alpha} \phi(k, \lambda) |b_k|, (k = 1, 2, 3, ...).$$
then from Theorem 2.2, we have $0 \leq x_k \leq 1 (k = 2, 3, 4, \ldots)$ and $0 \leq y_k \leq 1 (k = 2, 3, 4, \ldots)$, we define

$$x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k$$

and note that by Theorem 2.2, $x_1 \geq 0$. Consequently, we obtain

$$f(z) = \sum_{k=1}^{\infty} \{x_k h_k(z) + y_k g_k(z)\}$$

as required.

In our next theorem, we obtain the bounds for function in $TS_H(\alpha, \lambda, \mu, q)$ which yields a covering results for this class.

**Theorem 2.4.** Let $f \in TS_H(\alpha, \lambda, \mu, q)$ then

$$|f(z)| \leq (1 + |b_1|)r + \left(\frac{1 - \alpha}{1 + q - \alpha(1 + q\mu)} - \frac{1 + \alpha(1 - 2\mu)}{1 + q - \alpha(1 + q\mu)} |b_1|\right) \frac{2 - \lambda}{2} r^2$$

and

$$|f(z)| \geq (1 - |b_1|)r - \left(\frac{1 - \alpha}{1 + q - \alpha(1 + q\mu)} - \frac{1 + \alpha(1 - 2\mu)}{1 + q - \alpha(1 + q\mu)} |b_1|\right) \frac{2 - \lambda}{2} r^2 |z|$$

$$= r < 1.$$

**Proof.** We only prove the right hand inequality. The proof for left hand inequality is similar and will be omitted. Let $f \in TS_h(\alpha, \lambda, \mu, q)$. Then taking the absolute value of $f$ we have

$$|f(z)| \leq (1 + |b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^2$$

$$= (1 + |b_1|)r + \frac{1 - \alpha}{\{1 + q - \alpha(1 + q\mu)\} \phi(2, \lambda)} \times$$

$$\times \sum_{k=2}^{\infty} \left(\frac{\{1 + q - \alpha(1 + q\mu)\} \phi(2, \lambda)}{1 - \alpha} (|a_k| + |b_k|) r^2 \right)$$

$$\leq (1 + |b_1|)r + \frac{(1 - \alpha)(2 - \lambda)}{2\{1 + q - \alpha(1 + q\mu)\}} \sum_{k=2}^{\infty} \left(\frac{\{k\} q(1 - \alpha\mu) - \alpha(1 - \mu)}{1 - \alpha} |a_k| + \frac{\{k\} q(1 - \alpha\mu) + \alpha(1 - \mu)}{1 - \alpha} |b_k|\right) \phi(k, \lambda) r^2$$

$$\leq (1 + |b_1|)r + \frac{(1 - \alpha)(2 - \lambda)}{2\{1 + q - \alpha(1 + q\mu)\}} \left(1 - \frac{1 + \alpha(1 - 2\mu)}{1 - \alpha} |b_1|\right) r^2$$

$$\leq (1 + |b_1|)r + \left(\frac{1 - \alpha}{1 + q - \alpha(1 + q\mu)} - \frac{1 + \alpha(1 - 2\mu)}{1 + q - \alpha(1 + q\mu)} |b_1|\right) \frac{2 - \lambda}{2} r^2.$$

$\square$
The following covering result follows from the left hand inequality of Theorem 2.4.

**Corollary 2.5.** Let \( f \in TSH(\alpha, \lambda, \mu, q) \). Then
\[
\left\{ w : |w| < \left( 1 - \frac{(1 - \alpha)(2 - \lambda)}{2(1 + q - \alpha(1 + q\mu))} \right) \left( 1 - \frac{[1 + \alpha(1 - 2\mu)](2 - \lambda)}{2(1 + q - \alpha(1 + q\mu))} \right) |b_1| \right\} \subset f(U).
\]

3 Convolution and convex combination

In this section, we prove that the class \( TSH(\alpha, \lambda, \mu, q) \) is closed under convolution and convex combination. Now, we need the following definition of convolution of two harmonic functions.

**Definition 3.1.** Let the function \( f(z) \) and \( F(z) \) be defined by
\[
f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| z^k
\]
and
\[
F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| z^k.
\]
Then the convolution of \( f(z) \) and \( F(z) \) are denoted by \( f(z) \ast F(z) \) and defined by
\[
(f \ast F)(z) = f(z) \ast F(z)
\]
\[
(f \ast F)(z) = z - \sum_{k=2}^{\infty} |a_k| |A_k| z^k + \sum_{k=1}^{\infty} |b_k| |B_k| z^k. \quad (3.1)
\]

Using this definition we show that the class \( TSH(\alpha, \lambda, \mu, q) \) is closed under convolution.

**Theorem 3.2.** For \( 0 \leq \beta \leq \alpha < 1 \), let \( f \in TSH(\alpha, \lambda, \mu, q) \) and \( F \in TSH(\beta, \lambda, \mu, q) \). Then \( (f \ast F) \in TSH(\alpha, \lambda, \mu, q) \).

**Proof.** Let \( f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| z^k \) be in \( TSH(\alpha, \lambda, \mu, q) \) and \( F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| z^k \) be in \( TSH(\beta, \lambda, \mu, q) \). Then the convolution \( (f \ast F)(z) \) is given by (3.1). To prove that \( (f \ast F) \in TSH(\alpha, \lambda, \mu, q) \), from Theorem 2.2 it is sufficient to show that
\[
\sum_{k=2}^{\infty} \left\{ \frac{|k| q (1 - \alpha \mu) - \alpha (1 - \mu)}{1 - \alpha} \right\} |a_k A_k| \phi(k, \lambda) +
\sum_{k=1}^{\infty} \left\{ \frac{|k| q (1 - \alpha \mu) + \alpha (1 - \mu)}{1 - \alpha} \right\} |b_k B_k| \phi(k, \lambda) \leq 1 - \alpha.
\]
Since \( F \in TSH(\beta, \lambda, \mu, q) \) then by Theorem 2.2, we obtain \( |A_k| \leq 1 \), and \( |B_k| \leq 1 \).
Now
\[
\sum_{k=2}^{\infty} \frac{\{k\} q(1 - \alpha \mu) - \alpha (1 - \mu)}{1 - \alpha} \left| a_k A_k \right| \phi(k, \lambda) + \sum_{k=1}^{\infty} \frac{\{k\} q(1 - \alpha \mu) + \alpha (1 - \mu)}{1 - \alpha} \left| b_k B_k \right| \phi(k, \lambda)
\]
\[
\leq \sum_{k=2}^{\infty} \frac{\{k\} q(1 - \alpha \mu) - \alpha (1 - \mu)}{1 - \alpha} \left| a_k \right| \phi(k, \lambda) + \sum_{k=1}^{\infty} \frac{\{k\} q(1 - \alpha \mu) + \alpha (1 - \mu)}{1 - \alpha} \left| b_k \right| \phi(k, \lambda)
\]
\[
\leq 1, \quad \text{since } (f)(z) \in TS_H(\alpha, \lambda, \mu, q).
\]

Therefore \((f * F)(z) \in TS_H(\alpha, \lambda, \mu, q)\).

In our next theorem, we prove that the class \(TS_H(\alpha, \lambda, \mu, q)\) is closed under convex combination.

**Theorem 3.3.** The class \(TS_H(\alpha, \lambda, \mu, q)\) is closed under convex combination.

**Proof.** For \(\alpha = 1, 2, 3, \ldots\) let \(f_i(z) \in TS_H(\alpha, \lambda, \mu, q)\) where \(f_i(z)\) is of the form

\[
f_i(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| z^k.
\]

Then from Theorem 2.2, we have

\[
\sum_{k=2}^{\infty} \frac{\{k\} q(1 - \alpha \mu) - \alpha (1 - \mu)}{1 - \alpha} \left| a_k \right| \phi(k, \lambda) + \sum_{k=1}^{\infty} \frac{\{k\} q(1 - \alpha \mu) + \alpha (1 - \mu)}{1 - \alpha} \left| b_k \right| \phi(k, \lambda) \leq 1. \tag{3.2}
\]

For \(\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1\), the convex combination of \(f_i\) may be written as

\[
\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{i=1}^{\infty} \left( \sum_{k=2}^{\infty} t_i |a_{k_i}| \right) z^k + \sum_{i=1}^{\infty} \left( \sum_{k=1}^{\infty} t_i |b_{k_i}| \right) z^k.
\]

Then by the condition (3.2), we have

\[
\sum_{k=2}^{\infty} \frac{\{k\} q(1 - \alpha \mu) - \alpha (1 - \mu)}{1 - \alpha} \left( \sum_{i=1}^{\infty} t_i |a_{k_i}| \right) \phi(k, \lambda) + \sum_{k=1}^{\infty} \frac{\{k\} q(1 - \alpha \mu) + \alpha (1 - \mu)}{1 - \alpha} \left( \sum_{i=1}^{\infty} t_i |b_{k_i}| \right) \phi(k, \lambda)
\]

\[
\leq 1,
\]
A new subclass of harmonic univalent functions ...

\[ \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=2}^{\infty} \left\{ \frac{[k]q(1-\alpha \mu) - \alpha(1-\mu)}{1-\alpha} \right\} |a_k| \phi(k, \lambda) + \right. \\
\left. \sum_{k=1}^{\infty} \left\{ \frac{[k]q(1-\alpha \mu) + \alpha(1-\mu)}{1-\alpha} \right\} |b_k| \phi(k, \lambda) \right\} \\
\leq \sum_{i=1}^{\infty} t_i \leq 1. \]

Then by Theorem 2.2, we have \( \sum_{i=1}^{\infty} t_i f_i(z) \in TS_H(\alpha, \lambda, \mu, q). \)

4 A family of class preserving integral operators

**Definition 4.1.** Let \( f(z) = h(z) + \overline{g(z)} \) be defined by (1.1) then \( F(z) \) be defined by the relation

\[ F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t) dt + \frac{c+1}{z^c} \int_0^z t^{c-1} g(t) dt, \quad (c > -1). \] (4.1)

**Theorem 4.2.** Let \( f(z) = h(z) + \overline{g(z)} \in S_H \) be given by (1.6) and \( f(z) \in TS_H(\alpha, \lambda, \mu, q) \) \( 0 \leq \alpha < 1, 0 \leq \mu \leq 1, 0 \leq \lambda \leq 1 \) and \( 0 < q < 1 \). Then \( F(z) \) defined by (4.1) is also in the class \( TS_H(\alpha, \lambda, \mu, q) \).

**Proof.** From the representation of (4.1) it follows that

\[ F(z) = z - \sum_{k=2}^{\infty} \frac{c+1}{z^c} \frac{a_k}{c+k} z^k + \sum_{k=1}^{\infty} \frac{c+1}{z^c} \frac{b_k}{c+k} z^k. \] (4.2)

Since \( f \in TS_H(\alpha, \lambda, \mu, q) \), we have

\[ \sum_{k=2}^{\infty} \left\{ \frac{[k]q(1-\alpha \mu) - \alpha(1-\mu)}{1-\alpha} \right\} |a_k| \phi(k, \lambda) + \right. \\
\left. \sum_{k=1}^{\infty} \left\{ \frac{[k]q(1-\alpha \mu) + \alpha(1-\mu)}{1-\alpha} \right\} |b_k| \phi(k, \lambda) \leq 1. \]

Now

\[ \sum_{k=2}^{\infty} \left\{ \frac{[k]q(1-\alpha \mu) - \alpha(1-\mu)}{1-\alpha} \right\} \frac{c+1}{z^c} \frac{a_k}{c+k} \phi(k, \lambda) + \right. \\
\left. \sum_{k=1}^{\infty} \left\{ \frac{[k]q(1-\alpha \mu) + \alpha(1-\mu)}{1-\alpha} \right\} \frac{c+1}{z^c} \frac{b_k}{c+k} \phi(k, \lambda) \leq \sum_{k=2}^{\infty} \left\{ \frac{[k]q(1-\alpha \mu) - \alpha(1-\mu)}{1-\alpha} \right\} |a_k| \phi(k, \lambda) + \right. \\
\left. \sum_{k=1}^{\infty} \left\{ \frac{[k]q(1-\alpha \mu) + \alpha(1-\mu)}{1-\alpha} \right\} |b_k| \phi(k, \lambda) \leq \sum_{k=2}^{\infty} \left\{ \frac{[k]q(1-\alpha \mu) - \alpha(1-\mu)}{1-\alpha} \right\} |a_k| \phi(k, \lambda) + \right. \\
\left. \sum_{k=1}^{\infty} \left\{ \frac{[k]q(1-\alpha \mu) + \alpha(1-\mu)}{1-\alpha} \right\} |b_k| \phi(k, \lambda) \leq 1. \]
\[
\sum_{k=2}^{\infty} \left\{ \frac{[k]_q(1-\alpha\mu) - \alpha(1-\mu)}{1-\alpha} \right\} |a_k| \phi(k, \lambda)
\]
\[
\sum_{k=1}^{\infty} \left\{ \frac{[k]_q(1-\alpha\mu) + \alpha(1-\mu)}{1-\alpha} \right\} |b_k| \phi(k, \lambda) \leq 1.
\]

Then by Theorem 2.2, we have \( F(z) \in TS_H(\alpha, \lambda, \mu, q) \).

Definition 4.3. Let \( f = h + \overline{g} \) be defined by (1.1). Then the q-Jackson type integral operator \( F_q : H \rightarrow H \) is defined by the relation
\[
F_q(z) = \left[ \frac{[c]_q}{z^{c+1}} \int_0^z t^c h(t) d_q t + \frac{[c]_q}{z^{c+1}} \int_0^z t^c g(t) d_q t \right] (4.3)
\]
where \([c]_q\) is the q-number defined by (1.4) and \( H \) is the class of functions of the form (1.1), which are harmonic in \( U \).

Theorem 4.4. Let \( f(z) = h(z) + \overline{g(z)} \in S_H \) be given by (1.6) and \( f(z) \in TS_H(\alpha, \lambda, \mu, q) \) where \( 0 \leq \alpha < 1 \), \( 0 \leq \mu \leq 1 \), \( 0 \leq \lambda \leq 1 \) and \( 0 < q < 1 \). Then \( F_q(z) \) defined by (4.3) is in the class \( TS_H(\alpha, \lambda, \mu, q) \).

Proof. Let
\[
f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| z^k. \tag{4.4}
\]
Since \( f \in TS_H(\alpha, \lambda, \mu, q) \), then by Theorem 2.2, we have
\[
\sum_{k=2}^{\infty} \left\{ \frac{[k]_q(1-\alpha\mu) - \alpha(1-\mu)}{1-\alpha} \right\} |a_k| \phi(k, \lambda) + \sum_{k=1}^{\infty} \left\{ \frac{[k]_q(1-\alpha\mu) + \alpha(1-\mu)}{1-\alpha} \right\} |b_k| \phi(k, \lambda) \leq 1.
\]
From the representation of (4.3), we have
\[
F_q(z) = z - \sum_{k=2}^{\infty} \frac{[c]_q}{[k + c + 1]_q} |a_k| z^k + \sum_{k=1}^{\infty} \frac{[c]_q}{[k + c + 1]_q} |b_k| z^k.
\]
\[
[k + c + 1]_q - [c]_q = \sum_{i=0}^{k+c} q^i - \sum_{i=0}^{c-1} q^i = \sum_{i=c}^{k+c} q^i > 0
\]
\[
\Rightarrow [k + c + 1]_q > [c]_q \Rightarrow \frac{[c]_q}{[k + c + 1]_q} < 1.
\]
A new subclass of harmonic univalent functions ...

Now

\[
\sum_{k=2}^{\infty} \frac{[k]_q(1 - \alpha \mu) - \alpha(1 - \mu)}{1 - \alpha} \frac{[c]_q}{[c + k + 1]_q} |a_k| \phi(k, \lambda) + \\
\sum_{k=1}^{\infty} \frac{[k]_q(1 - \alpha \mu) + \alpha(1 - \mu)}{1 - \alpha} \frac{[c]_q}{[c + k + 1]_q} |b_k| \phi(k, \lambda)
\]

\[
\leq \sum_{k=2}^{\infty} \frac{[k]_q(1 - \alpha \mu) - \alpha(1 - \mu)}{1 - \alpha} |a_k| \phi(k, \lambda) + \\
\sum_{k=1}^{\infty} \frac{[k]_q(1 - \alpha \mu) + \alpha(1 - \mu)}{1 - \alpha} |b_k| \phi(k, \lambda) \leq 1.
\]

Therefore, by Theorem 2.2, we have \( F_q(z) \in TS_H(\alpha, \lambda, \mu, q) \).

\[\square\]

Acknowledgment :

The authors would like thank the referee for their insightful suggestions to improve the paper in the present form.

References


