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ON THE MERSENNE AND MERSENNE-LUCAS HYBRID QUATERNIONS

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Abstract

In this paper, we define Mersenne, Mersenne-Lucas hybrid quaternions. We give the Binet's formula, the generating functions, exponential generating functions and sum formula of these quaternions. We find some relations between Mersenne-Lucas hybrid quaternions, Jacobsthal hybrid quaternions, Jacobsthal-Lucas hybrid quaternions and Mersenne hybrid quaternions.

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1 Introduction

Mersenne numbers are named after Marin Mersenne, who studied these numbers in the 17th century. Mersenne numbers are denoted by M_n and have the form $M_n = 2^n - 1$ (A000225). The first few terms of the Mersenne sequence are

$$0, 1, 3, 7, 15, 31, 63, 127, 255, \dots,\ 2^n - 1, \dots\ .$$

The Mersenne numbers $\{M_n\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$$M_{\boldsymbol{n+2}} = 3M_{n+1} - 2M_n$$

with $M_0 = 0$ and $M_1 = 1$ [1]. The Binet formula of the Mersenne numbers are defined by the following [1]:

$$M_n = 2^n - 1.$$

Similarly, the Mersenne-Lucas numbers $\{m_n\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$$m_{\boldsymbol{n}} = 3m_{n-1} - 2m_{n-2} \tag{1}$$

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with $m_0 = 2$ and $m_1 = 3$ [11].

The Binet formula for Mersenne- Lucas numbers is defined by [11]

$$m_n = 2^n + 1.$$

Mersenne, Mersenne-Lucas numbers have been studied by many authors and associated with other number sequences [1, 3, 4, 8, 11, 13, 14].

Moreover, we know well that the Jacobsthal numbers J_n are defined by the recurrence sequence

$$J_{n+2} = J_{n+1} + 2J_n, \quad n \geq 0$$

with the initial conditions $J_0 = 0$ and $J_1 = 1$.

Similarly, the Jacobsthal-Lucas numbers are defined by

$$j_{n+2} = j_{n+1} + 2j_n, \quad n \geq 0$$

with the initial conditions $j_0 = 2$ and $j_1 = 1$.

Additionally, these exists the following relationships between Mersenne, Mersenne-Lucas, Jacobsthal and Jacobsthal-Lucas. (see [14, 15])

$$i) \;\; m_n = \begin{cases} 3J_n \quad , & if \; n \; is \; even \\ 3J_n+2, & if \; n \; is \; odd \end{cases}$$

$$ii) \;\; m_n = \begin{cases} j_n \quad , & if \; n \; is \; even \\ j_n+2, & if \; n \; is \; odd \end{cases}$$

$$iii) \;\; M_n = \begin{cases} 3J_n, & if \; n \; is \; even \\ j_n, & if \; n \; is \; odd \end{cases} .$$

Quaternions were first described by Irish mathematician William Rowan Hamilton in 1843 and applied to mathematics in three-dimensional space [5]. Quaternions are generalizations of complex numbers and are not commutative.Quaternion algebra finds its application in fields such as robotics, navigation, computer visualization and animation, apart from mathematics. Quaternions are also vital to the control systems that guide airplanes and rockets. Quaternions have generally the following form,

$$q = a + ib + jc + kd$$

where, i, j, k are basis quaternions and a, b, c, d are real numbers [5].

The product rule for quaternion units is shown in Table 1.

.	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

Table 1: The multiplication of quaternion units

Quaternions have been studied by many researchers and have been studied with other number sequences [7, 9, 10]. Özdemir introduced the hybrid numbers and give some properties of these numbers. [6]. The set of hybrid numbers is

$$K = \{a + bi + c\varepsilon + dh \ : a, b, c, d \in \mathbb{R}\}$$

The product rule for hybrid units is shown in Table 2.

.	i	ε	h
i	-1	$1-h$	$\varepsilon+i$
ε	$1+h$	0	$-\varepsilon$
h	$-\varepsilon-i$	ε	1

Table 2: The multiplication of hybrid units

The authors studied hybrid quaternion numbers in their work in [2] and applied it to many number sequences.

In this paper, we define Mersenne, Mersenne-Lucas hybrid quaternions and give some properties of them. We prove some theorems about Mersenne, Mersenne-Lucas hybrid quaternions. In addition, we find Binet formulas, generating functions, exponential generating functions, sum formulas of these numbers.

2 Preliminaries

The Jacobsthal hybrid number, $\{JH_n\}_{n=0}^{\infty}$ is defined as

$$JH_n = J_n + J_{n+1}i + J_{n+2}\varepsilon + J_{n+3}h, \quad n \geq 0,$$

where J_n is the nth Jacobsthal number [2].

Similarly, the Jacobsthal-Lucas hybrid number, $\{jH_n\}_{n=0}^{\infty}$ is defined as

$$jH_n = j_n + j_{n+1}i + j_{n+2}\varepsilon + j_{n+3}h, \quad n \geq 0,$$

where j_n is the nth Jacobsthal-Lucas number [2].

The Mersenne hybrid number, $\{MH_n\}_{n=0}^{\infty}$ is defined as

$$MH_n = M_n + M_{n+1}i + M_{n+2}\varepsilon + M_{n+3}h, \ n \geq 0,$$

where M_n is the nth Mersenne number [14].

Let $n \geq 0$ be integer, Mersenne-Lucas hybrid numbers $\{mh_n\}$ for $n = 0, \ldots, \infty$ are defined as,

$$mh_n = m_n + m_{n+1}i + m_{n+2}\varepsilon + m_{n+3}h$$

where m_n is nth Mersenne-Lucas number [8].

The Binet formulas of the Mersenne, Mersenne-Lucas hybrid numbers as follows

$$i)\ MH_n = [2^n(1 + 2i + 4\varepsilon + 8h) - (1 + i + \varepsilon + h)] \ \ (see[14]), \tag{2}$$

$$ii)\ mh_n = [2^n(1 + 2i + 4\varepsilon + 8h) + (1 + i + \varepsilon + h)] \ \ (see[15]). \tag{3}$$

The sum of the $\{mh_n\}_{n=0}^{\infty}$ as follows [15].

$$\sum_{\mathrm{k=0}}^{\mathrm{n}} mh_k = 2mh_n + n - 2 + (n-3)\, i + (n-5)\, \varepsilon + (n-9)\, h.$$

The sum of the $\{MH_n\}_{n=0}^{\infty}$ as follows [14].

$$\sum_{\mathrm{k=0}}^{\mathrm{n}} MH_k = 2MH_n - (n + (n+1)i + (n+3)\varepsilon + (n+7)\, h). \tag{4}$$

By the following identities between the Mersenne hybrid number, Jacobsthal hyrid numbers and Jacobsthal-Lucas hyrid numbers are provided [14]:

$$i)\ MH_n + MH_{n+1} = 3\left(JH_n + JH_{n+1}\right) - 2\left(1 + i + \varepsilon + h\right), \tag{5}$$
$$ii)\ MH_n + MH_{n+1} = jH_n + jH_{n+1} - 2\left(1 + i + \varepsilon + h\right). \tag{6}$$

By the following identities between Mersenne-Lucas hybrid number, Jacobsthal hyrid numbers and Jacobsthal-Lucas hyrid numbers are provided [15]:

$$i)\ mh_n + mh_{n+1} = 3\left(JH_n + JH_{n+1}\right) + 2\left(1 + i + \varepsilon + h\right),$$
$$ii)\ mh_n + mh_{n+1} = jH_n + jH_{n+1} + 2 + 3i + 5\varepsilon + 5h.$$

There is a relationship between Mersenne-Lucas hybrid numbers and Mersenne hyrid numbers [15]:

$$mh_n = 2MH_{n+1} - 3MH_n.$$

3 Mersenne, Mersenne-Lucas hybrid quaternion

In this section, we introduce some properties Mersenne, Mersenne-Lucas hybrid quaternion. We find some relations between Mersenne-Lucas hybrid quaternion, Jacobsthal hybrid quaternion, Jacobsthal-Lucas hybrid quaternion and Mersenne hybrid quaternion.

Definition 1. *The Mersenne hybrid quaternion is defined as follows:*

$$MHQ_n = MH_n + MH_{n+1}i + MH_{n+2}j + MH_{n+3}k$$

where i, j, k are quaternion units (see Table 1) and MH_n is nth Mersenne hybrid number [3].

The Mersenne hybrid quaternion can be written as follows:

$$MHQ_n = MQ_n + MQ_{n+1}i + MQ_{n+2}\varepsilon + MQ_{n+3}h$$

where i, ε, h are hybrid units(see Table 2) and MQ_n is nth the Mersenne quaternion number [3], which is defined by

$$MQ_n = M_n + iM_{n+1} + jM_{n+2} + kM_{n+3}h.$$

Definition 2. *The Mersenne-Lucas hybrid quaternion is defined as follows:*

$$mhq_n = mh_n + mh_{n+1}i + mh_{n+2}j + mh_{n+3}k$$

where i, i, k *are quaternion unit and* mh_n *is nth the Mersenne-Lucas hybrid number [3].*

The Mersenne-Lucas hybrid quaternion can be written as follows:

$$mhq_n = mq_n + mq_{n+1}i + mq_{n+2}\varepsilon + mq_{n+3}h$$

where i, ε, h are hybrid unit and mq_n is nth the Mersenne-Lucas quaternion number [3], which is defined by

$$mq_n = m_n + im_{n+1} + jm_{n+2} + km_{n+3}$$

We will use the Definition 1 and Definition 2 in this study.

Theorem 1. *The Binet's formula of the Mersenne and Mersenne-Lucas hybrid quaternion are defined by*

$$\textbf{\textit{a.}}\ MHQ_n = 2^n\ \underline{2}\overline{2} - \underline{1}\overline{1},$$
$$\textbf{\textit{b.}}\ mhq_n = 2^n\underline{2}\overline{2}\ + \underline{1}\overline{1},$$

where

$$\overline{2} = 1 + 2i + 4j + 8k,\ \underline{2} = 1 + 2i + 4\varepsilon + 8h,$$
$$\overline{1} = 1 + i + j + k,\ \underline{1} = 1 + i + \varepsilon + h.$$

Proof. By using Eq. (2), we have

$$\begin{aligned}
\textbf{\textit{a.}}\ MHQ_n &= MH_n + MH_{n+1}i + MH_{n+2}j + MH_{n+3}k \\
&= [2^n(1 + 2i + 4\varepsilon + 8h) - (1 + i + \varepsilon + h)] \\
&\quad + \left[2^{n+1}(1 + 2i + 4\varepsilon + 8h) - (1 + \ i + \varepsilon + h)\right] i \\
&\quad + \left[2^{n+2}(1 + 2i + 4\varepsilon + 8h) - (1 + i + \varepsilon + h)\right] j \\
&\quad + \left[2^{n+3}(1 + 2i + 4\varepsilon + 8h) - (1 + i + \varepsilon + h)\right] k \\
&= (2^n\underline{2} - \underline{1}) + \left(2^{n+1}\underline{2} - \underline{1}\right) i + \left(2^{n+2}\underline{2} - \underline{1}\right) j + (2^{n+3}\underline{2} - \underline{1})k \\
&= 2^n\underline{2} - \underline{1} + 2^{n+1}\underline{2}i - \underline{1}i + 2^{n+2}\underline{2}j - \underline{1}\ j + 2^{n+3}\underline{2}k - \underline{1}k \\
&= 2^n\underline{2}(1 + 2i + 4j + 8k) - \underline{1}(1 + i + j + k) \\
&= 2^n\ \underline{2}\overline{2} - \underline{1}\overline{1}.
\end{aligned}$$

Thus, the proof is completed.
The proof of ***b*** is done similarly to ***a***. □

Theorem 2. *The recurrence relation of the Mersenne, Mersenne-Lucas hybrid quaternion as follows:*

$$mhq_{n+2} = 3mhq_{n+1} - 2mhq_n$$

for $n \geq 0$.

Proof. We will use Eq. (1) for the proof. We have

$$\begin{aligned} mhq_{n+2} &= mhq_{n+2} + mhq_{n+3}i + mhq_{n+4}j + mhq_{n+5}k \\ &= 3mh_{n+1} - 2mh_n + (3mh_{n+2} - 2mh_{n+1})\,i \\ &\quad + (3mh_{n+3} - 2mh_{n+2})\,j + (3mh_{n+4} - 2mh_{n+3})k \\ &= 3(mh_{n+1} + mh_{n+2}i + mh_{n+3}j + mh_{n+4}k) \\ &\quad - 2(mh_n + mh_{n+1}i + mh_{n+2}j + mh_{n+3}j) \\ &= 3mhq_{n+1} - 2mhq_n. \end{aligned}$$

Thus, the proof is completed. □

Theorem 3. *The generating functions of the Mersenne, Mersenne-Lucas hybrid quaternion are*

$$\textbf{\textit{a}}.\ \sum_{n=0}^{\infty} MHQ_n x^n = \frac{MHQ_0 + x(MHQ_1 - 3MHQ_0)}{1 - 3x - 2x^2},$$

$$\textbf{\textit{b}}.\ \sum_{n=0}^{\infty} mhq_n x^n = \frac{mhq_0 + x(mhq_1 - 3mhq_0)}{1 - 3x - 2x^2}.$$

Proof. ***a.*** Suppose that the generating function of the Mersenne hybrid quaternion sequence has the form

$$G(x) = \sum_{n=0}^{\infty} MHQ_n x^n = MHQ_0 + MHQ_1 x + MHQ_2 x^2 + \cdots + MHQ_n x^n + \ldots$$

Then, we have

$$\begin{aligned} -3xG(x) &= \ MHQ_0 3x - MHQ_1 3x^2 - MHQ_2 3x^3 - \cdots - MHQ_n 3x^{n+1} - \ldots \\ 2x^2 G(x) &= \ MHQ_0 2x^2 + MHQ_1 2x^3 + MHQ_2 2x^4 + \cdots + MHQ_n 2x^{n+2} + \ldots \end{aligned}$$

It follows that

$$\begin{aligned} G(x)\left(1 - 3x - 2x^2\right) &= MHQ_0 + x\left(MHQ_1 - 3MHQ_0\right) \\ &\quad + x^2(MHQ_2 - 3MHQ_1 + 2MHQ_0) \\ &\quad + x^3\left(MHQ_3 - 3MHQ_2 + 2MHQ_1\right). \end{aligned}$$

If necessary calculations are taken, the desired will be achieved:

$$G(x) = \frac{MHQ_0 + x\left(MHQ_1 - 3MHQ_0\right)}{1 - 3x - 2x^2}.$$

Thus, the proof is completed. The proof for ***b*** is done similarly. □

Theorem 4. *The exponential generating functions of the Mersenne, Mersenne-Lucas hybrid quaternions are given by*

$$\textbf{a.}\quad \sum_{n=0}^{\infty} MHQ_n \frac{x^n}{n!} = \left(\underline{2}\overline{2}\ \ e^x - \underline{1}\overline{1}\ \right) e^x,$$

$$\textbf{b.}\quad \sum_{n=0}^{\infty} mhq_n \frac{x^n}{n!} = \left(\ \underline{2}\overline{2}\ e^x + \underline{1}\overline{1}\ \right) e^x.$$

Proof.

$$\begin{aligned}
\textbf{a.}\quad \sum_{n=0}^{\infty} MHQ_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \left(2^n \underline{2}\overline{2}\ \ - \underline{1}\overline{1}\ \right)\ \frac{x^n}{n!} \\
&= \underline{2}\overline{2} \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} - \underline{1}\overline{1}\ \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
&= \underline{2}\overline{2}\ e^{2x} - \underline{1}\overline{1}\ e^x \\
&= \left(\underline{2}\ \overline{2}\ e^x - \underline{1}\overline{1}\ \right) e^x.
\end{aligned}$$

Thus, the proof is completed.
The proof for ***b*** is done similarly. □

Theorem 5. *The sum of the Mersenne, Mersenne-Lucas hybrid quaternions are given by*

$$\textbf{a.}\quad \sum_{k=0}^{n} MHQ_k = 2MHQ_n - \underline{n} - Ai - Bj - Ck,$$

$$\textbf{b.}\quad \sum_{k=0}^{n} mhq_n = 2mhq_n + \overline{(n-2)} + A^*i + B^*j + C^*k,$$

where

1. $\underline{n} = n + (n+1)\, i + (n+3)\, \varepsilon + (n+7)\, h,$
2. $\overline{n} = n + (n-1)\, i + (n-3)\, \varepsilon + (n-7)\, h,$
3. $A = \underline{(n+1)} + MH_0, \quad B = \underline{(n+2)} + MH_0 + MH_1,$
   $C = \underline{(n+3)} MH_0 + MH_1 + MH_2,$
4. $A^* = \ \overline{(n-1)} - mh_0, \quad B^* = \overline{(n-2)} - mh_0 - mh_1,$
   $C^* = \overline{(n-3)} - mh_0 - mh_1 - mh_2.$

Proof. Let us write

$$\textbf{a.}\quad \sum_{k=0}^{n} MHQ_k = MHQ_0 + MHQ_1 + MHQ_2 + \cdots + MHQ_n.$$

So, we deduce

$$\begin{aligned}\sum_{k=0}^{n} MHQ_k &= (MH_0 + MH_1 i + MH_2 j + MH_3 k) + (MH_1 + MH_2 i \\ &\quad + MH_3 j + MH_4 k) + \cdots + (MH_n + MH_{n+1} i + MH_{n+2} j \\ &\quad + MQ_{n+3} k) \\ &= (MH_0 + MH_1 + \cdots + MH_n) + (MH_1 + \cdots + MH_{n+1}) i \\ &\quad + (MH_2 + \cdots + MH_{n+2})\, j + (MH_3 + \cdots + MH_{n+3})\, k.\end{aligned}$$

From Eq. (4), we have

$$\begin{aligned}\sum_{k=0}^{n} MHQ_k &= [2MH_n - (n + (n+1)\, i + (n+3)\, \varepsilon + (n+7)\, h)] \\ &\quad + [2MH_{n+1} - (n + 1 + (n+2)\, i + (n+4)\, \varepsilon + (n+8)\, h) - MH_0]\, i \\ &\quad + [2MH_{n+2} - (n + 2 + (n+3)\, i + (n+5)\, \varepsilon + (n+9)\, h) - MH_0 - MH_1]\, j \\ &\quad + [2MH_{n+3} - (n + 3 + (n+4)\, i + (n+6)\, \varepsilon + (n+10)\, h) - MH_0 - MH_1 \\ &\qquad - MH_2]\, k \\ &= 2\, (MH_n + MH_{n+1} i + MH_{n+2} j + MH_{n+3} k) - \underline{n} + (-\underline{(n+1)} i - MH_0) \\ &\quad + (-\underline{(n+2)} - MH_0 - MH_1) j + (-\underline{(n+3)} - MH_0 - MH_1 - MH_2) k \\ &= 2MHQ_n - \underline{n} - Ai - Bj - Ck.\end{aligned}$$

Thus, the proof is completed.
Similarly, the proof of ***b*** is can be done. □

Theorem 6. *The following equation is provided:*

$$MHQ_{n+1} = 2MHQ_n + \underline{1}\ \overline{1}.$$

Proof. From Theorem 1. ***a***, we have

$$\begin{aligned}2MHQ_n + \underline{1}\ \overline{1} &= 2(2^n\ \underline{2}\overline{2} - \underline{1}\overline{1}) + \underline{1}\ \overline{1} \\ &= 2^{n+1}\underline{2}\overline{2} - 2\underline{1}\overline{1} + \underline{1}\ \overline{1} \\ &= 2^{n+1}\underline{2}\overline{2} - \underline{1}\overline{1} \\ &= MHQ_{n+1}.\end{aligned}$$

Thus, the proof is completed. □

Theorem 7. *The following equations are provided:*

- ***a.*** *if n even,* $MHQ_n + MHQ_{n+1} = 3\left(JHQ_n + JHQ_{n+1}\right) - 2\ \underline{1}\ \overline{1}$,
- ***b.*** *if n odd,* $MHQ_n + MHQ_{n+1} = jHQ_n + jHQ_{n+1} - 2\ \underline{1}\ \overline{1}$,
- ***c.*** *if n even,* $mhq_n + mhq_{n+1} = 3\left(JHQ_n + JHQ_{n+1}\right) + 2\ \underline{1}\ \overline{1}$,
- ***d.*** *if n odd,* $mhq_n + mhq_{n+1} = jHQ_n + jHQ_{n+1} - 2\ \underline{1}\ \overline{1}$.

Proof. ***a.*** if n is even , by using Eq.(5), we have

$$\begin{aligned} MHQ_n + MHQ_{n+1} &= MH_n + MH_{n+1}i + MH_{n+2}j + MH_{n+3}k \\ &\quad + MH_{n+1} + MH_{n+2}i + MH_{n+3}j + MH_{n+4}k \\ &= (3\,(JH_n + JH_{n+1}) - 2\ \underline{1}\ ) + (3\,(JH_{n+1} + JH_{n+2}) - 2\ \underline{1}\ )\,i \\ &\quad + (3\,(JH_{n+2} + JH_{n+3}) - 2\ \underline{1}\ )\,j \\ &\quad + (3\,(JH_{n+3} + JH_{n+4}) - 2\ \underline{1}\ )\,k \\ &= 3JHQ_n + 3JHQ_{n+1} - 2\ \underline{1}\ (1 + i + j + k) \\ &= 3\left(JHQ_n + JHQ_{n+1}\right) - 2\ \underline{1}\ \overline{1}. \end{aligned}$$

Thus, the proof is completed.
The other options of the theorem can be shown similarly. □

4 Conclusion

In this work, we are discussed the Mersenne and Mersenne-Lucas hybrid quaternions and their properties. We obtained the Binet's formula, the generating functions, exponential generating functions and sum formulas of these numbers. Additionally, we find some relations between Mersenne-Lucas hybrid quaternion, Jacobsthal hybrid quaternion, Jacobsthal-Lucas hybrid quaternion and Mersenne hybrid quaternion.

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